Holomorphic Legendrian curves and Darboux charts

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Complex contact manifolds

**Kobayashi 1959** A complex contact manifold is a pair $(X, \xi)$ where:

- $X$ is a complex manifold of odd dimension $2n + 1 \geq 3$,
- $\xi$ is a holomorphic hyperplane subbundle of the tangent bundle $TX$ which is **maximally nonintegrable**, in the sense that the **O’Neill tensor**

$$O : \xi \times \xi \to TX/\xi \equiv L, \ (v, w) \mapsto [v, w] \mod \xi$$

(also called the **Frobenius obstruction**) is nondegenerate.

- Equivalently, every point $p \in X$ admits an open neighborhood $U \subset X$ and a holomorphic 1-form $\alpha$ on $U$ such that

$$\xi|_U = \ker \alpha, \quad \alpha \wedge (d\alpha)^n \neq 0.$$

The 1-form $\alpha$ is determined up to a nonvanishing holomorphic factor.

Such $\xi$ is a **holomorphic contact structure** on $X$, and $\alpha$ is a **holomorphic contact form**.

(On the other hand, $\alpha \wedge d\alpha = 0$ defines a hypersurface foliation.)
Darboux’s theorem and stability results

Contact complex manifolds \((X, \xi)\) and \((X', \xi')\) are contactomorphic if there exists a biholomorphism \(F : X \to X'\) satisfying

\[
dF_x(\xi_x) = \xi'_{F(x)} \quad \text{for all } x \in X.
\]

**Example (Model complex contact space)**

\[(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0), \quad \alpha_0 = dz + \sum_{j=1}^n x_j dy_j,\]

\[
d\alpha_0 = \sum_{j=1}^n dx_j \wedge dy_j, \quad \alpha_0 \wedge (d\alpha_0)^n = n! dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge dz.
\]

**Darboux 1882; Moser 1965** Every complex contact manifold \((X^{2n+1}, \xi)\) is locally contactomorphic to \((\mathbb{C}^{2n+1}, \xi_0)\).

**Gray 1959** If \((X, \xi)\) is a compact contact manifold, then any contact perturbation \(\xi'\) of \(\xi\) is contactomorphic to \(\xi\).
Contact Hamiltonians

These are holomorphic vector fields $V$ on $(X, \xi)$ whose flow $\phi_t$ satisfies

$$\phi_t^* \xi = \xi.$$ 

If $\xi = \ker \alpha$, the above is equivalent to $\phi_t^* \alpha = f_t \alpha$ for some functions $f_t$.

The **Reeb vector field** of $(X, \alpha)$ is the unique holomorphic vector field satisfying

$$V \lrcorner \alpha = \alpha(V) = 1, \quad V \lrcorner d\alpha = 0.$$ 

There is a bijective correspondence between holomorphic functions $h \in \mathcal{O}(X)$ and holomorphic contact Hamiltonians, given by

- $V \mapsto h := V \lrcorner \alpha \in \mathcal{O}(X)$;
- $\mathcal{O}(X) \ni h \mapsto V$, where $V$ is determined by the conditions

$$V \lrcorner \alpha = h, \quad V \lrcorner d\alpha = -dh + V(h)\alpha.$$ 

In particular, if $X$ is compact then the only contact Hamiltonians are the constant multiples $cV$ ($c \in \mathbb{C}$) of the Reeb vector field.
The normal bundle of a contact structure

A complex contact manifold \((X, \xi)\) is given by a complex contact atlas \(\{(U_j, \alpha_j)\}\) with \(\alpha_i = f_{i,j} \alpha_j\) on \(U_{i,j} = U_i \cap U_j\).

The 1-cocycle \(f_{i,j} \in \mathcal{O}^*(U_{i,j})\) determines a holomorphic line bundle \(L = TX/\xi\) (the normal bundle), and the collection \((\alpha_i)\) is a 1-form \(\alpha \in \Gamma(X, \Omega^1(L))\) given by the tautological projection

\[
TX \xrightarrow{\alpha} L = TX/\xi.
\]

From \(d(f \alpha) = df \wedge \alpha + f \alpha = f \alpha \pmod{\alpha}\) we see that

\(d\alpha\) is a section of \(\Lambda^2(\xi^*) \otimes L\).

Thus, letting \(K_X = \Lambda^{2n+1}(T^*X)\) (the canonical bundle), we see that

\(\alpha \wedge (d\alpha)^n \neq 0\) is a section of \(K_X \otimes L^\otimes(n+1)\).

This provides a holomorphic line bundle isomorphism

\[(n+1)L = L^\otimes(n+1) \cong K_X^{-1} = \det(TX)\]
Conversely, assume that \( L \) is a holomorphic line bundle on \( X^{2n+1} \) with
\[
L^{\otimes (n+1)} \cong K_X^{-1}.
\]
Given a holomorphic 1-form \( \alpha \in \Gamma(X, \Omega^1(L)) \), we have
\[
\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \Gamma(X, \emptyset).
\]
If \( X \) is \textbf{compact} then \( \Gamma(X, \emptyset) = \mathbb{C} \). The map
\[
\Gamma(X, \Omega^1(L)) \ni \alpha \mapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}
\]
is homogeneous of degree \( n + 1 \). This shows:

\textbf{LeBrun & Salamon 1994, LeBrun 1995} The set of all complex contact structures with the normal bundle \( L \) on a compact manifold \( X \) is a connected complex manifold, i.e., the complement of a degree \( n + 1 \) hypersurface in \( \mathbb{P}(\Gamma(X, \Omega^1(L))) \) (or empty).

By Gray’s theorem, all these structures are contactomorphic. If \( X \) is simply connected, then it admits at most one complex contact structure.
Example: A contact structure on $\mathbb{CP}^{2n+1}$

Let $z_1, \ldots, z_{2n+2}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$ 

Then, ker $\theta$ determines a contact structure on $\mathbb{CP}^{2n+1} = \mathbb{C}^{2n+2}_*/\mathbb{C}^*$. 

Let $\theta_j$ ($j = 1, \ldots, 2n + 2$) be the pull-back of $\theta$ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$ 

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots.$$ 

Then $(H_j, \theta_j)$ is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each $j$, and this collection defines a contact atlas on $X = \mathbb{CP}^{2n+1}$. We have

$$K^{-1}_X = \theta_X(2n+2), \quad L = K^{-1/(n+1)}_X = \theta_X(2),$$

$$\alpha \in \Gamma(\mathbb{CP}^{2n+1}, \Omega^1(2)).$$
Complex contact structure on $\mathbb{P}(T^*Z)$

Let $Z^{n+1}$ be a complex manifold. The holomorphic cotangent bundle $T^*Z$ carries the tautological 1-form $\theta$ given in any local coordinates $z_0, \ldots, z_n$ on $Z$, and the induced fiber coordinates $\zeta_0, \ldots, \zeta_n$ on $T^*_ZZ$, by

$$\theta = \zeta_0 dz_0 + \ldots + \zeta_n dz_n \quad (= pdq \text{ in classical notation}).$$

Then, $\ker \theta$ determines a contact structure $\xi$ on the projectivized cotangent bundle $X = \mathbb{P}(T^*Z)$ (the manifold of all hyperplanes in $TZ$). On the affine chart $\{\zeta_j = 1\}$ we have

$$\xi = \ker(dz_j + \sum_{i \neq j} \zeta_i dz_i).$$

Note that

$$d\theta = \omega = \sum_{j=1}^n d\zeta_j \wedge dz_j \quad (= dp \wedge dq)$$

is the canonical symplectic form on $T^*Z$ and $\theta = i_\nabla \omega$, where $\nabla = \sum_{j=0}^n \zeta_j \partial_{\zeta_j}$ is the Euler vector field.
Contact hypersurfaces in complex symplectic manifolds

Let \((Z, \omega)\) be a **holomorphic symplectic manifold** of dimension \(2n + 2 \geq 4\), i.e. \(\omega\) is a holomorphic 2-form on \(Z\) with

\[
d\omega = 0 \quad \text{and} \quad \omega^{n+1} \neq 0.
\]

A holomorphic vector field \(V\) on \(Z\) is a **Liouville vector field** for \(\omega\) if

\[
L_V \omega = \omega \iff d(i_V \omega) = \omega \iff \phi_t^* \omega = e^t \omega.
\]

Here \(\phi_t\) is the flow of \(V\) and \(L_V\) is the Lie derivative. Let

\[
\theta = i_V \omega = V\| \omega; \quad d\theta = \omega.
\]

If \(X \subset Z\) is a complex hypersurface transverse to \(V\), then \(\alpha = \theta|_{TX}\) is a contact form on \(X\). Indeed:

\[
\theta \wedge (d\theta)^n = i_V \omega \wedge \omega^n = \frac{1}{n+1} i_V (\omega^{n+1})
\]

is a volume form on \(X\) provided that \(V\) is transverse to \(X\).

The converse: **symplectization** of a contact manifold \((X, \alpha)\):

\[
Z = \mathbb{C}_t \times X, \quad \omega = d(e^t \alpha) = e^t (dt \wedge \alpha + d\alpha), \quad i_{\partial_t} \omega|_{t=0} = \alpha.
\]
Contact Fano manifolds as closed adjoint orbits

**Boothby 1961, Wolf 1965, Beauville 1998** Let $G$ be a simple complex Lie group with Lie algebra $\mathfrak{g}$. The adjoint action of $G$ on $\mathbb{P}(\mathfrak{g})$ has a unique closed orbit $X_\mathfrak{g}$ which is contained in the closure of every other orbit; $X_\mathfrak{g}$ is a contact Fano manifold. The simplest example is $\mathbb{C}\mathbb{P}^{2n+1}$.

**Conjecture** Every projective contact manifold is a Fano manifold as above, or a projectivized cotangent bundle.

**Ye 1994** This holds true for projective threefolds.

**Demailly 2002** If a compact Kähler manifold $X$ admits a contact structure, then $K_X$ is not pseudo-effective (hence not nef), so $\kappa(X) = -\infty$. If in addition $b_2(X) = 1$ then $X$ is projective and hence $K_X$ is negative, i.e., $X$ is a Fano manifold.

Together with the results by Kebekus, Peternell and Sommese (2000) it follows that a projective contact manifold is either Fano with $b_2 = 1$, or a projectivized cotangent bundle.

**It remains to classify Fano manifolds with $b_2 = 1$.**
A smooth map $F: M \to (X, \xi)$ is said to be isotropic if

$$dF_p(T_pM) \subset \xi_{F(p)}, \quad p \in M.$$  

An isotropic immersion is Legendrian if $\dim_{\mathbb{R}} M = 2n$ is maximal.

If $\xi = \ker \alpha$ then $F: M \to X$ is isotropic iff $F^*\alpha = 0$.

It turns out that Legendrian submanifolds are necessarily complex:

**Lemma**

*If $\dim X = 2n + 1$ and $F$ is an isotropic immersion, then $\dim_{\mathbb{R}} M \leq 2n$; if $\dim_{\mathbb{R}} M = 2n$ then $F(M)$ is an immersed complex submanifold of $X$.***

**Corollary**

*A Stein complex contact manifold $(X, \xi)$ does not contain any (smooth) compact Legendrian submanifolds.*
How many Legendrian submanifolds are there?

Example (Legendrians in model contact space)

Let \((\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)\) with \(\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j\). Given a holomorphic function \(z = z(y_1, \ldots, y_n)\), the formula

\[
dz - \sum_{j=1}^{n} \frac{\partial z}{\partial y_j} dy_j = 0
\]

does not show that \(y \mapsto (-\frac{\partial z}{\partial y}, y, z(y))\) is a Legendrian submanifold.

**Bryant 1981** Every compact Riemann surface embeds as a complex Legendrian curve in \(\mathbb{CP}^3\). **Main idea:** consider meromorphic maps

\[
M \to \mathbb{C}^3_{(x,y,z)} \subset \mathbb{CP}^3, \quad t \mapsto (-\dot{z}(t)/\dot{y}(t), y(t), z(t)).
\]

Find \((y, z)\) such that this is an (Legendrian) embedding \(M \hookrightarrow \mathbb{CP}^3\).
Proper Legendrian curves in \((\mathbb{C}^{2n+1}, \xi_0)\)

**Theorem (Alarcón, F., López, Compositio Math., in press)**

Let \(M\) be an open Riemann surface and \(K \subset M\) be a compact set in \(M\) whose complement has no relatively compact connected components.

Then, every holomorphic Legendrian curve \(F: K \to \mathbb{C}^{2n+1}\) can be approximated uniformly on \(K\) by proper holomorphic Legendrian embeddings

\[
\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}) : M \hookrightarrow \mathbb{C}^{2n+1}.
\]

Furthermore, given a pair of indices \(\{i, j\} \subset \{1, 2, \ldots, 2n + 1\}\) with \(i \neq j\), we may choose \(\tilde{F}\) such that

\[
(\tilde{F}_i, \tilde{F}_j) : M \to \mathbb{C}^2 \text{ is a proper map.}
\]
Proof: The front projection

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha_0 = dz + xdy$.

If $(x, y, z): M \to \mathbb{C}^3$ is Legendrian, then its **front projection** $(y, z): M \to \mathbb{C}^2$ is a zig-zag diagram, with $x = -dz/dy$.

We find proper holomorphic maps $(y, z): M \to \mathbb{C}^2$ such that any critical point of $y$ is also a critical point of $z$ of order one more.
Proof: The Lagrangian projection

Consider $(\mathbb{C}^3, \alpha_0)$ with $\alpha_0 = dz + xdy$, $d\alpha_0 = dx \wedge dy$.

A holomorphic map $(x, y, z): M \rightarrow \mathbb{C}^3$ is Legendrian if and only if $xdy$ is an exact 1-form (i.e., with vanishing periods over all closed curves in $M$), and

$$z = -\int xdy.$$

The construction of proper exact immersions $(x, y): M \rightarrow \mathbb{C}^2$ proceeds by inductively enlarging their domain. Let $\rho: M \rightarrow \mathbb{R}^+_{\geq 0}$ be a strongly subharmonic Morse exhaustion function. We must consider two cases:

**The noncritical case:** Let $D \subset D'$ be Runge domains in $M$ of the form

$$D = \{ \rho < c \}, \quad D' = \{ \rho < c' \}, \quad d\rho \neq 0 \text{ on } D' \setminus D.$$

**The critical case:** $\rho$ has a single critical point $p \in D' \setminus \overline{D}$. The (only) nontrivial case is when the Morse index of $p$ is equals one (critical points of Morse index zero are local minima of $\rho$).
Proof, 2: The period map

**The noncritical case:** Let $C_1, \ldots, C_\ell \subset D$ be closed curves forming a basis of the homology group $H_1(D; \mathbb{Z}) \cong H_1(D'; \mathbb{Z}) = \mathbb{Z}^\ell$ such that $\bigcup_{j=1}^\ell C_j$ is Runge in $M$. Consider the **period map**

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell): \mathcal{A}^1(D)^2 \to \mathbb{C}^\ell$$

$$\mathcal{P}_j(x, y) = \int_{C_j} x \, dy, \quad x, y \in \mathcal{A}^1(D), \ j = 1, \ldots, \ell.$$ 

We may assume that $y \in \mathcal{A}^1(D)$ is nonconstant. We find a holomorphic spray $X(\cdot, \zeta): \overline{D} \to \mathbb{C}$ ($\zeta \in \mathbb{C}^\ell$) of class $\mathcal{A}^1(D)$ and of the form

$$X(u, \zeta) = x(u) + \sum_{k=1}^\ell \zeta_k g_k(u), \quad u \in \overline{D}, \ \zeta \in \mathbb{C}^\ell.$$ 

such that $X(\cdot, 0) = x$ and

$$\left. \frac{\partial}{\partial \zeta} \right|_{\zeta=0} \mathcal{P}(X(\cdot, \zeta), y): \mathbb{C}^\ell \to \mathbb{C}^\ell$$

is an isomorphism.
Proof, 3: Sprays and Runge’s theorem

By Runge’s theorem we can find holomorphic maps

\[ \tilde{x}(\cdot, \cdot) : M \times \mathbb{C}^\ell \to \mathbb{C}, \quad \tilde{y} : M \to \mathbb{C} \]

approximating \( X, y \) (respectively) in \( \mathcal{C}^1(\overline{D}) \).

Since \( \mathcal{P}(X(\cdot, 0), y) = 0 \), the period domination condition implies (by the implicit function theorem) that there is \( \zeta_0 \in \mathbb{C}^\ell \) close to 0 such that

\[ \mathcal{P}(\tilde{x}(\cdot, \zeta_0), \tilde{y}) = 0. \]

Hence, the 1-form \( \tilde{x}(\cdot, \zeta_0) d\tilde{y} \) is exact on \( \overline{D}' \). Fix a point \( p_0 \in D \) and set

\[ \tilde{z}(p) = z(p_0) - \int_{p_0}^p \tilde{x}(\cdot, \zeta_0) d\tilde{y}, \quad p \in D'. \]

The Legendrian curve

\[ (\tilde{x}(\cdot, \zeta_0), \tilde{y}, \tilde{z}) : \overline{D}' \to \mathbb{C}^3 \]

approximates \((x, y, z)\) in \( \mathcal{C}^1(\overline{D}) \). This establishes the noncritical case.
**Proof, 4: The critical case**

**The critical case:** This amounts to a change of topology of the sublevel set. The new bigger domain $D' \subset M$ deformation retracts onto $\overline{D \cup E}$, where $E$ is a smooth arc attached to $\overline{D}$ with its endpoints $a, b \in bD$.

Let $(x, y, z): \overline{D} \to \mathbb{C}^3$ be a Legendrian curve. We extend the functions $x, y$ to smooth functions $\tilde{x}, \tilde{y}: \overline{D \cup E} \to \mathbb{C}$ such that

$$\int_E \tilde{x} \tilde{y} = z(b) - z(a).$$

This ensures that the extended function

$$\tilde{z}(p) = z(a) + \int_a^p \tilde{x} \tilde{y}, \quad p \in \overline{D \cup E} \subset M$$

is well defined and matches the function $z$ on $\overline{D}$.

Hence, $(\tilde{x}, \tilde{y}, \tilde{z}): \overline{D \cup E} \to \mathbb{C}^3$ is a **generalized Legendrian curve**.

Now, use period dominating sprays and Mergelyan approximation theorem to conclude the proof similarly as before.
Proof, 5: How to ensure properness of \((x, y): M \to \mathbb{C}^2\)

Assume \(\max\{|x|, |y|\} > m\) on \(bD\). Subdivide \(bD\) into arcs \(\alpha_{l,a}\) such that on each of them, one of the functions \(|x|, |y|\) is \(> m\). Assume that \(|x| > m\) on \(\alpha_{l,a}\). Extend \(x\) smoothly to the arcs \(\gamma_{l,a}\) and \(\gamma_{l,a+1}\) such that \(|x| > m\), and \(|x| > m + 1\) at the outer endpoints of these two arcs. Apply Mergelyan to approximate \(x\) on \(\overline{D} \cup \gamma_{l,a} \cup \gamma_{l,a+1}\) by \(\tilde{x} \in \mathcal{O}(M)\).

Choose a disc \(Y_{l,a} \subset \Omega_{l,a}\) such that \(|\tilde{x}| > m\) on \(\Omega_{l,a} \setminus Y_{l,a}\). Use Mergelyan to approximate \(y\) on \(\overline{D}\) by \(\tilde{y} \in \mathcal{O}(M)\) such that \(|\tilde{y}| > m + 1\) on \(Y_{l,a}\). \(\Omega_{l,a}\). Then, \(\max\{|\tilde{x}|, |\tilde{y}|\} > m + 1\) on \(bD'\) and \(> m\) on \(\overline{D'} \setminus D\).
Rough shape of the space of Legendrian curves in $\mathbb{C}^{2n+1}$

Let $M$ be an open Riemann surface. Denote by $\mathcal{L}(M, \mathbb{C}^{2n+1})$ the space of all holomorphic Legendrian immersions $M \to (\mathbb{C}^{2n+1}, \xi_0)$. Let $pr: \mathbb{C}^{2n+1} \to \mathbb{C}^{2n}$ be the Lagrangian projection $(x, y, z) \mapsto (x, y)$.

Consider the sequence of maps

$$
\mathcal{L}(M, \mathbb{C}^{2n+1}) \xrightarrow{pr} \mathcal{I}_*(M, \mathbb{C}^{2n}) \xleftarrow{l} \mathcal{I}(M, \mathbb{C}^{2n}) \xrightarrow{\phi} \mathcal{O}(M, \mathbb{C}^{2n}_*) \xrightarrow{\psi} \mathcal{C}(M, S^{4n-1}),
$$

where $\mathcal{I}_*(M, \mathbb{C}^{2n})$ is the space of exact holomorphic immersions $M \to \mathbb{C}^{2n}$, $\mathcal{I}(M, \mathbb{C}^{2n})$ is the set of all holomorphic immersions,

$$
\phi(x, y) = (dx/\theta, dy/\theta): M \to \mathbb{C}^{2n}_*, \quad (x, y) \in \mathcal{I}(M, \mathbb{C}^{2n}),
$$

and $\psi$ is induced by the retraction $\mathbb{C}^{2n}_* \to S^{4n-1}$.

**Theorem (Lárusson and F., CAG and Math. Z. 2017)**

All maps in the above sequence are weak homotopy equivalences, and homotopy equivalence when $M$ has finite topological type.
Homotopy groups of $\mathcal{L}(M, C^{2n+1})$

The proof combines methods explained above and the parametric version of Gromov’s convex integration lemma.

Since $M$ has the homotopy type of a bouquet of $\ell$ circles, where $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$, we get

**Corollary (Lárusson and F., Math. Z. 2017)**

The space $\mathcal{L}(M, C^{2n+1})$ is weakly homotopy equivalent to the free $\ell$-loop space $\mathcal{L}_\ell S^{4n-1}$, and is homotopy equivalent to it if $\ell < \infty$.

It follows that

$$\pi_k(\mathcal{L}(M, C^{2n+1})) = \pi_k(S^{4n-1}) \times \pi_{k+1}(S^{4n-1})^\ell.$$ 

In particular, $\mathcal{L}(M, C^{2n+1})$ is $(4n - 3)$-connected.
The situation may be radically different for nonstandard contact structures on $\mathbb{C}^{2n+1}$. The **Kobayashi pseudometric** associated to a contact structure is defined by using **holomorphic Legendrian discs**.


For any $n \geq 1$ there exists a holomorphic contact structure $\xi$ on $\mathbb{C}^{2n+1}$ which is **Kobayashi hyperbolic** and isotopic to $\xi_0$. In particular, every holomorphic Legendrian curve from $\mathbb{C}$ or $\mathbb{C}^*$ to $(\mathbb{C}^{2n+1}, \xi)$ is constant.

**Idea of proof:** We take $\alpha = \Phi^* \alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j$ and $\Phi: \mathbb{C}^{2n+1} \rightarrow \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image $\Omega$ avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b \mathbb{D}^{2n}_{(x,y)} \times C_N \mathbb{D}_z.$$ 

Assuming that $C_N \geq n 2^{3N+1}$ for all $N \in \mathbb{N}$, $\mathbb{C}^{2n+1} \setminus K$ is $\alpha_0$-hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^* \alpha_0)$ is hyperbolic.
Complete bounded Legendrian curves in \((\mathbb{C}^{2n+1}, \xi_0)\)

**Martín-Umehara-Yamada 2014**  Do there exist complete bounded holomorphic Legendrian curves in \(\mathbb{C}^3\)? Can they have Jordan boundaries? (Analogue of the **Calabi-Yau problem** in the theory of minimal surface.)

**Theorem (Alarcón, F., López, Compositio Math.)**

Let \(M\) be a compact bordered Riemann surface. Every holomorphic Legendrian curve \(M \to \mathbb{C}^{2n+1}\) can be uniformly approximated by topological embeddings \(F: M \to \mathbb{C}^{2n+1}\) such that \(F|_\overset{\circ}{M}: \overset{\circ}{M} \to \mathbb{C}^{2n+1}\) is a complete holomorphic Legendrian embedding.

Besides the methods explained above, we use the following

**Riemann-Hilbert lemma for Legendrian curves:** given a Legendrian immersion \(f: M \to \mathbb{C}^{2n+1}\) and a continuous family of Legendrian discs \(F(u, \cdot): \overline{D} \to \mathbb{C}^{2n+1}\) with \(F(u, 0) = f(u)\) for all \(u \in bM\), there is a Legendrian approximate solution \(H: M \to \mathbb{C}^{2n+1}\) to the Riemann-Hilbert boundary value problem.
Theorem (Alarcón & F., preprint 2017)

Let $(X, \xi)$ be a complex contact manifold of dimension $2n + 1 \geq 3$.

Assume that $M$ is an open Riemann surface or a contractible Stein manifold of dimension $m \leq n$, $\theta_1, \ldots, \theta_m$ are holomorphic 1-forms on $M$ providing a framing of $T^*M$, and $f : M \to X$ is a holomorphic isotropic immersion.

Then there are a neighborhood $\Omega \subset M \times \mathbb{C}^{2n+1-m}$ of $M \times \{0\}$ and a holomorphic immersion $F : \Omega \to X$ such that $F|_{M \times \{0\}} = f$ and

$$F^*\xi = \ker \left( dz - \sum_{j=1}^{m} y_j \theta_j - \sum_{i=m+1}^{n} y_i dx_i \right),$$

where $(x_{m+1}, \ldots, x_n, y_1, \ldots, y_n, z)$ are complex coordinates on $\mathbb{C}^{2n+1-m}$. 

Darboux charts around isotropic Stein submanifolds
A special case

Assume that there is a holomorphic immersion \((x_1, \ldots, x_m): M \to \mathbb{C}^m\).
(This always holds if \(M\) is an open Riemann surface.) Choosing \(\theta_j = dx_j\) for \(j = 1, \ldots, m\), we get the normal form

\[
F^*\zeta = \ker \left( dz - \sum_{j=1}^{n} y_j dx_j \right).
\]

By using the shear automorphism

\[
z \mapsto z + \sum_{j=1}^{n} x_j y_j,
\]

we see that the above structure is contactomorphic to the one given by

\[
\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.
\]

This theorem is proved by standardizing the contact structure \(F^*\zeta\) along \(R \times \{0\}^{2n}\) and applying Moser’s method.
Lemma

Assume that \((X, \alpha)\) is complex contact manifold with a locally closed isotropic Stein submanifold \(M \subset X\). Let \(\alpha_t\) be a homotopy of holomorphic 1 forms on \(X\), with \(\alpha_0 = \alpha\), such that for all \(t \in [0, 1]\),

\[ \alpha = \alpha_t \text{ on } TX|_M \text{ and } \alpha_t \text{ is contact near } M. \]

Then there exist a neighborhood \(\Omega \subset X\) of \(M\) and a holomorphic flow \(\phi_t : \Omega \to X\) \((t \in [0, 1])\), fixing \(M\) pointwise, such that

\[ \phi_t^*(\alpha_t) = \alpha_0 \text{ for all } t \in [0, 1]. \]

If in addition we have

\[ d\alpha_t = d\alpha_0 \text{ on } TX|_M \text{ for all } t \in [0, 1], \]

then \(\phi_t\) be chosen such that \(T\phi_t = \text{Id} \text{ on } TX|_M\) for every \(t \in [0, 1]\).
Two consequences of the existence of Darboux charts

Corollary (Alarcón & F. 2017)

Let $M$ be a compact bordered Riemann surface. Every holomorphic Legendrian immersion $M \rightarrow (X, \xi)$ can be uniformly approximated by topological embeddings $\bar{f}: M \rightarrow X$ such that

$$\bar{f}|_{\bar{M}}: \bar{M} \rightarrow X$$ is a complete Legendrian embedding.

Corollary (Deformation theory for Legendrian curves)

The space of all small Legendrian deformations of a Legendrian immersion $f: M \rightarrow (X, \xi)$ can be identified with an open set in a complex Banach space which can be explicitly described (as in the model contact space $(\mathbb{C}^{2n+1}, \xi_0)$).
A few open problems

1. How many contact structures are there on $\mathbb{C}^3$? On $\mathbb{C}^{2n+1}$? How could one distinguish them?

2. Does every Stein manifold $\mathcal{X}^{2n+1}$ satisfying LeBrun-Salamon condition (i.e., such that the canonical bundle $K_{\mathcal{X}}$ has $(n+1)$-st root) admit a contact structure?
   
   Note that a generic holomorphic 1-form on a Stein manifold is contact on the complement of a complex hypersurface.

3. Does the Runge approximation theorem hold for holomorphic contact structures? In particular, is it possible to approximate a holomorphic contact form on a convex set in $\mathbb{C}^{2n+1}$ by a contact form on all of $\mathbb{C}^{2n+1}$?

4. Does every Stein contact manifold $(\mathcal{X}, \xi)$ admit proper holomorphic Legendrian curves normalized by any bordered Riemann surface? We have a positive answer for pseudoconvex domains in the model contact space $(\mathbb{C}^{2n+1}, \alpha_0)$. 