H-principle for complex contact structures on Stein manifolds

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Arnold: “Contact geometry is all geometry”

I have recently become interested in **holomorphic directed (Pfaffian) systems**, in particular, **holomorphic contact systems**.

Contact geometry is intimately connected with symplectic geometry, Riemannian geometry, CR geometry, and many other areas. It has been used to prove big results in differential topology.

**Cerf 1964** Every diffeomorphism of \( S^3 = \partial B^4 \) extends to a diffeomorphism of \( B^4 \).

**Eliashberg 1989, 1992** discovered a different proof based on his classification of contact structures on \( S^3 \):

- **tight**: the standard contact structure is unique up to isotopy
- **overtwisted**: countable infinity of distinct ones, classified homotopically (first shown to exist by Bennequin in 1982)

**Martinet 1971** Every orientable closed 3-manifold admits a contact structure (in fact an overtwisted one).
A complex contact manifold is a pair $(X, \xi)$ where

- $X$ is a complex manifold of odd dimension $2n + 1 \geq 3$, and
- $\xi$ is a holomorphic hyperplane subbundle of the tangent bundle $TX$ which is maximally nonintegrable, in the sense that the following bilinear pairing is nondegenerate:

$$O : \xi \otimes \xi \to TX/\xi = L, \quad (v, w) \mapsto [v, w] \mod \xi$$

- Equivalently, every point $p \in X$ has a neighborhood $U \subset X$ such that $\xi|_U = \ker \alpha$, where $\alpha$ is a holomorphic 1-form on $U$ satisfying

$$\alpha \wedge (d\alpha)^n \neq 0.$$ 

Such $\xi$ is a holomorphic contact subbundle on $X$, and a holomorphic 1-form $\alpha$ satisfying the above condition is a holomorphic contact form.
Darboux’s theorem and stability results

Contact manifolds \((X, ξ)\) and \((X', ξ')\) are **contactomorphic** if there exists a diffeomorphism (biholomorphism) \(F: X \to X'\) satisfying

\[
dF_x(ξ_x) = ξ'_F(x) \text{ for all } x \in X.
\]

**Example (model contact space)**

\[
X \in \{\mathbb{R}^{2n+1}, \mathbb{C}^{2n+1}\}, \quad ξ_0 = \ker α_0, \quad α_0 = dz + \sum_{j=1}^{n} x_j dy_j.
\]

**Darboux 1882; Engel 1989; Cartan 1901; Moser 1965** Every contact manifold \((X^{2n+1}, ξ)\) is locally contactomorphic to this model.

**Gray 1959** If \(X\) is a compact manifold then contact structures in an isotopy \(\{ξ_t\}_{t \in [0,1]}\) are pairwise contactomorphic, i.e., there is an isotopy \(\{f_t\}_{t \in [0,1]} \subset \text{Diff}(X)\) such that \((df_t)(ξ_0) = ξ_t\).
The normal bundle of a contact structure

**Le Brun & Salamon 1994** A contact subbundle $\xi \subset TX$ is given by a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L)) = H^0(X, T^*X \otimes L)$ with values in the holomorphic line bundle $L = TX/\xi$ (the **normal bundle** of $\xi$):

$$0 \longrightarrow \xi \longrightarrow TX \overset{\alpha}{\longrightarrow} L \longrightarrow 0.$$ 

If $f$ is a holomorphic function then $d(f\alpha) = df \wedge \alpha + fd\alpha$, so

$$d\alpha|_{\xi}$$

is a section of $\Lambda^2(\xi^*) \otimes L$.

Letting $K_X = \Lambda^{2n+1}(T^*X)$ (the canonical bundle of $X$), it follows that

$$\alpha \wedge (d\alpha)^n \neq 0$$

is a trivialisation of $K_X \otimes L^{\otimes(n+1)}$.

This provides a holomorphic line bundle isomorphism

$$L^{\otimes(n+1)} \cong K_X^{-1} = \Lambda^{2n+1}(TX).$$
The space $\text{Cont}(X)$ of holomorphic contact structures

Conversely, assume $X^{2n+1}$ is a complex manifold with $H^1(X, \mathbb{Z}_{n+1}) = 0$ and $c_1(TX)$ divisible by $n + 1$. Then there exists the line bundle

$$L = K_X^{(-1)/(n+1)}, \quad L^{\otimes (n+1)} \cong K_X^{-1}.$$

Given a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$, consider

$$\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \mathcal{O}(X).$$

If $X$ is compact then $\mathcal{O}(X) = \mathbb{C}$. If the constant $\alpha \wedge (d\alpha)^n \in \mathbb{C}$ is nonzero then $\alpha$ is a contact form on $X$. The map

$$\Gamma(X, \Omega^1(L)) \ni \alpha \longmapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}$$

is homogeneous of degree $n + 1$.

Hence, the space $\text{Cont}(X)$ is either empty or the complement of a degree $n + 1$ hypersurface in $\mathbb{P}(\Gamma(X, \Omega^1(L)))$. 
Example: A (unique) contact structure on $\mathbb{CP}^{2n+1}$

Let $z_1, \ldots, z_{2n+2}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$  

Then, $\theta$ defines a contact structure on $\mathbb{CP}^{2n+1}$. Let $\theta_j$ be the pull-back of $\theta$ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$  

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots.$$  

Then $(H_j, \theta_j)$ is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each $j$, and this collection forms a contact atlas on $X = \mathbb{CP}^{2n+1}$. We have

$$K_X^{-1} = \mathcal{O}_X(2n+2), \quad L = K_X^{-1/(n+1)} = \mathcal{O}_X(2),$$  

$$\alpha = [\theta] \in \Gamma(\mathbb{CP}^{2n+1}, \Omega^1(2)).$$

This contact structure is unique by Gray-Le Brun-Salamon theorem.
Compact complex contact manifolds are very special

**Le Brun, Salamon 1994** Any two complex contact structures on a simply connected compact complex manifold are contactomorphic.

**Demailly 2002** If a compact Kähler manifold $X$ admits a complex contact structure, then $\kappa_X = -\infty$.

**Examples of projective complex contact manifolds:**
(a) $\mathbb{P} T^* Z$, where $Z$ is projective.
(b) Unique closed orbit $X_G$ of the adjoint action of a simple complex Lie group $G$ on $\mathbb{P} g$. Then $X_G$ is Fano (i.e., $K_X^{-1}$ is ample), e.g. $\mathbb{P}^{2n+1}$.

**Conjecture:** These are the only examples.

**Ye 1994** True in dimension 3. Uses the minimal model program.

**Kebekus et al. (2000), Demailly 2002** A contact compact Kähler manifold $X$ not of type $\mathbb{P} T^* Z$ is Fano with $b_2 = 1$.

**Equivalent conjecture (Wolf)** $X$ as above is homogeneous.
Contact structures on Stein manifolds

Assume now that \( X \) is a **Stein manifold** of dimension \( 2n + 1 \geq 3 \). For a generic holomorphic 1-form \( \alpha \) on \( X \), the equation

\[
\alpha \wedge (d\alpha)^n = 0
\]

defines a (possibly empty) complex hypersurface \( \Sigma_\alpha \subset X \), and \( \alpha \) is a contact form on the Stein manifold \( X \setminus \Sigma_\alpha \).

This observation shows that there exist a plethora of Stein contact manifolds, but it does not answer the question whether a given Stein manifold (or a given diffeomorphism class of Stein manifolds) admits a contact structure. More precisely, when is a complex hyperplane subbundle \( \xi \subset TX \) satisfying the necessary condition

\[
\Lambda^{2n} \xi \cong L^n = (TX/\xi)^n
\]

homotopic to a holomorphic contact subbundle?

**How many nonequivalent contact structures are there on \( \mathbb{C}^3 \)?**

No one seems to have a slightest clue.
A hyperbolic contact structure on $\mathbb{C}^{2n+1}$

The **Kobayashi pseudometric** associated to a holomorphic contact structure is defined by using holomorphic Legendrian discs.


For any $n \geq 1$ there exists a holomorphic contact structure $\xi$ on $\mathbb{C}^{2n+1}$ which is **Kobayashi hyperbolic** and isotopic to $\xi_0$. In particular, every holomorphic Legendrian curve $\mathbb{C} \to (\mathbb{C}^{2n+1}, \xi)$ is constant.

**Idea of proof:** We take $\alpha = \Phi^*\alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j$ and $\Phi: \mathbb{C}^{2n+1} \to \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image $\Omega$ avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^{2n}(x,y) \times C_N \mathbb{D}_z.$$

Assuming that $C_N \geq n2^{3N+1}$ for all $N \in \mathbb{N}$,

$\mathbb{C}^{2n+1} \setminus K$ is $\alpha_0$-hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^*\alpha_0)$ is hyperbolic.
Formal contact structures

**Definition**

Let $X$ be a complex manifold of dimension $2n + 1 \geq 3$. A formal complex contact structure on $X$ is a pair $(\alpha, \beta)$, where

- $\alpha$ is a smooth $(1, 0)$-form on $X$ with values in a line bundle $L \to X$,
- $\beta$ is a smooth $(2, 0)$-form on $\xi = \ker \alpha$ with values in $L$, and
- $\alpha \wedge \beta^n \neq 0$ at each point of $X$.

We denote by

$$\text{Cont}_{\text{for}}(X)$$

the space of all formal complex contact structures on $X$.

The existence of a formal contact structure on $X$ implies the same conditions

$$K_X \otimes L^{n+1} \cong X \times \mathbb{C} \cong \Lambda^{2n} \xi^* \otimes L^n.$$
The Main Theorem

We have the natural inclusion

\[ \text{Cont}(X) \hookrightarrow \text{Cont}_{\text{for}}(X), \quad \alpha \mapsto (\alpha, d\alpha|_{\xi = \ker \alpha}). \]

**Theorem**

Let \( X \) be a Stein manifold of odd dimension. Given \((\alpha_0, \beta_0) \in \text{Cont}_{\text{for}}(X)\), there are a homotopy \((\alpha_t, \beta_t) \in \text{Cont}_{\text{for}}(X)\) \((t \in [0, 1])\) and a Stein domain \( \Omega \subset X \), diffeotopic to \( X \), such that

\[ \alpha_1|_{\Omega} \in \text{Cont}(\Omega) \quad \text{and} \quad \beta_1|_{\ker \alpha_1} = d\alpha_1 \text{ on } \Omega. \]

Furthermore, if \( Q \subset P \) are compact Hausdorff spaces and \( \{(\alpha_p, \beta_p)\}_{p \in P} \in \text{Cont}_{\text{for}}(X) \) is a continuous family such that

\[ \forall p \in Q : \quad \alpha_p \in \text{Cont}(X) \quad \text{and} \quad \beta_p = d\alpha_p|_{\ker \alpha_p}, \]

then there are a Stein domain \( \Omega \subset X \) diffeotopic to \( X \) and a homotopy \((\alpha_{p,t}, \beta_{p,t}) \in \text{Cont}_{\text{for}}(\Omega)\) \((p \in P, \ t \in [0, 1])\) which is fixed for all \( p \in Q \) such that \( \alpha_{p,1} \in \text{Cont}(\Omega) \) and \( \beta_p = d\alpha_p|_{\ker \alpha_p} \) for every \( p \in P \).
Why restricting to a Stein domain $\Omega \subset X$?

**Problem**

Given a holomorphic contact form $\alpha$ on an open neighbourhood of a compact convex set $K \subset \mathbb{C}^{2n+1}$, is it possible to approximate $\alpha$ uniformly on $K$ by holomorphic contact forms on $\mathbb{C}^{2n+1}$?

Is this also possible for any continuous family of holomorphic contact forms $\alpha_p$ with parameter $p \in P$ in a compact Hausdorff space?

The corresponding problem for holomorphic foliations is also open and very challenging. **These issues do not appear in the smooth case.**

**Theorem**

*If the above problem has an affirmative answer, then the inclusion $\text{Cont}(X) \hookrightarrow \text{Cont}_{\text{for}}(X)$ is a weak homotopy equivalence. This holds true for germs of contact structures along any (stratified) totally real submanifold $M \subset X$.***
If $X$ is a Stein manifold of dimension 3, then the connected components of $\text{Cont}_{\text{for}}(X)$ are classified by the following pairs of data:

(i) an isomorphism class of a complex line bundle $L$ on $X$ satisfying $L^2 \cong (K_X)^{-1}$ (equivalently, a cohomology class $c \in H^2(X; \mathbb{Z})$ such that $2c = c_1(TX)$), and

(ii) a choice of a homotopy class of trivialisations of $K_X \otimes L^2 \cong \Lambda^2 \zeta^* \otimes L \cong X \times \mathbb{C}$, that is, an element of the 1st cohomology group $[X, \mathbb{C}^*] = [X, S^1] = H^1(X; \mathbb{Z})$.

In particular, if $H^1(X; \mathbb{Z}) = 0$ and $H^2(X; \mathbb{Z}) = 0$ then the space $\text{Cont}_{\text{for}}(X)$ is connected; this holds for $X = \mathbb{C}^3$. 

Let $X$ be a Stein threefold with $H^1(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) = 0$. Is the space $\text{Cont}(X)$ connected? In particular, is $\text{Cont}(\mathbb{C}^3)$ connected?
A holomorphic contact bundle $\xi$ on $X$ is determined by a holomorphic 1-form $\alpha$ up to a nonvanishing factor $f \in \mathcal{O}(X, \mathbb{C}^*)$. Since

$$f\alpha \land d(f\alpha)^n = f^{n+1}\alpha \land d\alpha,$$

this changes the trivialisation of $K_X \otimes L^{n+1}$ by the factor $f^{n+1}$ where $\dim X = 2n + 1$ (by $f^2$ if $\dim X = 3$).

**Corollary**

A homotopy class of holomorphic contact bundles on a Stein 3-fold $X$ is uniquely determined by a pair $(c, d)$, where

$$c \in H^2(X; \mathbb{Z}), \quad 2c = c_1(TX); \quad d \in H^1(X; \mathbb{Z})/2H^1(X; \mathbb{Z}).$$

Every such pair $(c, d)$ is represented by a holomorphic contact bundle on a Stein domain $\Omega \subset X$ diffeotopic to $X$. 


Example: Lines bundles on a Grauert tube around $S^2$

Let $Y$ be a Grauert tube around the 2-sphere $S^2$; we may take

$$Y = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}.$$  

Since $TY|_{S^2} = TS^2 \oplus TS^2$ is trivial, $TY$ is holomorphically trivial.

Let $\pi: X \to Y$ be a holomorphic line bundle; these correspond to the elements of $H^2(Y; \mathbb{Z}) = H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Considering $Y$ as the zero section of $X$, we can view $X$ as the normal bundle $N_{Y,X}$ of $Y$ in $X$. The adjunction formula gives

$$K_X|_Y \cong K_Y \otimes (N_{Y,X})^{-1} = X^{-1}.$$

For each $X$ with even Chern number $c_1(X) \in H^2(Y; \mathbb{Z}) = \mathbb{Z}$, $(K_X)^{-1} = \det TX$ has a unique square root $L$ with $c_1(L) = \frac{1}{2} c_1(X)$.

Hence, there is a holomorphic $L$-valued contact form on a neighbourhood of $S^2$ in $X$. Is there one on all of $X$?
Example: $X = \mathbb{C}^* \times \mathbb{C}^2$

Let $X$ be a 3-dimensional Stein tube around an embedded circle $S^1 \subset X$. In this case

$$H^2(X; \mathbb{Z}) = H^2(S^1; \mathbb{Z}) = 0, \quad H^1(X; \mathbb{Z}) = H^1(S^1; \mathbb{Z}) = \mathbb{Z}.$$ 

Hence, the homotopy classes of holomorphic contact forms along $S^1 \subset X$ are classified by $k \in \mathbb{Z}$. We can see them explicitly on $X = \mathbb{C}^* \times \mathbb{C}^2$:

$$\alpha_k = \begin{cases} 
 dz + \frac{1}{k+1}x^{k+1}dy & \text{if } k \neq -1, \\
 \frac{1}{\sqrt{2}} \left( \frac{1}{x} dz + x dy \right) & \text{if } k = -1.
\end{cases}$$

Then

$$\alpha_k \wedge d\alpha_k = x^k dx \wedge dy \wedge dz, \quad k \in \mathbb{Z},$$

so this family provides all homotopy classes of framings of $X \times \mathbb{C}$.

The contact bundle $\xi_k = \ker \alpha_k$ on $X = \mathbb{C}^* \times \mathbb{C}^2$ is homotopic to $\xi_0$ if $k$ is even, and to $\xi_1 \cong \xi_{-1}$ if $k$ is odd. The bundles $\xi_0$ and $\xi_1$ are not homotopic to each other through contact bundles.
These contact forms come from covering maps

Note that the form $\alpha_k$ for $k \neq -1$ is the pullback of $\alpha_0 = dz + xdy$ (the standard contact form on $\mathbb{C}^3$) by the covering map $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^2$, $(x, y, z) \mapsto (x^{k+1}/(k+1), y, z)$.

In order to understand $\alpha_{-1}$, consider the contact form on $\mathbb{C}^3$ given by

$$\beta = \cos x \cdot dz + \sin x \cdot dy.$$ 

It defines the standard contact structure on $\mathbb{C}^3$, because it is the pullback of $dz - ydx$ by the automorphism

$$(x, y, z) \mapsto (x, y \cos x - z \sin x, y \sin x + z \cos x).$$

Let $F : \mathbb{C}^3 \to \mathbb{C}^* \times \mathbb{C}^2$, $F(x, y, z) = (e^{ix}, y, z)$. Then, $\beta = F^* \alpha'$, where $\alpha'$ is the contact form on $\mathbb{C}^* \times \mathbb{C}^2$ given by

$$\alpha' = \frac{1}{2} \left( x + \frac{1}{x} \right) dz + \frac{1}{2i} \left( x - \frac{1}{x} \right) dy, \quad \alpha' \wedge d\alpha' = \frac{1}{ix} dx \wedge dy \wedge dz.$$ 

Then, $\alpha_{-1}$ is homotopic to $\alpha'$ through the family of contact forms

$$\sigma_t = \frac{1}{\sqrt{2(1+t^2)}} \left( \left( tx + \frac{1}{x} \right) dz + \left( x - \frac{t}{x} \right) e^{-i\pi t/2} dy \right), \quad t \in [0, 1].$$
Example: \( X = (\mathbb{C}^*)^3 \)

The domain \( X = (\mathbb{C}^*)^3 \) is a Grauert tube around the standard totally real 3-torus \( T^3 = (S^1)^3 \hookrightarrow \mathbb{C}^3 \). We have

\[
H^2(X; \mathbb{Z}) = H^2(T^3; \mathbb{Z}) = \mathbb{Z}^3, \quad H^1(X; \mathbb{Z}) = H^1(T^3; \mathbb{Z}) = \mathbb{Z}^3.
\]

Clearly, \( K_X \) is trivial, and since \( H^2(X; \mathbb{Z}) \) is a free abelian group, the only square root \( L \) of \( K_X \) is the trivial bundle.

Consider the following family of contact forms for \( (k, l, m) \in \mathbb{Z}^3 \):

\[
\alpha_{k, l, m} = \begin{cases} 
    z^m dz + \frac{1}{k+1} x^{k+1} y^l dy & \text{if } k \neq -1, \\
    \frac{1}{2x} z^m dz + x y^l dy & \text{if } k = -1,
\end{cases}
\]

A calculation shows that

\[
\alpha_{k, l, m} \wedge d\alpha_{k, l, m} = x^k y^l z^m dx \wedge dy \wedge dz,
\]

so this family provides all homotopy classes of framings of \( X \times \mathbb{C} \).
Outline of proof of the main theorem

We first consider the problem around totally real submanifolds $M \subset X$. The model case is $\mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1}$. Consider smooth $(1,0)$-forms $
abla = \sum_{j=1}^{2n+1} a_j(z) dz_j$ whose coefficients $a_j(z)$ are $\bar{\partial}$-flat on a compact domain $D \subset \mathbb{R}^{2n+1}$. The contact condition

$$\nabla \wedge (d\nabla)^n \neq 0 \quad \text{on} \quad D$$

determines a partial differential relation $\mathcal{R}$ of first order. We verify that

- $\mathcal{R}$ is ample in the coordinate directions, and hence its sections satisfy all forms of the h-principle (M. Gromov 1973).

- A formal contact structure $(\nabla, \beta)$ with $\bar{\partial}$-flat coefficients on $D$ is a nonholonomic section of $\mathcal{R}$. Hence, if $\beta = d\nabla$ holds on $bD$ then $(\nabla, \beta)$ can be deformed by a homotopy $(\nabla_t, \beta_t)$, fixed near $bD$, to a holonomic section $(\nabla_1, d\nabla_1)$ on $D$.

- A sufficiently good holomorphic approximation of $\nabla_1$ is a holomorphic contact form on a neighbourhood of $D$ in $\mathbb{C}^{2n+1}$.

- The general case is solved by induction on a triangulation of $M$, reducing to the model case by $\bar{\partial}$-flat changes of coordinates.
Outline of proof of the main theorem, 2

- The skeleton (core) of a Stein manifold $X$ is an embedded CW complex in $X$ made of totally real (Lagrangian) cells. It comprises all the topology of $X$, and it has Stein neighbourhoods diffeotopic to $X$.

- The inductive step in the proof amounts to attaching an embedded totally real disc $M$ to a compact strongly pseudoconvex domain $W \subset X$ such that $M \cap W = bM$ is a Legendrian sphere in $bW$ and the attachment of $M$ to $W$ is transverse along $bM$.

- In the inductive step, we have a formal contact structure $(\alpha, \beta)$ on $X$ such that $\alpha$ is holomorphic on a neighbourhood of $W$ and $\beta|_{\zeta} = d\alpha|_{\zeta}$ there, where $\zeta = \ker \alpha$.

- By the special case, we can change $(\alpha, \beta)$ along $M$ to an almost holomorphic contact structure, keeping it fixed near $bM$. Mergelyan approximation on $W \cup M$ then gives a holomorphic contact form $\tilde{\alpha}$ on a neighbourhood of $W \cup M$. Proceed by induction.

- If one could approximate holomorphic contact forms on compact convex sets in $\mathbb{C}^{2n+1}$ by entire contact forms, then one could construct a holomorphic contact form on all of $X$. 
Open problems

1. How many contact structures are there on $\mathbb{C}^3$? On $\mathbb{C}^{2n+1}$? How to distinguish them?

2. Is there an analogue of the tight/overtwisted phenomenon from smooth contact geometry?

3. Does every Stein manifold $X^{2n+1}$ whose canonical bundle $K_X$ has $(n+1)$-st root admit a (formal) contact structure?

4. Does the Runge approximation theorem hold for holomorphic contact structures? In particular, does it hold on convex sets in $\mathbb{C}^{2n+1}$?

5. Does every Stein contact manifold $(X, \xi)$ contain proper Legendrian curves normalized by bordered Riemann surfaces?

Bryant (1982) Every compact Riemann surface embeds as a holomorphic Legendrian curve in $\mathbb{C} \mathbb{P}^3$.

Alarcón, F., López (2017) Every open Riemann surface is a properly embedded Legendrian curve in $(\mathbb{C}^3, \alpha_0)$. Every bordered Riemann surface is a complete Legendrian curve with Jordan boundary in $(\mathbb{C}^3, \alpha_0)$.

Lárusson, F., 2018 Results in projectivised cotangent bundles.