H-principle for complex contact structures on Stein manifolds

Franc Forstnerič

Univerza *v Ljubljani*

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Arnold: "Contact geometry is all geometry"

I have recently become interested in **holomorphic directed (Pfaffian) systems**, in particular, **holomorphic contact systems**.

Contact geometry is intimately connected with symplectic geometry, Riemannian geometry, CR geometry, and many other areas.
It has been used to prove big results in differential topology.

Cerf 1964 Every diffeomorphism of $S^3 = \partial B^4$ extends to a diffeomorphism of B^4.

Eliashberg 1989, 1992 discovered a different proof based on his classification of contact structures on S^3:

- *tight*: the standard contact structure is unique up to isotopy
- *overtwisted*: countable infinity of distinct ones, classified homotopically (first shown to exist by Bennequin in 1982)

Martinet 1971 Every orientable closed 3-manifold admits a contact structure (in fact an overtwisted one).

Complex contact manifolds

Kobayashi 1959 A **complex contact manifold** is a pair (X, ξ) where

- X is a complex manifold of odd dimension $2n+1 \geq 3$, and
- ξ is a holomorphic hyperplane subbundle of the tangent bundle TX which is **maximally nonintegrable**, in the sense that the following bilinear pairing is nondegenerate:

$$O : \xi \otimes \xi \to TX/\xi = L, \quad (v, w) \mapsto [v, w] \mod \xi$$

- Equivalently, every point $p \in X$ has a neighborhood $U \subset X$ such that

$$\xi|_U = \ker \alpha,$$

where α is a holomorphic 1-form on U satisfying

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Such ξ is a **holomorphic contact subbundle** on X, and a holomorphic 1-form α satisfying the above condition is a **holomorphic contact form**.

Darboux's theorem and stability results

Contact manifolds (X, ξ) and (X', ξ') are **contactomorphic** if there exists a diffeomorphism (biholomorphism) $F\colon X \to X'$ satisfying

$$dF_x(\xi_x) = \xi'_{F(x)} \quad \text{for all } x \in X.$$

Example (model contact space)

$$X \in \{\mathbb{R}^{2n+1}, \mathbb{C}^{2n+1}\}, \quad \xi_0 = \ker \alpha_0, \quad \alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.$$

Darboux 1882; Engel 1989; Cartan 1901; Moser 1965 Every contact manifold (X^{2n+1}, ξ) is locally contactomorphic to this model.

Gray 1959 If X is a compact manifold then contact structures in an isotopy $\{\xi_t\}_{t\in[0,1]}$ are pairwise contactomorphic, i.e., there is an isotopy $\{f_t\}_{t\in[0,1]} \subset \mathrm{Diff}(X)$ such that $(df_t)(\xi_0) = \xi_t$.

The normal bundle of a contact structure

Le Brun & Salamon 1994 A contact subbundle $\xi \subset TX$ is given by a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L)) = H^0(X, T^*X \otimes L)$ with values in the holomorphic line bundle $L = TX/\xi$ (the **normal bundle** of ξ):

$$0 \longrightarrow \xi \longrightarrow TX \xrightarrow{\alpha} L \longrightarrow 0.$$

If f is a holomorphic function then $d(f\alpha) = df \wedge \alpha + fd\alpha$, so

$$d\alpha|_\xi \text{ is a section of } \Lambda^2(\xi^*) \otimes L.$$

Letting $K_X = \Lambda^{2n+1}(T^*X)$ (the canonical bundle of X), it follows that

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ is a trivialisation of } K_X \otimes L^{\otimes(n+1)}.$$

This provides a holomorphic line bundle isomorphism

$$L^{\otimes(n+1)} \cong K_X^{-1} = \Lambda^{2n+1}(TX).$$

The space $\mathrm{Cont}(X)$ of holomorphic contact structures

Conversely, assume X^{2n+1} is a complex manifold with $H^1(X, \mathbb{Z}_{n+1}) = 0$ and $c_1(TX)$ divisible by $n+1$. Then there exists the line bundle

$$L = K_X^{-1/(n+1)}, \quad L^{\otimes(n+1)} \cong K_X^{-1}.$$

Given a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$, consider

$$\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \mathscr{O}(X).$$

If X is **compact** then $\mathscr{O}(X) = \mathbb{C}$. If the constant $\alpha \wedge (d\alpha)^n \in \mathbb{C}$ is nonzero then α is a contact form on X. The map

$$\Gamma(X, \Omega^1(L)) \ni \alpha \longmapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}$$

is homogeneous of degree $n+1$.

Hence, the space $\mathrm{Cont}(X)$ is either empty or the complement of a degree $n+1$ hypersurface in $\mathbb{P}(\Gamma(X, \Omega^1(L)))$.

Example: A (unique) contact structure on $\mathbb{CP}^{2n+1}$

Let $z_1, \ldots, z_{2n+2}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$

Then, θ defines a contact structure on $\mathbb{CP}^{2n+1}$. Let θ_j be the pull-back of θ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots .$$

Then (H_j, θ_j) is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each j, and this collection forms a contact atlas on $X = \mathbb{CP}^{2n+1}$. We have

$$K_X^{-1} = \mathscr{O}_X(2n+2), \quad L = K_X^{-1/(n+1)} = \mathscr{O}_X(2),$$

$$\alpha = [\theta] \in \Gamma(\mathbb{CP}^{2n+1}, \Omega^1(2)).$$

This contact structure is unique by Gray-Le Brun-Salamon theorem.

Compact complex contact manifolds are very special

Le Brun, Salamon 1994 Any two complex contact structures on a simply connected compact complex manifold are contactomorphic.

Demailly 2002 If a compact Kähler manifold X admits a complex contact structure, then $\kappa_X = -\infty$.

Examples of projective complex contact manifolds:

(a) $\mathbb{P}T^*Z$, where Z is projective.

(b) Unique closed orbit $X_\mathfrak{g}$ of the adjoint action of a simple complex Lie group G on $\mathbb{P}\mathfrak{g}$. Then $X_\mathfrak{g}$ is **Fano** (i.e., K_X^{-1} is ample), e.g. $\mathbb{P}^{2n+1}$.

Conjecture: These are the only examples.

Ye 1994 True in dimension 3. Uses the minimal model program.

Kebekus et al. (2000), Demailly 2002 A contact compact Kähler manifold X not of type $\mathbb{P}T^*Z$ is Fano with $b_2 = 1$.

Equivalent conjecture (Wolf) X as above is homogeneous.

Contact structures on Stein manifolds

Assume now that X is a **Stein manifold** of dimension $2n+1 \geq 3$. For a generic holomorphic 1-form α on X, the equation

$$\alpha \wedge (d\alpha)^n = 0$$

defines a (possibly empty) complex hypersurface $\Sigma_\alpha \subset X$, and α is a contact form on the Stein manifold $X \setminus \Sigma_\alpha$.

This observation shows that there exist a plethora of Stein contact manifolds, but it does not answer the question whether a given Stein manifold (or a given diffeomorphism class of Stein manifolds) admits a contact structure. More precisely, when is a complex hyperplane subbundle $\xi \subset TX$ satisfying the necessary condition

$$\Lambda^{2n}\xi \cong L^n = (TX/\xi)^n$$

homotopic to a holomorphic contact subbundle?

How many nonequivalent contact structures are there on $\mathbb{C}^3$? Noone seems to have a slightest clue.

A hyperbolic contact structure on $\mathbb{C}^{2n+1}$

The **Kobayashi pseudometric** associated to a holomorphic contact structure is defined by using holomorphic Legendrian discs.

Theorem (F., J. Geom. Anal. 2017)

For any $n \geq 1$ there exists a holomorphic contact structure ξ on $\mathbb{C}^{2n+1}$ which is **Kobayashi hyperbolic** *and isotopic to ξ_0. In particular, every holomorphic Legendrian curve $\mathbb{C} \to (\mathbb{C}^{2n+1}, \xi)$ is constant.*

Idea of proof: We take $\alpha = \Phi^* \alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$ and $\Phi \colon \mathbb{C}^{2n+1} \to \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image Ω avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b \mathbb{D}^{2n}_{(x,y)} \times C_N \overline{\mathbb{D}}_z .$$

Assuming that $C_N \geq n 2^{3N+1}$ for all $N \in \mathbb{N}$,

$\mathbb{C}^{2n+1} \setminus K$ is α_0-hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^* \alpha_0)$ is hyperbolic.

Formal contact structures

Definition

Let X be a complex manifold of dimension $2n+1 \geq 3$.
A **formal complex contact structure** on X is a pair (α, β), where

- α is a smooth $(1,0)$-form on X with values in a line bundle $L \to X$,
- β is a smooth $(2,0)$-form on $\xi = \ker\alpha$ with values in L, and
- $\alpha \wedge \beta^n \neq 0$ at each point of X.

We denote by

$$\mathrm{Cont}_{\mathrm{for}}(X)$$

the space of all formal complex contact structures on X.

The existence of a formal contact structure on X implies the same conditions

$$K_X \otimes L^{n+1} \cong X \times \mathbb{C} \cong \Lambda^{2n}\xi^* \otimes L^n.$$

The Main Theorem

We have the natural inclusion

$$\mathrm{Cont}(X) \hookrightarrow \mathrm{Cont}_{\mathrm{for}}(X), \qquad \alpha \mapsto (\alpha, d\alpha|_{\xi=\ker\alpha}).$$

Theorem

Let X be a Stein manifold of odd dimension. Given $(\alpha_0, \beta_0) \in \mathrm{Cont}_{\mathrm{for}}(X)$, there are a homotopy $(\alpha_t, \beta_t) \in \mathrm{Cont}_{\mathrm{for}}(X)$ $(t \in [0,1])$ and a Stein domain $\Omega \subset X$, diffeotopic to X, such that

$$\alpha_1|_\Omega \in \mathrm{Cont}(\Omega) \quad \textit{and} \quad \beta_1|_{\ker\alpha_1} = d\alpha_1 \ \ \textit{on} \ \ \Omega.$$

Furthermore, if $Q \subset P$ are compact Hausdorff spaces and $\{(\alpha_p, \beta_p)\}_{p\in P} \in \mathrm{Cont}_{\mathrm{for}}(X)$ is a continus family such that

$$\forall p \in Q: \quad \alpha_p \in \mathrm{Cont}(X) \ \ \textit{and} \ \ \beta_p = d\alpha_p|_{\ker\alpha_p},$$

then there are a Stein domain $\Omega \subset X$ diffeotopic to X and a homotopy $(\alpha_{p,t}, \beta_{p,t}) \in \mathrm{Cont}_{\mathrm{for}}(\Omega)$ $(p \in P,\ t \in [0,1])$ which is fixed for all $p \in Q$ such that $\alpha_{p,1} \in \mathrm{Cont}(\Omega)$ and $\beta_p = d\alpha_p|_{\ker\alpha_p}$ for every $p \in P$.

Why restricting to a Stein domain $\Omega \subset X$?

Problem

Given a holomorphic contact form α on an open neighbourhood of a compact convex set $K \subset \mathbb{C}^{2n+1}$, is it possible to approximate α uniformly on K by holomorphic contact forms on $\mathbb{C}^{2n+1}$?

Is this also possible for any continuous family of holomorphic contact forms α_p with parameter $p \in P$ in a compact Hausdorff space?

The corresponding problem for holomorphic foliations is also open and very challenging. **These issues do not appear in the smooth case.**

Theorem

If the above problem has an affirmative answer, then the inclusion $\mathrm{Cont}(X) \hookrightarrow \mathrm{Cont}_{\mathrm{for}}(X)$ is a weak homotopy equivalence.

This holds true for germs of contact structures along any (stratified) totally real submanifold $M \subset X$.

Connected components of $\mathrm{Cont}_{\mathrm{for}}(X)$ on Stein threefolds

Lemma

If X is a Stein manifold of dimension 3, then the connected components of $\mathrm{Cont}_{\mathrm{for}}(X)$ are classified by the following pairs of data:

(i) *an isomorphism class of a complex line bundle L on X satisfying $L^2 \cong (K_X)^{-1}$ (equivalently, a cohomology class $c \in H^2(X;\mathbb{Z})$ such that $2c = c_1(TX)$), and*

(ii) *a choice of a homotopy class of trivialisations of $K_X \otimes L^2 \cong \Lambda^2\xi^* \otimes L \cong X \times \mathbb{C}$, that is, an element of the 1st cohomology group $[X, \mathbb{C}^*] = [X, S^1] = H^1(X;\mathbb{Z})$.*

In particular, if $H^1(X;\mathbb{Z}) = 0$ and $H^2(X;\mathbb{Z}) = 0$ then the space $\mathrm{Cont}_{\mathrm{for}}(X)$ is connected; this holds for $X = \mathbb{C}^3$.

Problem

Let X be a Stein threefold with $H^1(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = 0$. Is the space $\mathrm{Cont}(X)$ connected? In particular, is $\mathrm{Cont}(\mathbb{C}^3)$ connected?

Homotopy classification of contact bundles

A holomorphic contact bundle ξ on X is determined by a holomorphic 1-form α up to a nonvanishing factor $f \in \mathscr{O}(X, \mathbb{C}^*)$. Since

$$f\alpha \wedge d(f\alpha)^n = f^{n+1}\alpha \wedge d\alpha,$$

this changes the trivialisation of $K_X \otimes L^{n+1}$ by the factor f^{n+1} where $\dim X = 2n+1$ (by f^2 if $\dim X = 3$).

Corollary

A homotopy class of holomorphic contact bundles on a Stein 3-fold X is uniquely determined by a pair (c, d), where

$$c \in H^2(X;\mathbb{Z}), \quad 2c = c_1(TX); \qquad d \in H^1(X;\mathbb{Z})/2H^1(X;\mathbb{Z}).$$

Every such pair (c, d) is represented by a holomorphic contact bundle on a Stein domain $\Omega \subset X$ diffeotopic to X.

Example: Lines bundles on a Grauert tube around S^2

Let Y be a Grauert tube around the 2-sphere S^2; we may take

$$Y = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}.$$

Since $TY|_{S^2} = TS^2 \oplus TS^2$ is trivial, TY is holomorphically trivial.

Let $\pi\colon X \to Y$ be a holomorphic line bundle; these correspond to the elements of $H^2(Y; \mathbb{Z}) = H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Considering Y as the zero section of X, we can view X as the normal bundle $N_{Y,X}$ of Y in X. The adjunction formula gives

$$K_X|_Y \cong K_Y \otimes (N_{Y,X})^{-1} = X^{-1}.$$

For each X with **even Chern number** $c_1(X) \in H^2(Y; \mathbb{Z}) = \mathbb{Z}$, $(K_X)^{-1} = \det TX$ has a unique square root L with $c_1(L) = \frac{1}{2}c_1(X)$.

Hence, there is a holomorphic L-valued contact form on a neighbourhood of S^2 in X. Is there one on all of X?

Example: $X = \mathbb{C}^* \times \mathbb{C}^2$

Let X be a 3-dimensional Stein tube around an embedded circle $S^1 \subset X$. In this case

$$H^2(X;\mathbb{Z}) = H^2(S^1;\mathbb{Z}) = 0, \quad H^1(X;\mathbb{Z}) = H^1(S^1;\mathbb{Z}) = \mathbb{Z}.$$

Hence, the homotopy classes of holomorphic contact forms along $S^1 \subset X$ are classified by $k \in \mathbb{Z}$. We can see them explicitly on $X = \mathbb{C}^*_x \times \mathbb{C}^2_{(y,z)}$:

$$\alpha_k = \begin{cases} dz + \frac{1}{k+1} x^{k+1} dy & \text{if } k \neq -1, \\ \frac{1}{\sqrt{2}} \left(\frac{1}{x} dz + x dy \right) & \text{if } k = -1. \end{cases}$$

Then

$$\alpha_k \wedge d\alpha_k = x^k dx \wedge dy \wedge dz, \qquad k \in \mathbb{Z},$$

so this family provides all homotopy classes of framings of $X \times \mathbb{C}$.

The contact bundle $\xi_k = \ker \alpha_k$ on $= \mathbb{C}^* \times \mathbb{C}^2$ is homotopic to ξ_0 if k is even, and to $\xi_1 \cong \xi_{-1}$ is k is odd. The bundles ξ_0 and ξ_1 are not homotopic to each other through contact bundles.

These contact forms come from covering maps

Note that the form α_k for $k \neq -1$ is the pullback of $\alpha_0 = dz + xdy$ (the standard contact form on $\mathbb{C}^3$) by the covering map $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^2$, $(x, y, z) \mapsto (x^{k+1}/(k+1), y, z)$.
In order to understand α_{-1}, consider the contact form on $\mathbb{C}^3$ given by

$$\beta = \cos x \cdot dz + \sin x \cdot dy.$$

It defines the standard contact structure on $\mathbb{C}^3$, because it is the pullback of $dz - ydx$ by the automorphism

$$(x, y, z) \to (x, y \cos x - z \sin x, y \sin x + z \cos x).$$

Let $F : \mathbb{C}^3 \to \mathbb{C}^* \times \mathbb{C}^2$, $F(x, y, z) = (\mathrm{e}^{\mathrm{i}x}, y, z)$. Then, $\beta = F^*\alpha'$, where α' is the contact form on $\mathbb{C}^* \times \mathbb{C}^2$ given by

$$\alpha' = \frac{1}{2}\left(x + \frac{1}{x}\right) dz + \frac{1}{2\mathrm{i}}\left(x - \frac{1}{x}\right) dy, \quad \alpha' \wedge d\alpha' = \frac{1}{\mathrm{i}x} dx \wedge dy \wedge dz.$$

Then, α_{-1} is homotopic to α' through the family of contact forms

$$\sigma_t = \frac{1}{\sqrt{2(1+t^2)}}\left(\left(tx + \frac{1}{x}\right) dz + \left(x - \frac{t}{x}\right) \mathrm{e}^{-\mathrm{i}\pi t/2} dy\right), \quad t \in [0, 1].$$

Example: $X = (\mathbb{C}^*)^3$

The domain $X = (\mathbb{C}^*)^3$ is a Grauert tube around the standard totally real 3-torus $T^3 = (S^1)^3 \hookrightarrow \mathbb{C}^3$. We have

$$H^2(X;\mathbb{Z}) = H^2(T^3;\mathbb{Z}) = \mathbb{Z}^3, \quad H^1(X;\mathbb{Z}) = H^1(T^3;\mathbb{Z}) = \mathbb{Z}^3.$$

Clearly, K_X is trivial, and since $H^2(X;\mathbb{Z})$ is a free abelian group, the only square root L of K_X is the trivial bundle.

Consider the following family of contact forms for $(k,l,m) \in \mathbb{Z}^3$:

$$\alpha_{k,l,m} = \begin{cases} z^m dz + \frac{1}{k+1} x^{k+1} y^l dy & \text{if } k \neq -1, \\ \frac{1}{2x} z^m dz + x y^l dy & \text{if } k = -1, \end{cases}$$

A calculation shows that

$$\alpha_{k,l,m} \wedge d\alpha_{k,l,m} = x^k y^l z^m dx \wedge dy \wedge dz,$$

so this family provides all homotopy classes of framings of $X \times \mathbb{C}$.

Outline of proof of the main theorem

We first consider the problem around totally real submanifolds $M \subset X$. The model case is $\mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1}$. Consider smooth $(1,0)$-forms $\alpha = \sum_{j=1}^{2n+1} a_j(z) dz_j$ whose coefficients $a_j(z)$ are $\bar\partial$-flat on a compact domain $D \subset \mathbb{R}^{2n+1}$. The contact condition

$$\alpha \wedge (d\alpha)^n \neq 0 \quad \text{on} \quad D$$

determines a partial differential relation $\mathcal{R}$ of first order. We verify that

- $\mathcal{R}$ is ample in the coordinate directions, and hence its sections satisfy all forms of the h-principle (**M. Gromov 1973**).
- A formal contact structure (α, β) with $\bar\partial$-flat coefficients on D is a nonholonomic section of $\mathcal{R}$. Hence, if $\beta = d\alpha$ holds on bD then (α, β) can be deformed by a homotopy (α_t, β_t), fixed near bD, to a holonomic section $(\alpha_1, d\alpha_1)$ on D.
- A sufficiently good holomorphic approximation of α_1 is a holomorphic contact form on a neighbourhood of D in $\mathbb{C}^{2n+1}$.
- The general case is solved by induction on a triangulation of M, reducing to the model case by $\bar\partial$-flat changes of coordinates.

Outline of proof of the main theorem, 2

- The skeleton (core) of a Stein manifold X is an embedded CW complex in X made of totally real (Lagrangian) cells. It comprises all the topology of X, and it has Stein neighbourhoods diffeotopic to X.
- The inductive step in the proof amounts to attaching an embedded totally real disc M to a compact strongly pseudoconvex domain $W \subset X$ such that $M \cap W = bM$ is a Legendrian sphere in bW and the attachment of M to W is transverse along bM.
- In the inductive step, we have a formal contact structure (α, β) on X such that α is holomorphic on a neighbourhood of W and $\beta|_\xi = d\alpha|_\xi$ there, where $\xi = \ker \alpha$.
- By the special case, we can change (α, β) along M to an almost holomorphic contact structure, keeping it fixed near bM. Mergelyan approximation on $W \cup M$ then gives a holomorphic contact form $\widetilde{\alpha}$ on a neighbourhood of $W \cup M$. Proceed by induction.
- If one could approximate holomorphic contact forms on compact convex sets in $\mathbb{C}^{2n+1}$ by entire contact forms, then one could construct a holomorphic contact form on all of X.

Open problems

1. How many contact structures are there on $\mathbb{C}^3$? On $\mathbb{C}^{2n+1}$? How to distinguish them?
2. Is there an analogue of the tight/overtwisted phenomenon from smooth contact geometry?
3. Does every Stein manifold X^{2n+1} whose canonical bundle K_X has $(n+1)$-st root admit a (formal) contact structure?
4. Does the Runge approximation theorem hold for holomorphic contact structures? In particular, does it hold on convex sets in $\mathbb{C}^{2n+1}$?
5. Does every Stein contact manifold (X, ξ) contain proper Legendrian curves normalized by bordered Riemann surfaces?

Bryant (1982) Every compact Riemann surface embeds as a holomorphic Legendrian curve in $\mathbb{CP}^3$.
Alarcón, F., López (2017) Every open Riemann surface is a properly embedded Legendrian curve in $(\mathbb{C}^3, \alpha_0)$.
Every bordered Riemann surface is a complete Legendrian curve with Jordan boundary in $(\mathbb{C}^3, \alpha_0)$.
Lárusson, F., 2018 Results in projectivised cotangent bundles.