The h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$

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Abstract

Let $M$ be an open Riemann surface.

Alarcón and Forstnerič, 2015 (Crelle’s Journal, in press):
Every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$. 
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Every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$.

This is a basic h-principle. We now upgrade it to a parametric h-principle:

Theorem (F. Lárusson & F. Forstnerič, Feb. 2016)
For any $n \geq 3$, the inclusion

$$\iota : \mathcal{RN}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$$

of the space of real parts of all nonflat null holomorphic immersions $M \to \mathbb{C}^n$ into the space of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ satisfies the parametric h-principle with approximation; in particular, it is a weak homotopy equivalence (WHE).
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If $M$ has finitely generated homology group $H_1(M; \mathbb{Z})$, then

$\mathcal{RN}_\ast(M, \mathbb{C}^n)$ is a deformation retract of $\mathcal{M}_\ast(M, \mathbb{R}^n)$. 
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- convex integration theory (Gromov)
- absolute neighborhood retracts (Borsuk, Whitehead, Milnor,...)
Weierstrass representation of minimal surfaces in $\mathbb{R}^n$

Let $M$ be an open Riemann surface and $n \geq 3$. The following are equivalent for a conformal immersion $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$: 

$u$ parametrizes a minimal surface $u(M) \subset \mathbb{R}^n$. 

$u$ has identically vanishing mean curvature vector. 

$u$ is harmonic: $\triangle u = 0$. 

$\Phi = \partial u = (\phi_1, \ldots, \phi_n)$ is a holomorphic 1-form satisfying 

$\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2\right) = 0$. 

Conversely, if $\Phi = (\phi_1, \ldots, \phi_n)$ is as above and 

$\int_{\gamma} \Re(\Phi) = 0$ for all $\gamma \in H_1(M; \mathbb{Z})$, 

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The null quadric

\[ \mathbb{A} = \mathbb{A}^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}. \]
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Every conformal minimal immersion \( M \to \mathbb{R}^n \) \((n \geq 3)\) is of the form

\[ u(p) = u(p_0) + \int_{p_0}^{p} \Re (f \theta), \quad p_0, p \in M \]

where \( \theta \) is a nowhere vanishing holomorphic 1-form on \( M \),

\[ f = \frac{2 \partial u}{\theta} = (f_1, \ldots, f_n) : M \to \mathcal{A}_* = \mathcal{A} \setminus \{0\} \subset \mathbb{C}^n \]

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If the complex periods of \( f\theta \) vanish, then

\[ F(p) = \int_p^p f \theta \in \mathbb{C}^n, \quad p \in M \]

is a \textbf{holomorphic null curve} in \( \mathbb{C}^n \) with \( u = \Re F \). Equivalently:

\[ \text{Flux}(u)(\gamma) := \int_\gamma \Im(f \theta) = 0 \quad \forall \gamma \in H_1(M; \mathbb{Z}). \]
A diagram of spaces and maps

\[ \mathcal{N}_*(M, \mathbb{C}^n) : \text{nonflat holomorphic null curves } M \rightarrow \mathbb{C}^n \]
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The map \( \phi \) is given by \( F \mapsto \frac{\partial F}{\theta} \), and \( \psi \) is given by \( u \mapsto 2 \frac{\partial u}{\theta} \).

Our main theorem: each of the maps \( \iota, \phi, \psi \) is a weak homotopy equivalence (WHE).

The projection \( F \mapsto \text{Re } F \) of a null curve to its real part is clearly a homotopy equivalence.

Since \( A^\ast \) is an Oka manifold, the inclusion \( \mathcal{O}(M, A^\ast) \to \mathcal{C}(M, A^\ast) \) is a WHE by the Oka-Grauert principle.
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$\mathcal{N}_*(M, \mathbb{C}^n)$: nonflat holomorphic null curves $M \rightarrow \mathbb{C}^n$

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- Our main theorem: each of the maps $\iota$, $\phi$, $\psi$ is a weak homotopy equivalence (WHE).
- The projection $F \mapsto \Re F$ of a null curve to its real part is clearly a homotopy equivalence.
- Since $\mathcal{A}_*$ is an **Oka manifold**, the inclusion $\mathcal{O}(M, \mathcal{A}_*) \hookrightarrow \mathcal{C}(M, \mathcal{A}_*)$ is a WHE by the **Oka-Grauert principle**.
Connected components of the space $M_*(M, \mathbb{R}^n)$

Recall that $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell$ with $\ell \in \mathbb{Z}_+ \cup \{\infty\}$. 
Connected components of the space $\mathcal{M}_*(M, \mathbb{R}^n)$

Recall that $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell$ with $\ell \in \mathbb{Z}_+ \cup \{\infty\}$.

The punctured null quadric $\mathcal{A}^{n-1}_* \subset \mathbb{C}^n$ is simply connected when $n \geq 4$, while $\pi_1(\mathcal{A}^2_*) \cong \mathbb{Z}_2$ in view of the two-sheeted universal covering

$$\pi: \mathbb{C}_*^2 = \mathbb{C}^2 \setminus \{0\} \to \mathcal{A}^2_*, \quad \pi(u, v) = (u^2 - v^2, i(u^2 + v^2), 2uv).$$
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Hence, the path components of the space $\mathcal{C}(M, \mathcal{A}_*^2)$ are in one-to-one correspondence with group homomorphisms $H_1(M; \mathbb{Z}) \to \mathbb{Z}_2$ (i.e., the elements of $(\mathbb{Z}_2)^\ell$), and $\mathcal{C}(M, \mathcal{A}^{n-1}_*)$ is path connected if $n \geq 4$. 
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**Corollary**

*Let $M$ be a connected open Riemann surface with $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell$. Then the path connected components of $\mathcal{M}_*(M, \mathbb{R}^3)$ and $\mathcal{N}_*(M, \mathbb{C}^3)$ are in one-to-one correspondence with the elements of $(\mathbb{Z}_2)^\ell$. If $n \geq 4$ then $\mathcal{M}_*(M, \mathbb{R}^n)$ and $\mathcal{N}_*(M, \mathbb{C}^n)$ are path connected.*
Path connected components of the space $\mathcal{M}(M, \mathbb{R}^n)$

**Theorem (Alarcón, Forstnerič, Lopez, April 2016)**

Let $M$ be an open connected Riemann surface. The natural inclusion $\mathcal{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathcal{M}(M, \mathbb{R}^n)$ of the space of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ into the space of all conformal minimal immersions induces a bijection of path components of the two spaces.

Open problem: Is the inclusion $\mathcal{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathcal{M}(M, \mathbb{R}^n)$ a WHE?
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This follows by combining the following two results:

- Given a flat conformal minimal immersion $X: M \to \mathbb{R}^n$ ($n \geq 3$), there exists an isotopy $X_t: M \to \mathbb{R}^n$ ($t \in [0,1]$) of conformal minimal immersions such that $X_0 = X$ and $X_1$ is nonflat.
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- Let $\theta$ be a nowhere vanishing holomorphic 1-form on $M$. For every homomorphism $p: H_1(M; \mathbb{Z}) \to \mathbb{Z}_2 = H_1(\mathbb{A}_*; \mathbb{Z})$ there exists a flat conformal minimal immersion $X: M \to \mathbb{R}^3$ with $H_1(\partial X / \theta) = p$. 

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**Open problem:** Is the inclusion $\mathcal{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathcal{M}(M, \mathbb{R}^n)$ a WHE?
Let $M$ be $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ or an annulus, with $\theta = dz$. There are two homotopy classes of continuous or holomorphic maps $f : M \to \mathbb{A}_*$. 

Let $\pi : \mathbb{C}_*^2 \to \mathbb{A}_*$ be the universal covering map as above. Note that $f$ is nullhomotopic if and only if it factors through $\pi$.

Consider the **Weierstrass representation:**

$$f_1 = (1 - g^2)\eta, \quad f_2 = i(1 + g^2)\eta, \quad f_3 = 2g\eta,$$

where $g$ is meromorphic and $\eta$ is holomorphic on $M$. Assume for simplicity that $g$ is holomorphic or, equivalently, that $\eta$ has no zeros.

Then, $f$ factors through $\pi$ if and only if $\eta$ has a square root on $M$.

Indeed, if $\eta$ has a square root then $f = \pi(\sqrt{\eta}, g\sqrt{\eta})$; conversely, if $f = \pi(u, v)$ for some holomorphic map $(u, v) : M \to \mathbb{C}_*^2$, then $u^2 = \eta$. 

Examples in dimension $n = 3$

1. **A flat null curve:** $M = \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ and $f : \mathbb{C}_* \to \mathbb{A}_* \subset \mathbb{C}^3$ is the map $f(\zeta) = \zeta(1, i, 0)$. In this case, $g = 0$ and $\eta(\zeta) = \zeta$ does not have a square root on $M$. Thus, the flat null curve

$$F(\zeta) = \frac{1}{2}(\zeta^2, i\zeta^2, 0), \quad \zeta \in \mathbb{C}_*$$

has derivative in the nontrivial isotopy class.

2. **The catenoid:** $M = \mathbb{C}_*$, $g(\zeta) = \zeta$, and $\eta(\zeta) = 1/\zeta^2$. Since $\eta$ has a square root on $M$, we are in the trivial isotopy class. The same holds for the helicoid which is parameterized by $\mathbb{C}$.

3. **Henneberg’s surface:**

$$M = \mathbb{C} \setminus \{0, 1, -1, i, -i\}, \quad g(\zeta) = \zeta, \quad \eta(\zeta) = 1 - \zeta^{-4}.$$

On a small punctured disc centered at one of the points $1, -1, i, \text{ or } -i$, $\eta$ does not have a square root, so we are in the nontrivial isotopy class.

On the punctured disc $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$, the function $\eta$ has a square root, so we are in the trivial isotopy class.
Meeks’s minimal Möbius strip

4. Double sheeted covering of Meeks’s minimal Möbius strip:

\[ M = \mathbb{C}_*, \quad g(\zeta) = \zeta^2 \frac{\zeta + 1}{\zeta - 1}, \quad \eta(\zeta) = i \frac{(\zeta - 1)^2}{\zeta^4}. \]

Note that \( \eta \) has a square root on \( M \). Despite the pole of \( g \) at 1, we get a factorization through \( \pi \) and we are in the trivial isotopy class.

Let \( F = u + iv : \mathbb{C}_* \to \mathbb{C}^3 \) be the null curve with this Weierstrass data. Then \( u \) is invariant with respect to the fixed-point-free antiholomorphic involution

\[ J(\zeta) = -1/\bar{\zeta} \quad \text{on} \quad \zeta \in \mathbb{C}_*, \]

and hence it induces a conformal minimal immersion \( \mathbb{C}_*/J \to \mathbb{R}^3 \).

This is Meeks’s properly immersed minimal Möbius strip in \( \mathbb{R}^3 \) with finite total curvature \(-6\pi\).
Meeks’s minimal Möbius strip

W.H. Meeks, 1981:

Illustration © Antonio Alarcón.
Theorem

Assume that $M$ is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \subseteq M$ is a smoothly bounded Runge domain, and $u: M \times P \to \mathbb{R}^n \ (n \geq 3)$ is a continuous map satisfying the following:

(a) $u_p^t = u(\cdot, p): M \to \mathbb{R}^n$ is a nonflat CMI for every $p \in P$;
(b) $u_p^t|_D: D \to \mathbb{R}^n$ has vanishing flux for every $p \in P$;
(c) Flux($u_p$) = 0 for every $p \in Q$.

Given $\epsilon > 0$, there exists a homotopy $u_t: M \times P \to \mathbb{R}^n \ (t \in [0, 1])$ such that each map $u_t^p = u_t(\cdot, p): M \to \mathbb{R}^n$ is a nonflat CMI satisfying

(1) $u_t^p = u_p^t$ for every $(p, t) \in (P \times \{0\}) \cup (Q \times [0, 1])$;
(2) $|u_t^p(x) - u_p^t(x)| < \epsilon$ for all $x \in D$ and $(p, t) \in P \times [0, 1]$;
(3) $u_t^p|_D$ has vanishing flux for every $(p, t) \in P \times [0, 1]$;
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Parametric h-principle for $\mathcal{RM}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$
Theorem

Assume that $M$ is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \Subset M$ is a smoothly bounded Runge domain, and $u: M \times P \to \mathbb{R}^n$ $(n \geq 3)$ is a continuous map satisfying the following:

(a) $u_p = u(\cdot, p): M \to \mathbb{R}^n$ is a nonflat CMI for every $p \in P$;
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Given $\epsilon > 0$, there exists a homotopy $u^t: M \times P \to \mathbb{R}^n$ ($t \in [0, 1]$) such that each map $u^t_p := u^t(\cdot, p): M \to \mathbb{R}^n$ is a nonflat CMI satisfying
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Parametric h-principle for $\mathcal{RM}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$
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The WHE-principle for $\mathcal{R}_N^*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$

This is the parametric h-principle with approximation for the inclusion

$$\mathcal{R}_N^*(M, \mathbb{C}^n) = \{ u \in \mathcal{M}_*(M, \mathbb{R}^n) : \text{Flux}(u) = 0 \} \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n).$$
The WHE-principle for $\mathcal{R}M_*(M, \mathbb{C}^n) \hookrightarrow M_*(M, \mathbb{R}^n)$

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Assuming that this result holds, we now give:
The WHE-principle for $\mathcal{RN}_*(M, \mathbb{C}^n) \hookrightarrow M_*(M, \mathbb{R}^n)$

This is the parametric h-principle with approximation for the inclusion

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**Proof of the WHE-principle.**
The WHE-principle for $\mathcal{RN}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$

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**Proof of the WHE-principle.**

Let $k \in \mathbb{Z}_+$. Applying the h-principle with $P = S^k$ (the real $k$-sphere) and $Q = \emptyset$ shows that the inclusion induced map

$$\pi_k(\mathcal{RN}_*(M, \mathbb{C}^n)) \longrightarrow \pi_k(\mathcal{M}_*(M, \mathbb{R}^n)),$$

is surjective.
The WHE-principle for $\mathcal{RN}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$

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Applying the h-principle with $P = \overline{B}^{k+1}$ (the closed ball in $\mathbb{R}^{k+1}$) and $Q = S^k = b\overline{B}^{k+1}$ shows that the above map is also injective.
The WHE-principle for $\mathcal{RM}_*(M, \mathbb{C}^n) \hookrightarrow \mathcal{M}_*(M, \mathbb{R}^n)$

This is the parametric h-principle with approximation for the inclusion

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**Proof of the WHE-principle.**

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Thus, it is an isomorphism for every $k \in \mathbb{Z}_+$. \qed
Proof of the h-principle for $\mathcal{H}\mathcal{N}_*(M, \mathbb{C}^n) \hookrightarrow M_*(M, \mathbb{R}^n)$

Pick a smooth strongly subharmonic Morse exhaustion function $\rho : M \to \mathbb{R}$ and exhaust $M$ by sublevel sets

$$D_j = \{x \in M : \rho(x) < c_j\}, \quad j \in \mathbb{N}$$

where $c_1 < c_2 < c_3 < \ldots$ is an increasing sequence of regular values of $\rho$ such that $\lim_{j \to \infty} c_j = \infty$ and each interval $[c_j, c_{j+1}]$ contains at most one critical value of the function $\rho$.

We may assume that $D = D_1$. 
Proof of the h-principle for $\mathcal{R}N_\ast(M, C^n) \hookrightarrow M_\ast(M, \mathbb{R}^n)$

Pick a smooth strongly subharmonic Morse exhaustion function $\rho: M \to \mathbb{R}$ and exhaust $M$ by sublevel sets

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We may assume that $D = D_1$.

Let $\epsilon > 0$ be as in the theorem. Pick a sequence $\epsilon_j > 0$ with $\sum_{j=1}^\infty \epsilon_j < \epsilon$. Set

$$u_{p,1}^t := u_p|_{\overline{D}_1}, \quad (p, t) \in P \times [0, 1].$$
The recursive scheme

We recursively construct a sequence of homotopies of CMI’s

\[ u^t_{p,j} : \overline{D}_j \rightarrow \mathbb{R}^n, \quad (p, t) \in P \times [0, 1], \quad j \in \mathbb{N} \]

satisfying the following conditions for every \( j = 2, 3, \ldots \):
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(a) \( u^t_{p,j} = u_p|_{\overline{D}_j} \) for \( (p, t) \in (P \times \{0\}) \cup (Q \times [0, 1]) \);
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(b) \( \|u^t_{p,j} - u^t_{p,j-1}\|_{\overline{D}_{j-1}} < \epsilon_j \) for all \((p, t) \in P \times [0, 1];\)
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(c) \( \text{Flux}(u^t_{p,j}|_{\overline{D}_{j-1}}) = \text{Flux}(u^t_{p,j-1}) \) for every \( (p, t) \in P \times [0, 1] \);
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These conditions imply that the limit

\[ u^t_p = \lim_{j \rightarrow \infty} u^t_{p,j} : M \rightarrow \mathbb{R}^n \quad ((p, t) \in P \times [0, 1]) \]

exists and satisfies the conclusion of the theorem.
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$$u^t_{p,j} : \overline{D}_j \rightarrow \mathbb{R}^n, \quad (p, t) \in P \times [0, 1], \quad j \in \mathbb{N}$$

satisfying the following conditions for every $j = 2, 3, \ldots$:

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(c) $\text{Flux}(u^t_{p,j}|_{\overline{D}_{j-1}}) = \text{Flux}(u^t_{p,j-1})$ for every $(p, t) \in P \times [0, 1]$;

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Indeed, Conditions (1)–(4) follow from (a)–(d), respectively.
The noncritical case

(a) The noncritical case: $\rho$ has no critical values in $[c_j, c_{j+1}]$. 
The noncritical case

(a) The noncritical case: $\rho$ has no critical values in $[c_j, c_{j+1}]$.

Choose a Runge homology basis $\mathcal{B} = \{\gamma_i: i = 1, \ldots, \ell\}$ for $H_1(D_j; \mathbb{Z})$ such that $\mathcal{B}' = \{\gamma_1, \ldots, \gamma_m\}$ for some $m \in \{0, \ldots, \ell\}$ is a homology basis for $H_1(D; \mathbb{Z})$. 
The noncritical case

(a) The noncritical case: $\rho$ has no critical values in $[c_j, c_{j+1}]$.

Choose a Runge homology basis $B = \{\gamma_i: i = 1, \ldots, \ell\}$ for $H_1(\overline{D}_j; \mathbb{Z})$ such that $B' = \{\gamma_1, \ldots, \gamma_m\}$ for some $m \in \{0, \ldots, \ell\}$ is a homology basis for $H_1(\overline{D}; \mathbb{Z})$.

Then, $B$ is also a homology basis for $H_1(\overline{D}_{j+1}; \mathbb{Z})$. 
(a) **The noncritical case:** $\rho$ has no critical values in $[c_j, c_{j+1}]$.

Choose a Runge homology basis $\mathcal{B} = \{\gamma_i: i = 1, \ldots, \ell\}$ for $H_1(\overline{D}_j; \mathbb{Z})$ such that $\mathcal{B}' = \{\gamma_1, \ldots, \gamma_m\}$ for some $m \in \{0, \ldots, \ell\}$ is a homology basis for $H_1(\overline{D}; \mathbb{Z})$.

Then, $\mathcal{B}$ is also a homology basis for $H_1(\overline{D}_{j+1}; \mathbb{Z})$.

Denote by $\mathcal{P}$ the **period map** associated to $\mathcal{B}$:

$$\mathcal{P}(f) = \left( \int_{\gamma_i} f \theta \right)_{i=1,\ldots,\ell} \in (\mathbb{C}^n)^{\ell}, \quad f \in \mathcal{A}(\overline{D}_j, \mathcal{A}_*) .$$

Also, $\mathcal{P}' : \mathcal{A}(\overline{D}, \mathcal{A}_*) \to (\mathbb{C}^n)^m$ is the period map with respect to $\mathcal{B}'$. 
The noncritical case, continued

Consider the continuous family of nonflat holomorphic maps

\[ f_p^t := 2\partial u_{p,j}^t / \theta : D_j \to \mathbb{A}_*, \quad p \in P, \ t \in [0, 1]. \]
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\[ f^t_p := 2\partial u^t_{p,j}/\theta : \overline{D}_j \to \mathbb{A}_*, \quad p \in P, \ t \in [0, 1]. \]

Conditions on \( u^t_{p,j} : \overline{D}_j \to \mathbb{R}^n \) imply the following:

\[
\begin{align*}
\Re \mathcal{P}(f^t_p) &= 0, \quad (p, t) \in P \times [0, 1]; \\
\mathcal{P}'(f^t_p|_{\overline{D}}) &= 0, \quad (p, t) \in P \times [0, 1]; \\
\mathcal{P}(f^1_p) &= 0, \quad p \in P.
\end{align*}
\]
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\end{align*}
\]

We embed \( f^t_p \) as the core \( f^t_p = f^t_{p,0} \) of a period dominating spray
\[ f^t_{p,\zeta} : \overline{D_j} \longrightarrow \mathbb{A}_*, \quad \zeta \in B \subset \mathbb{C}^N, \; p \in P, \; t \in [0, 1], \]
Consider the continuous family of nonflat holomorphic maps

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Conditions on \( u^t_{p,j} : \overline{D}_j \rightarrow \mathbb{R}^n \) imply the following:

\[
\Re P(f^t_p) = 0, \quad (p, t) \in P \times [0, 1];
\]
\[
P'(f^t_p|_{\overline{D}}) = 0, \quad (p, t) \in P \times [0, 1];
\]
\[
P(f^1_p) = 0, \quad p \in P.
\]

We embed \( f^t_p \) as the core \( f^t_p = f^t_{p,0} \) of a period dominating spray

\[ f^t_{p,\zeta} : \overline{D}_j \longrightarrow \mathbb{A}_*, \quad \zeta \in B \subset \mathbb{C}^N, \; p \in P, \; t \in [0, 1], \]

i.e., the period map

\[
B \ni \zeta \longmapsto P(f^t_{p,\zeta}) = \left( \int_{\gamma_i} f^t_{p,\zeta} \theta \right)_{i=1,\ldots,\ell} \in (\mathbb{C}^n)^l
\]

is submersive at \( \zeta = 0 \) for every \((p, t) \in P \times [0, 1]\).
The noncritical case, continued

Since $\mathcal{A}_*$ is an **Oka manifold** and $\overline{D}_j$ is a deformation retract of $\overline{D}_{j+1}$, the **parametric Oka property** allows us to approximate the spray $f_{p,\zeta}^t : \overline{D}_j \to \mathcal{A}_*$ by a holomorphic spray

$$g_{p,\zeta}^t : \overline{D}_{j+1} \longrightarrow \mathcal{A}_*, \quad (p, t) \in P \times [0, 1], \; \zeta \in rB$$

for some $r \in (1/2, 1)$. 
The noncritical case, continued

Since $\mathcal{A}_*$ is an **Oka manifold** and $\overline{D}_j$ is a deformation retract of $\overline{D}_{j+1}$, the **parametric Oka property** allows us to approximate the spray $f^t_{p,\zeta}: \overline{D}_j \to \mathcal{A}_*$ by a holomorphic spray

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for some $r \in (1/2, 1)$. If the approximation is sufficiently close, the implicit function theorem gives (in view of the period domination property of the spray $f^t_{p,\zeta}$) a continuous map

$$\zeta: P \times [0, 1] \to rB \subset \mathbb{C}^N,$$

vanishing on $(P \times \{0\}) \cup (Q \times [0, 1])$, such that the homotopy of holomorphic maps

$$\tilde{f}^t_p := g^t_{p,\zeta(p,t)}: \overline{D}_{j+1} \to \mathcal{A}_*, \quad (p, t) \in P \times [0, 1]$$

satisfies the following period conditions:

$$\mathcal{P}(\tilde{f}^t_p) = \mathcal{P}(f^t_p), \quad (p, t) \in P \times [0, 1].$$
Assume that the set $\overline{D}_j$ (and hence $\overline{D}_{j+1}$) is connected.
The noncritical case, conclusion

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Then, \( u_{p,j+1}^t : \overline{D}_{j+1} \to \mathbb{R}^n \) is a continuous family of conformal minimal immersions satisfying conditions \((a_{j+1})-(d_{j+1})\).
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Then, $u^t_{p,j+1} : \overline{D}_{j+1} \to \mathbb{R}^n$ is a continuous family of conformal minimal immersions satisfying conditions $(a_{j+1})-(d_{j+1})$.

In particular,

$$\mathcal{P}(\tilde{f}^1_p) = \mathcal{P}(f^1_p) = 0 \text{ for } p \in P$$

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If $\overline{D}_j$ is disconnected, we apply the same argument on the components.
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The critical case

(a) The critical case: $\rho$ contains a unique critical point $x_0 \in D_{j+1} \setminus \overline{D}_j$. Then, $\overline{D}_{j+1}$ deformation retracts onto a compact set $S = \overline{D}_j \cup E$, where $E \subset M \setminus D_j$ is an embedded arc attached with both endpoints to $\overline{D}_j$. 
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It suffices to construct an isotopy $u^t_{p,j+1}$ satisfying $(a_{j+1})-(d_{j+1})$ on a neighborhood of $S$ and apply the noncritical case to extend it to $\overline{D}_{j+1}$. 

The key to the proof is to find smooth extension of the map $f_t = 2\partial u^t_{p,j+1}$ across the arc $E$ with the correct integral in order to ensure the required period conditions on $S = \overline{D}_j \cup E$. This is accomplished by the following lemma.
(a) **The critical case:** \( \rho \) contains a unique critical point \( x_0 \in D_{j+1} \setminus \overline{D}_j \). Then, \( \overline{D}_{j+1} \) deformation retracts onto a compact set \( S = \overline{D}_j \cup E \), where \( E \subset M \setminus D_j \) is an embedded arc attached with both endpoints to \( \overline{D}_j \).

It suffices to construct an isotopy \( u_{p,j+1}^t \) satisfying \((a_{j+1})-(d_{j+1})\) on a neighborhood of \( S \) and apply the noncritical case to extend it to \( \overline{D}_{j+1} \).

**The key to the proof is to find smooth extension of the map**

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f_p^t = 2\partial u_{p,j}^t/\theta : \overline{D}_j \to \mathcal{A}_*
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Lemma

Let $Q \subset P$ be compact Hausdorff spaces and $\sigma_p : [0, 1] \to \mathbb{A}_*$ be a family of paths depending continuously on the parameter $p \in P$. 
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Lemma

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Given a continuous family $\alpha^t_p \in \mathbb{C}^n$ ($p \in P$, $t \in [0, 1]$) such that

$$\alpha^t_p = \alpha_p \quad \text{for all} \ (p, t) \in (P \times \{0\}) \cup (Q \times [0, 1]),$$
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Paths with given integrals in the null quadric $\mathbb{A}_*$

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(iii) $\int_0^1 \sigma^t_p(s) \, ds = \alpha^t_p \quad \text{for all } p \in P \text{ and } t \in [0, 1]$. 
Main ingredient: Gromov’s convex integration lemma

Gromov, 1973: Let $\Omega$ be an open connected set in a Banach space $B$. Fix a path $\sigma_0 : [0, 1] \to \Omega$, and let $\Gamma$ be the set of all paths $\sigma : [0, 1] \to \Omega$ which are homotopic to $\sigma_0$ with fixed ends $\sigma(0) = \sigma_0(0)$, $\sigma(1) = \sigma_0(1)$. 
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The main idea: We can represent any given vector $\alpha \in \text{Co}(\Omega)$ as

$$\alpha = \sum_{i=1}^{N} p_i \delta_i$$

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Construct a path $\sigma \in \Gamma$ which spends approximately the time $\delta_i$ at $p_i$ for each $i$ and goes quickly from one point to the next. Then,

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This shows that $\mathcal{I}(\Gamma)$ is open, convex, and dense in $\text{Co}(\Omega)$; hence it equals $\text{Co}(\Omega)$. A similar argument applies in the parametric case.
How is this lemma used?

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Let $\Omega \subset C^n$ be a thin tubular neighborhood of $A_{r,R}$. We apply Gromov’s lemma with the pair $\Omega \subset \text{Co}(\Omega)$ to get a deformation $(\sigma_p^t)$ which enjoys properties (i), (ii), and with (iii) replaced by an approximate condition

$$\left| \int_0^1 \sigma_p^t(s) \, ds - \alpha_p^t \right| < \epsilon, \quad p \in P, \ t \in [0, 1].$$

The small error is caused by projecting the paths from $\Omega$ to $A_{r,R}$. This is applied on a segment $I \subset E$ of the arc $E$. The error is corrected by using period dominating sprays on another disjoint segment $I' \subset E$.

This completes the proof of the main theorem.
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This completes the proof of the main theorem.
We now know that the mapping spaces on the following diagram all have the same weak homotopy type as the space $\mathcal{H}$ of continuous maps from a wedge of circles to $\mathcal{A}_\ast$. The projection $\mathbb{R}^\ast : \mathbb{C}_n \to \mathbb{R}^n$ gives the punctured null quadric $\mathcal{A}_\ast$ the structure of a fibre bundle with fibre $S^{n-2}$ over $\mathbb{R}^n \{0\} \cong S^{n-1}$. Thus, the structure of $\mathcal{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathcal{H}$ can be calculated in terms of homotopy groups of spheres. We leave this for another day.
Summary

We now know that the mapping spaces on the following diagram all have the same weak homotopy type as the space $\mathcal{H}$ of continuous maps from a wedge of circles to $\mathbb{A}_*$. 

\[
\begin{array}{ccc}
\mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{\phi} & \Theta_*(M, \mathbb{A}_*) \\
\downarrow{\mathbb{R}} & & \uparrow{\psi} \\
\mathbb{R}\mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{l} & \mathcal{M}_*(M, \mathbb{R}^n)
\end{array}
\]
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$$
\begin{array}{cccc}
\mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{\phi} & \mathcal{O}_*(M, \mathcal{A}_*) & \xrightarrow{\psi} & \mathcal{C}(M, \mathcal{A}_*) \\
\mathcal{R} & & & & \\
\mathcal{R} \mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{l} & \mathcal{M}_*(M, \mathbb{R}^n) \\
\end{array}
$$

The projection $\mathcal{R} : \mathbb{C}^n \rightarrow \mathbb{R}^n$ gives the punctured null quadric $\mathcal{A}_*$ the structure of a fibre bundle with fibre $S^{n-2}$ over $\mathbb{R}^n \setminus \{0\} \sim S^{n-1}$. 
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\begin{array}{cccc}
\mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{\phi} & \mathcal{O}_*(M, \mathcal{A}_*) & \rightarrow \mathcal{O}(M, \mathcal{A}_*) \rightarrow \mathcal{C}(M, \mathcal{A}_*) \\
\mathcal{R} \downarrow & & \downarrow \psi & \\
\mathcal{R}\mathcal{N}_*(M, \mathbb{C}^n) & \xrightarrow{\iota} & \mathcal{M}_*(M, \mathbb{R}^n)
\end{array}
\]

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**Thus, the structure of $\mathcal{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathcal{H}$ can be calculated in terms of homotopy groups of spheres.**
We now know that the mapping spaces on the following diagram all have the same weak homotopy type as the space $\mathcal{H}$ of continuous maps from a wedge of circles to $A_*$. 

\[
\begin{array}{cccccc}
\mathcal{N}_*(M, C^n) & \xrightarrow{\phi} & \mathcal{O}_*(M, A_*) & \xleftarrow{\psi} & \mathcal{O}(M, A_*) & \xrightarrow{\psi} & \mathcal{C}(M, A_*) \\
\mathcal{R} & \downarrow & & & & & \\
\mathcal{R}\mathcal{N}_*(M, C^n) & \xleftarrow{\iota} & \mathcal{M}_*(M, R^n)
\end{array}
\]

The projection $\mathcal{R} : C^n \rightarrow R^n$ gives the punctured null quadric $A_*$ the structure of a fibre bundle with fibre $S^{n-2}$ over $R^n \setminus \{0\} \sim S^{n-1}$.

Thus, the structure of $\mathcal{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathcal{H}$ can be calculated in terms of homotopy groups of spheres.

We leave this for another day.
Theorem

Let $M$ be a connected open Riemann surface of finite topological type: $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$ with $\ell \in \mathbb{Z}_+$. Let $n \geq 3$. Then the metrizable spaces

$$\mathcal{M}_\ast(M, \mathbb{R}^n), \mathcal{N}_\ast(M, \mathbb{C}^n), \mathcal{O}(M, \mathbb{A}_\ast), \mathcal{C}(M, \mathbb{A}_\ast)$$

are absolute neighborhood retracts (ANR).
Strong homotopy equivalences

**Theorem**

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**Corollary**

Let $M$ be as above and $n \geq 3$. Then, the six maps in the above diagram are homotopy equivalences.
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Let $M$ be a connected open Riemann surface of finite topological type: $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$ with $\ell \in \mathbb{Z}_+$. Let $n \geq 3$. Then the metrizable spaces

\[ \mathcal{M}_*(M, \mathbb{R}^n), \mathcal{N}_*(M, \mathbb{C}^n), \mathcal{O}(M, \mathbb{A}_*), \mathcal{C}(M, \mathbb{A}_*) \]

are absolute neighborhood retracts (ANR).

Corollary

Let $M$ be as above and $n \geq 3$. Then, the six maps in the above diagram are homotopy equivalences. Moreover, the inclusion $\iota$ and the injections

\[ \psi : \{ u \in \mathcal{M}_*(M, \mathbb{R}^n) : u(p) = 0 \} \hookrightarrow \mathcal{O}_*(M, \mathbb{A}_*), \]

\[ \phi : \{ F \in \mathcal{N}_*(M, \mathbb{C}^n) : F(p) = 0 \} \hookrightarrow \mathcal{O}_*(M, \mathbb{A}_*) \]

are inclusions of strong deformation retracts.
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Whitehead 1949: If connected topological space $X$ and $Y$ have the homotopy type of a CW complex, then a weak homotopy equivalence $X \to Y$ is a homotopy equivalence.
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**Lárusson 2015:** If $M$ is a Stein manifold of finite type and $Z$ is an Oka manifold, then $O(M, Z)$ is an ANR, and $O(M, Z) \hookrightarrow C(M, Z)$ is the inclusion of a deformation retract.
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**Theorem (Dugundji-Lefschetz)**

*A metric space $X$ is an ANR if and only if the following holds:*

For every open cover $U$ of $X$ there is a refinement $V$ of $U$ such that if $P$ is a simplicial complex with a subcomplex $Q$ containing all the vertices of $P$, then every continuous map $\phi_0: Q \to X$ such that for each simplex $\sigma$ of $P$, $\phi_0(\sigma \cap Q) \subset V$ for some $V \in V$ extends to a continuous map $\phi: P \to X$ such that for each simplex $\sigma$ of $P$, $\phi(\sigma) \subset U$ for some $U \in U$. 
Dugundji-Lefschetz characterization of ANRs

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The spaces $\mathcal{M}_*(M, \mathbb{R}^n)$, $\mathcal{N}_*(M, \mathbb{C}^n)$ are ANR’s

Assuming that $M$ is a Riemann surface with finitely generated $H_1(M; \mathbb{Z})$, the Dugundji-Lefschetz property holds for the mapping spaces

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\[ \begin{align*}
\text{Alarcón, Forstneriˇc, 2014:} & \quad \text{The space } \mathcal{N}_r(M, \mathbb{C}^n) \text{ of all nonflat holomorphic null curves } M \to \mathbb{C}^n \text{ of class } C^r(M) \text{ is a complex Banach manifold.} \\
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Let $M$ be a compact bordered Riemann surface and $r \in \mathbb{N}$.

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In particular, with $M$ as above these spaces are locally contractible.
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