The h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$

Franc Forstnerič

University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Academy of Sciences and Arts of the Republic of Slovenia

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Abstract

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Every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$.

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This is a basic h-principle. We now upgrade it to a parametric h-principle:

Theorem (F. Lárusson & F. Forstnerič, Feb. 2016)

For any $n \geq 3$, the inclusion

$$\iota : \Re\mathfrak{N}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$$

of the space of real parts of all nonflat null holomorphic immersions $M \to \mathbb{C}^n$ into the space of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ satisfies the parametric h-principle with approximation; in particular, it is a weak homotopy equivalence (WHE).

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If M has finitely generated homology group $H_1(M; \mathbb{Z})$, then $\Re\mathfrak{N}_*(M, \mathbb{C}^n)$ is a deformation retract of $\mathfrak{M}_*(M, \mathbb{R}^n)$.

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- absolute neighborhood retracts (Borsuk, Whitehead, Milnor,...)

Weierstrass representation of minimal surfaces in $\mathbb{R}^n$

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Conversely, if $\Phi = (\phi_1, \ldots, \phi_n)$ is as above and

$$\int_\gamma \Re(\Phi) = 0 \quad \text{for all} \quad \gamma \in H_1(M; \mathbb{Z}),$$

then $u = \int \Re\Phi : M \to \mathbb{R}^n$ is a conformal minimal immersion.

The null quadric

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$$u(p) = u(p_0) + \int_{p_0}^{p} \Re\,(f\theta)\,, \quad p_0, p \in M$$

where θ is a nowhere vanishing holomorphic 1-form on M,

$$f = 2\partial u/\theta = (f_1, \dots, f_n) \colon M \to \mathfrak{A}_* = \mathfrak{A} \setminus \{0\} \subset \mathbb{C}^n$$

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If the complex periods of $f\theta$ vanish, then

$$F(p) = \int^{p} f\theta \in \mathbb{C}^n, \quad p \in M$$

is a **holomorphic null curve** in $\mathbb{C}^n$ with $u = \Re F$. Equivalently:

$$\mathrm{Flux}(u)(\gamma) := \int_\gamma \Im(f\theta) = 0 \quad \forall \gamma \in H_1(M; \mathbb{Z}).$$

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The map ϕ is given by $F \mapsto \partial F/\theta$, and ψ is given by $u \mapsto 2\partial u/\theta$.

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- Our main theorem: **each of the maps ι, ϕ, ψ is a weak homotopy equivalence (WHE).**
- The projection $F \mapsto \Re F$ of a null curve to its real part is clearly a homotopy equivalence.
- Since $\mathfrak{A}_*$ is an **Oka manifold**, the inclusion $\mathscr{O}(M, \mathfrak{A}_*) \hookrightarrow \mathscr{C}(M, \mathfrak{A}_*)$ is a WHE by the **Oka-Grauert principle**.

Connected components of the space $\mathfrak{M}_*(M, \mathbb{R}^n)$

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The punctured null quadric $\mathfrak{A}_*^{n-1} \subset \mathbb{C}^n$ is simply connected when $n \geq 4$, while $\pi_1(\mathfrak{A}_*^2) \cong \mathbb{Z}_2$ in view of the two-sheeted universal covering

$$\pi\colon \mathbb{C}_*^2 = \mathbb{C}^2 \setminus \{0\} \to \mathfrak{A}_*^2, \quad \pi(u,v) = \big(u^2 - v^2, \mathrm{i}(u^2+v^2), 2uv\big).$$

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Hence, the path components of the space $\mathscr{C}(M, \mathfrak{A}_*^2)$ are in one-to-one correspondence with group homomorphisms $H_1(M;\mathbb{Z}) \to \mathbb{Z}_2$ (i.e., the elements of $(\mathbb{Z}_2)^\ell$), and $\mathscr{C}(M, \mathfrak{A}_*^{n-1})$ is path connected if $n \geq 4$.

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Corollary

Let M be a connected open Riemann surface with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^\ell$. Then the path connected components of $\mathfrak{M}_*(M,\mathbb{R}^3)$ and $\mathfrak{N}_*(M,\mathbb{C}^3)$ are in one-to-one correspondence with the elements of $(\mathbb{Z}_2)^\ell$.

If $n \geq 4$ then $\mathfrak{M}_*(M,\mathbb{R}^n)$ and $\mathfrak{N}_*(M,\mathbb{C}^n)$ are path connected.

Path connected components of the space $\mathfrak{M}(M, \mathbb{R}^n)$

Theorem (Alarcón, Forstnerič, Lopez, April 2016)

Let M be an open connected Riemann surface. The natural inclusion $\mathfrak{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n)$ of the space of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ into the space of all conformal minimal immersions induces a bijection of path components of the two spaces.

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This follows by combining the following two results:

- Given a flat conformal minimal immersion $X \colon M \to \mathbb{R}^n$ $(n \geq 3)$, there exists an isotopy $X_t \colon M \to \mathbb{R}^n$ $(t \in [0,1])$ of conformal minimal immersions such that $X_0 = X$ and X_1 is nonflat.

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- Let θ be a nowhere vanishing holomorphic 1-form on M. For every homomorphism $\mathfrak{p}\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}_2 = H_1(\mathfrak{A}_*;\mathbb{Z})$ there exists a flat conformal minimal immersion $X\colon M \to \mathbb{R}^3$ with $H_1(\partial X/\theta) = \mathfrak{p}$.

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Open problem: Is the inclusion $\mathfrak{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n)$ a WHE?

Examples in dimension $n = 3$

Let M be $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ or an annulus, with $\theta = dz$. There are two homotopy classes of continuous or holomorphic maps $f\colon M \to \mathfrak{A}_*$.

Let $\pi\colon \mathbb{C}_*^2 \to \mathfrak{A}_*$ be the universal covering map as above. Note that f is nullhomotopic if and only if it factors through π.

Consider the **Weierstrass representation:**

$$f_1 = (1 - g^2)\eta, \quad f_2 = \mathfrak{i}(1 + g^2)\eta, \quad f_3 = 2g\eta,$$

where g is meromorphic and η is holomorphic on M. Assume for simplicity that g is holomorphic or, equivalently, that η has no zeros.

Then, f factors through π if and only if η has a square root on M.

Indeed, if η has a square root then $f = \pi(\sqrt{\eta}, g\sqrt{\eta})$; conversely, if $f = \pi(u, v)$ for some holomorphic map $(u, v) : M \to \mathbb{C}_*^2$, then $u^2 = \eta$.

Examples in dimension $n = 3$

1. A flat null curve: $M = \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ and $f\colon \mathbb{C}_* \to \mathfrak{A}_* \subset \mathbb{C}^3$ is the map $f(\zeta) = \zeta(1, \mathrm{i}, 0)$. In this case, $g = 0$ and $\eta(\zeta) = \zeta$ does not have a square root on M. Thus, the flat null curve

$$F(\zeta) = \frac{1}{2}(\zeta^2, \mathrm{i}\zeta^2, 0), \quad \zeta \in \mathbb{C}_*$$

has derivative in the nontrivial isotopy class.

2. The catenoid: $M = \mathbb{C}_*$, $g(\zeta) = \zeta$, and $\eta(\zeta) = 1/\zeta^2$. Since η has a square root on M, we are in the trivial isotopy class.
The same holds for the **helicoid** which is parameterized by $\mathbb{C}$.

3. Henneberg's surface:

$$M = \mathbb{C} \setminus \{0, 1, -1, \mathrm{i}, -\mathrm{i}\}, \quad g(\zeta) = \zeta, \quad \eta(\zeta) = 1 - \zeta^{-4}.$$

On a small punctured disc centered at one of the points 1, -1, i, or $-\mathrm{i}$, η does not have a square root, so we are in the nontrivial isotopy class.

On the punctured disc $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$, the function η has a square root, so we are in the trivial isotopy class.

4. Double sheeted covering of Meeks's minimal Möbius strip:

$$M = \mathbb{C}_*, \quad g(\zeta) = \zeta^2 \frac{\zeta+1}{\zeta-1}, \quad \eta(\zeta) = \mathfrak{i}\, \frac{(\zeta-1)^2}{\zeta^4}.$$

Note that η has a square root on M. Despite the pole of g at 1, we get a factorization through π and we are in the trivial isotopy class.

Let $F = u + \mathfrak{i}v \colon \mathbb{C}_* \to \mathbb{C}^3$ be the null curve with this Weierstrass data. Then u is invariant with respect to the fixed-point-free antiholomorphic involution

$$\mathfrak{I}(\zeta) = -1/\bar{\zeta} \quad \text{on} \quad \zeta \in \mathbb{C}_*,$$

and hence it induces a conformal minimal immersion $\mathbb{C}_*/\mathfrak{I} \to \mathbb{R}^3$.

This is Meeks's properly immersed minimal Möbius strip in $\mathbb{R}^3$ with finite total curvature -6π.

Meeks's minimal Möbius strip

W.H. Meeks, 1981:

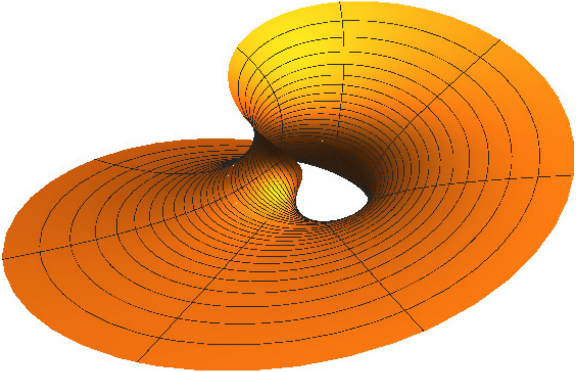

Illustration © **Antonio Alarcón**.

Parametric h-principle for $\Re\mathfrak{N}_*(M,\mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M,\mathbb{R}^n)$

Theorem

Assume that M is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \Subset M$ is a smoothly bounded Runge domain, and $u\colon M \times P \to \mathbb{R}^n$ $(n \geq 3)$ is a continuous map satisfying the following:

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(a) $u_p = u(\cdot\,, p) \colon M \to \mathbb{R}^n$ *is a nonflat CMI for every* $p \in P$*;*

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(c) $\mathrm{Flux}(u_p) = 0$ *for every* $p \in Q$.

Parametric h-principle for $\mathfrak{R}\mathfrak{N}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$

Theorem

Assume that M is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \Subset M$ is a smoothly bounded Runge domain, and $u\colon M \times P \to \mathbb{R}^n$ $(n \geq 3)$ is a continuous map satisfying the following:

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The WHE-principle for $\mathfrak{RN}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$

This is the parametric h-principle with approximation for the inclusion

$$\mathfrak{RN}_*(M, \mathbb{C}^n) = \{u \in \mathfrak{M}_*(M, \mathbb{R}^n) : \mathrm{Flux}(u) = 0\} \longrightarrow \mathfrak{M}_*(M, \mathbb{R}^n).$$

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Let $k \in \mathbb{Z}_+$. Applying the h-principle with $P = \mathbb{S}^k$ (the real k-sphere) and $Q = \emptyset$ shows that the inclusion induced map

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Applying the h-principle with $P = \overline{\mathbb{B}}^{k+1}$ (the closed ball in $\mathbb{R}^{k+1}$) and $Q = \mathbb{S}^k = b\mathbb{B}^{k+1}$ shows that the above map is also injective.

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Thus, it is an isomorphism for every $k \in \mathbb{Z}_+$. □

Proof of the h-principle for $\Re\mathfrak{M}_*(M,\mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M,\mathbb{R}^n)$

Pick a smooth strongly subharmonic Morse exhaustion function $\rho\colon M \to \mathbb{R}$ and exhaust M by sublevel sets

$$D_j = \{x \in M \colon \rho(x) < c_j\}, \quad j \in \mathbb{N}$$

where $c_1 < c_2 < c_3 < \ldots$ is an increasing sequence of regular values of ρ such that $\lim_{j\to\infty} c_j = \infty$ and each interval $[c_j, c_{j+1}]$ contains at most one critical value of the function ρ.

We may assume that $D = D_1$.

Proof of the h-principle for $\mathfrak{RM}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$

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We may assume that $D = D_1$.

Let $\epsilon > 0$ be as in the theorem. Pick a sequence $\epsilon_j > 0$ with $\sum_{j=1}^{\infty} \epsilon_j < \epsilon$. Set

$$u^t_{p,1} := u_p|_{\overline{D}_1}, \quad (p,t) \in P \times [0,1].$$

The recursive scheme

We recursively construct a sequence of homotopies of CMI's

$$u^t_{p,j} \colon \overline{D}_j \longrightarrow \mathbb{R}^n, \quad (p,t) \in P \times [0,1], \quad j \in \mathbb{N}$$

satisfying the following conditions for every $j = 2, 3, \ldots$:

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These conditions imply that the limit

$$u^t_p = \lim_{j\to\infty} u^t_{p,j} \colon M \to \mathbb{R}^n \quad ((p,t) \in P \times [0,1])$$

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Indeed, Conditions (1)–(4) follow from (a_j)–(d_j), respectively.

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Choose a Runge homology basis $\mathcal{B} = \{\gamma_i \colon i = 1, \ldots, \ell\}$ for $H_1(\overline{D}_j; \mathbb{Z})$ such that $\mathcal{B}' = \{\gamma_1, \ldots, \gamma_m\}$ for some $m \in \{0, \ldots, \ell\}$ is a homology basis for $H_1(\overline{D}; \mathbb{Z})$.

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Then, $\mathcal{B}$ is also a homology basis for $H_1(\overline{D}_{j+1}; \mathbb{Z})$.

Denote by $\mathcal{P}$ the **period map** associated to $\mathcal{B}$:

$$\mathcal{P}(f) = \left(\int_{\gamma_i} f\theta \right)_{i=1,\ldots,l} \in (\mathbb{C}^n)^\ell, \qquad f \in \mathscr{A}(\overline{D}_j, \mathfrak{A}_*).$$

Also, $\mathcal{P}' \colon \mathscr{A}(\overline{D}, \mathfrak{A}_*) \to (\mathbb{C}^n)^m$ is the period map with respect to $\mathcal{B}'$.

The noncritical case, continued

Consider the continuous family of nonflat holomorphic maps

$$f_p^t := 2\partial u_{p,j}^t/\theta \colon \overline{D}_j \longrightarrow \mathfrak{A}_*, \quad p \in P, \ t \in [0,1].$$

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$$\begin{aligned} \Re\mathcal{P}(f_p^t) &= 0, \quad (p,t) \in P \times [0,1]; \\ \mathcal{P}'(f_p^t|_{\overline{D}}) &= 0, \quad (p,t) \in P \times [0,1]; \\ \mathcal{P}(f_p^1) &= 0, \quad p \in P. \end{aligned}$$

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We embed f_p^t as the core $f_p^t = f_{p,0}^t$ of a **period dominating spray**

$$f_{p,\zeta}^t \colon \overline{D}_j \longrightarrow \mathfrak{A}_*, \quad \zeta \in B \subset \mathbb{C}^N, \ p \in P, \ t \in [0,1],$$

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i.e., the period map

$$B \ni \zeta \longmapsto \mathcal{P}(f_{p,\zeta}^t) = \left(\int_{\gamma_i} f_{p,\zeta}^t \theta \right)_{i=1,\ldots,\ell} \in (\mathbb{C}^n)^l$$

is submersive at $\zeta = 0$ for every $(p,t) \in P \times [0,1]$.

The noncritical case, continued

Since $\mathfrak{A}_*$ is an **Oka manifold** and $\overline{D}_j$ is a deformation retract of $\overline{D}_{j+1}$, the **parametric Oka property** allows us to approximate the spray $f^t_{p,\zeta}\colon \overline{D}_j \to \mathfrak{A}_*$ by a holomorphic spray

$$g^t_{p,\zeta}\colon \overline{D}_{j+1} \longrightarrow \mathfrak{A}_*, \quad (p,t) \in P \times [0,1],\ \zeta \in rB$$

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for some $r \in (1/2, 1)$. If the approximation is sufficiently close, the implicit function theorem gives (in view of the period domination property of the spray $f^t_{p,\zeta}$) a continuous map

$$\zeta\colon P \times [0,1] \longrightarrow rB \subset \mathbb{C}^N,$$

vanishing on $(P \times \{0\}) \cup (Q \times [0,1])$, such that the homotopy of holomorphic maps

$$\tilde{f}^t_p := g^t_{p,\zeta(p,t)}\colon \overline{D}_{j+1} \longrightarrow \mathfrak{A}_*, \quad (p,t) \in P \times [0,1]$$

satisfies the following period conditions:

$$\mathcal{P}(\tilde{f}^t_p) = \mathcal{P}(f^t_p), \qquad (p,t) \in P \times [0,1].$$

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Then, $u^t_{p,j+1} \colon \overline{D}_{j+1} \to \mathbb{R}^n$ is a continuous family of conformal minimal immersions satisfying conditions (a_{j+1})–(d_{j+1}).

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If $\overline{D}_j$ is disconnected, we apply the same argument on the components.

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The key to the proof is to find smooth extension of the map

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This is accomplished by the following lemma.

Paths with given integrals in the null quadric $\mathfrak{A}_*$

Lemma

Let $Q \subset P$ be compact Hausdorff spaces and $\sigma_p \colon [0,1] \to \mathfrak{A}_*$ be a family of paths depending continuously on the parameter $p \in P$.

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Main ingredient: Gromov's convex integration lemma

Gromov, 1973: Let Ω be an open connected set in a Banach space B. Fix a path $\sigma_0 : [0,1] \to \Omega$, and let Γ be the set of all paths $\sigma : [0,1] \to \Omega$ which are homotopic to σ_0 with fixed ends $\sigma(0) = \sigma_0(0)$, $\sigma(1) = \sigma_0(1)$.

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The main idea: We can represent any given vector $\alpha \in \mathrm{Co}(\Omega)$ as $\alpha = \sum_{i=1}^N p_i \delta_i$ with $p_i \in \Omega$ and $\sum_{i=1}^N \delta_i = 1$ for some big N.

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Construct a path $\sigma \in \Gamma$ which spends approximately the time δ_i at p_i for each i and goes quickly from one point to the next. Then,

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This shows that $\mathfrak{I}(\Gamma)$ is open, convex, and dense in $\mathrm{Co}(\Omega)$; hence it equals $\mathrm{Co}(\Omega)$. A similar argument applies in the parametric case.

How is this is lemma used?

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Let $\Omega \subset \mathbb{C}^n$ be a thin tubular neighborhood of $\mathfrak{A}_{r,R}$. We apply Gromov's lemma with the pair $\Omega \subset \mathrm{Co}(\Omega)$ to get a deformation (σ_p^t) which enjoys properties (i), (ii), and with (iii) replaced by an approximate condition

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This completes the proof of the main theorem.

Summary

We now know that the mapping spaces on the following diagram all have the same weak homotopy type as the space $\mathfrak{H}$ of continuous maps from a wedge of circles to $\mathfrak{A}_*$.

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Thus, the structure of $\mathfrak{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathfrak{H}$ can be calculated in terms of homotopy groups of spheres.

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We leave this for another day.

Strong homotopy equivalences

Theorem

Let M be a connected open Riemann surface of finite topological type: $H_1(M;\mathbb{Z}) = \mathbb{Z}^\ell$ with $\ell \in \mathbb{Z}_+$. Let $n \geq 3$. Then the metrizable spaces

$$\mathfrak{M}_*(M, \mathbb{R}^n),\ \mathfrak{N}_*(M, \mathbb{C}^n),\ \mathscr{O}(M, \mathfrak{A}_*),\ \mathscr{C}(M, \mathfrak{A}_*)$$

are absolute neighborhood retracts (ANR).

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Corollary

Let M be as above and $n \geq 3$. Then, the six maps in the above diagram are homotopy equivalences.
Moreover, the inclusion ι and the injections

$$\psi : \{u \in \mathfrak{M}_*(M,\mathbb{R}^n) : u(p) = 0\} \longrightarrow \mathscr{O}_*(M,\mathfrak{A}_*),$$

$$\phi : \{F \in \mathfrak{N}_*(M,\mathbb{C}^n) : F(p) = 0\} \longrightarrow \mathscr{O}_*(M,\mathfrak{A}_*)$$

are inclusions of strong deformation retracts.

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Smrekar & Yamashita 2009: Let M be a finitely dominated countable CW complex and let Z be a locally finite countable CW complex. Then the space $\mathscr{C}(M, Z)$ of continuous maps $M \to Z$ is ANR.

Absolute neighbourhood retracts (ANR)

Borsuk, 1932: A metric space X is said to be ANR if whenever it is embedded as a closed subspace of a metric space Y, some neighborhood of X in Y retracts onto X.

Whitehead 1949: If connected topological space X and Y have the homotopy type of a CW complex, then a weak homotopy equivalence $X \to Y$ is a homotopy equivalence.

Milnor, 1959: An ANR has the homotopy type of a CW complex. Hence, by Whitehead, a weak homotopy equivalence between a pair of ANR's is a homotopy equivalence.

Smrekar & Yamashita 2009: Let M be a finitely dominated countable CW complex and let Z be a locally finite countable CW complex. Then the space $\mathscr{C}(M, Z)$ of continuous maps $M \to Z$ is ANR.

Lárusson 2015: If M is a Stein manifold of finite type and Z is an Oka manifold, then $\mathscr{O}(M, Z)$ is an ANR, and $\mathscr{O}(M, Z) \hookrightarrow \mathscr{C}(M, Z)$ is the inclusion of a deformation retract.

Dugundji-Lefschetz characterization of ANRs

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Theorem (Dugundji-Lefschetz)

A metric space X is an ANR if and only if the following holds:

For every open cover $\mathscr{U}$ of X there is a refinement $\mathscr{V}$ of $\mathscr{U}$ such that if P is a simplicial complex with a subcomplex Q containing all the vertices of P, then every continuous map $\phi_0 : Q \to X$ such that

$$\text{for each simplex } \sigma \text{ of } P,\ \phi_0(\sigma \cap Q) \subset V \text{ for some } V \in \mathscr{V}$$

extends to a continuous map $\phi : P \to X$ such that

$$\text{for each simplex } \sigma \text{ of } P,\ \phi(\sigma) \subset U \text{ for some } U \in \mathscr{U}.$$

The spaces $\mathfrak{M}_*(M, \mathbb{R}^n)$, $\mathfrak{N}_*(M, \mathbb{C}^n)$ are ANR's

Assuming that M is a Riemann surface with finitely generated $H_1(M; \mathbb{Z})$, the Dugundji-Lefschetz property holds for the mapping spaces

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In particular, with M as above these spaces are locally contractible.