Holomorphic Legendrian curves and Darboux charts

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Complex contact manifolds

Kobayashi 1959 A **complex contact manifold** is a pair (X, ξ) where:

- X is a complex manifold of odd dimension $2n + 1 \geq 3$,
- ξ is a holomorphic hyperplane subbundle of the tangent bundle TX which is **maximally nonintegrable**, in the sense that the **O'Neill tensor**

$$O : \xi \times \xi \to TX/\xi = L, \quad (v, w) \mapsto [v, w] \mod \xi$$

  (also called the **Frobenius obstruction**) is nondegenerate.
- Equivalently, every point $p \in X$ admits an open neighborhood $U \subset X$ and a holomorphic 1-form α on U such that

$$\xi|_U = \ker \alpha, \quad \alpha \wedge (d\alpha)^n \neq 0.$$

  The 1-form α is determined up to a nonvanishing holomorphic factor.

Such ξ is a **holomorphic contact structure** on X, and α is a **holomorphic contact form**.

(On the other hand, $\alpha \wedge d\alpha = 0$ defines a hypersurface foliation.)

Darboux's theorem and stability results

Contact complex manifolds (X, ξ) and (X', ξ') are **contactomorphic** if there exists a biholomorphism $F\colon X \to X'$ satisfying

$$dF_x(\xi_x) = \xi'_{F(x)} \text{ for all } x \in X.$$

Example (Model complex contact space)

$$(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0), \quad \alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j,$$

$$d\alpha_0 = \sum_{j=1}^{n} dx_j \wedge dy_j, \quad \alpha_0 \wedge (d\alpha_0)^n = n! dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge dz.$$

Darboux 1882; Moser 1965 Every complex contact manifold (X^{2n+1}, ξ) is locally contactomorphic to $(\mathbb{C}^{2n+1}, \xi_0)$.

Gray 1959 If (X, ξ) is a compact contact manifold, then any contact perturbation ξ' of ξ is contactomorphic to ξ.

Contact Hamiltonians

These are holomorphic vector fields V on (X, ξ) whose flow ϕ_t satisfies

$$\phi_t^* \xi = \xi.$$

If $\xi = \ker \alpha$, the above is equivalent to $\phi_t^* \alpha = f_t \alpha$ for some functions f_t.

The **Reeb vector field** of (X, α) is the unique holomorphic vector field satisfying

$$V \rfloor \alpha = \alpha(V) = 1, \quad V \rfloor d\alpha = 0.$$

There is a bijective correspondence between holomorphic functions $h \in \mathscr{O}(X)$ and holomorphic contact Hamiltonians, given by

- $V \mapsto h := V \rfloor \alpha \in \mathscr{O}(X)$;
- $\mathscr{O}(X) \ni h \mapsto V$, where V is determined by the conditions

$$V \rfloor \alpha = h, \quad V \rfloor d\alpha = -dh + V(h)\alpha.$$

In particular, if X is compact then the only contact Hamiltonians are the constant multiples cV $(c \in \mathbb{C})$ of the Reeb vector field.

The normal bundle of a contact structure

A complex contact manifold (X, ξ) is given by a **complex contact atlas** $\{(U_j, \alpha_j)\}$ with $\alpha_i = f_{i,j}\alpha_j$ on $U_{i,j} = U_i \cap U_j$.

The 1-cocycle $f_{i,j} \in \mathscr{O}^*(U_{i,j})$ determines a holomorphic line bundle $L = TX/\xi$ **(the normal bundle)**, and the collection (α_i) is a 1-form $\alpha \in \Gamma(X, \Omega^1(L))$ given by the tautological projection

$$TX \xrightarrow{\alpha} L = TX/\xi.$$

From $d(f\alpha) = df \wedge \alpha + fd\alpha = fd\alpha \pmod{\alpha}$ we see that

$$d\alpha \text{ is a section of } \Lambda^2(\xi^*) \otimes L.$$

Thus, letting $K_X = \Lambda^{2n+1}(T^*X)$ (the canonical bundle), we see that

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ is a section of } K_X \otimes L^{\otimes(n+1)}.$$

This provides a holomorphic line bundle isomorphism

$$(n+1)L = L^{\otimes(n+1)} \cong K_X^{-1} = \det(TX).$$

The space of complex contact structures

Conversely, assume that L is a holomorphic line bundle on X^{2n+1} with

$$L^{\otimes(n+1)} \cong K_X^{-1}.$$

Given a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$, we have

$$\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \Gamma(X, \mathscr{O}).$$

If X is **compact** then $\Gamma(X, \mathscr{O}) = \mathbb{C}$. The map

$$\Gamma(X, \Omega^1(L)) \ni \alpha \mapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}$$

is homogeneous of degree $n+1$. This shows:

LeBrun & Salamon 1994, LeBrun 1995 The set of all complex contact structures with the normal bundle L on a compact manifold X is a connected complex manifold, i.e., the complement of a degree $n+1$ hypersurface in $\mathbb{P}(\Gamma(X, \Omega^1(L)))$ (or empty).

By Gray's theorem, all these structures are contactomorphic. If X is simply connected, then it admits at most one complex contact structure.

Example: A contact structure on $\mathbb{CP}^{2n+1}$

Let $z_1, \ldots, z_{2n+2}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$

Then, $\ker\theta$ determines a contact structure on $\mathbb{CP}^{2n+1} = \mathbb{C}_*^{2n+2}/\mathbb{C}^*$.

Let θ_j ($j = 1, \ldots, 2n+2$) be the pull-back of θ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots .$$

Then (H_j, θ_j) is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each j, and this collection defines a contact atlas on $X = \mathbb{CP}^{2n+1}$. We have

$$K_X^{-1} = \mathscr{O}_X(2n+2), \quad L = K_X^{-1/(n+1)} = \mathscr{O}_X(2),$$

$$\alpha \in \Gamma(\mathbb{CP}^{2n+1}, \Omega^1(2)).$$

Complex contact structure on $\mathbb{P}(T^*Z)$

Let Z^{n+1} be a complex manifold. The holomorphic cotangent bundle T^*Z carries the tautological 1-form θ given in any local coordinates $z_0, \ldots, z_n$ on Z, and the induced fiber coordinates $\zeta_0, \ldots, \zeta_n$ on $T^*_z Z$, by

$$\theta = \zeta_0 dz_0 + \ldots + \zeta_n dz_n \quad (= \mathbf{p} d\mathbf{q} \text{ in classical notation}).$$

Then, $\ker\theta$ determines a contact structure ξ on the projectivized cotangent bundle $X = \mathbb{P}(T^*Z)$ (the manifold of all hyperplanes in TZ). On the affine chart $\{\zeta_j = 1\}$ we have

$$\xi = \ker\Big(dz_j + \sum_{i\neq j} \zeta_i dz_i\Big).$$

Note that

$$d\theta = \omega = \sum_{j=1}^{n} d\zeta_j \wedge dz_j \quad (= d\mathbf{p} \wedge d\mathbf{q})$$

is the **canonical symplectic form** on T^*Z and $\theta = i_V\omega$, where $V = \sum_{j=0}^{n} \zeta_j \partial_{\zeta_j}$ is the **Euler vector field**.

Contact hypersurfaces in complex symplectic manifolds

Let (Z, ω) be a **holomorphic symplectic manifold** of dimension $2n+2 \geq 4$, i.e. ω is a holomorphic 2-form on Z with

$$d\omega = 0 \quad \text{and} \quad \omega^{n+1} \neq 0.$$

A holomorphic vector field V on Z is a **Liouville vector field** for ω if

$$L_V\omega = \omega \iff d(i_V\omega) = \omega \iff \phi_t^*\omega = e^t\omega.$$

Here ϕ_t is the flow of V and L_V is the Lie derivative. Let

$$\theta = i_V\omega = V\rfloor\omega; \quad d\theta = \omega.$$

If $X \subset Z$ is a complex hypersurface transverse to V, then $\alpha = \theta|_{TX}$ is a contact form on X. Indeed:

$$\theta \wedge (d\theta)^n = i_V\omega \wedge \omega^n = \frac{1}{n+1} i_V(\omega^{n+1})$$

is a volume form on X provided that V is transverse to X.

The converse: **symplectization** of a contact manifold (X, α):

$$Z = \mathbb{C}_t \times X, \quad \omega = d(e^t\alpha) = e^t\left(dt \wedge \alpha + d\alpha\right), \quad i_{\partial_t}\omega|_{t=0} = \alpha.$$

Contact Fano manifolds as closed adjoint orbits

Boothby 1961, Wolf 1965, Beauville 1998 Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$. The adjoint action of G on $\mathbb{P}(\mathfrak{g})$ has a unique closed orbit $X_{\mathfrak{g}}$ which is contained in the closure of every other orbit; **$X_{\mathfrak{g}}$ is a contact Fano manifold.** The simplest example is $\mathbb{CP}^{2n+1}$.

Conjecture Every projective contact manifold is a Fano manifold as above, or a projectivized cotangent bundle.

Ye 1994 This holds true for projective threefolds.

Demailly 2002 If a compact Kähler manifold X admits a contact structure, then K_X is not pseudo-effective (hence not nef), so $\kappa(X) = -\infty$. If in addition $b_2(X) = 1$ then X is projective and hence K_X is negative, i.e., X is a **Fano manifold**.

Together with the results by **Kebekus, Peternell and Sommese (2000)** it follows that **a projective contact manifold is either Fano with $b_2 = 1$, or a projectivized cotangent bundle**.

It remains to classify Fano manifolds with $b_2 = 1$.

Isotropic and Legendrian submanifolds

A smooth map $F\colon M \to (X, \xi)$ is said to be **isotropic** if

$$dF_p(T_pM) \subset \xi_{F(p)}, \quad p \in M.$$

An isotropic immersion is **Legendrian** if $\dim_{\mathbb{R}} M = 2n$ is maximal.

If $\xi = \ker \alpha$ then $F\colon M \to X$ is isotropic iff $F^*\alpha = 0$.

It turns out that Legendrian submanifolds are necessarily complex:

Lemma

If $\dim X = 2n + 1$ *and* F *is an isotropic immersion, then* $\dim_R M \leq 2n$*; if* $\dim_{\mathbb{R}} M = 2n$ *then* $F(M)$ *is an immersed complex submanifold of* X*.*

Corollary

A Stein complex contact manifold (X, ξ) *does not contain any (smooth) compact Legendrian submanifolds.*

How many Legendrian submanifolds are there?

Example (Legendrians in model contact space)

Let $(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)$ with $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$. Given a holomorphic function $z = z(y_1, \ldots, y_n)$, the formula

$$dz - \sum_{j=1}^{n} \frac{\partial z}{\partial y_j} dy_j = 0$$

shows that $y \mapsto (-\partial z/\partial y, y, z(y))$ is a Legendrian submanifold.

Bryant 1981 Every compact Riemann surface embeds as a complex Legendrian curve in $\mathbb{CP}^3$. **Main idea:** consider meromorphic maps

$$M \to \mathbb{C}^3_{(x,y,z)} \subset \mathbb{CP}^3, \quad t \mapsto (-\dot{z}(t)/\dot{y}(t), y(t), z(t)).$$

Find (y, z) such that this is an (Legendrian) embedding $M \hookrightarrow \mathbb{CP}^3$.

Proper Legendrian curves in $(\mathbb{C}^{2n+1}, \xi_0)$

Theorem (Alarcón, F., López, Compositio Math., in press)

Let M be an open Riemann surface and $K \subset M$ be a compact set in M whose complement has no relatively compact connected components.

Then, every holomorphic Legendrian curve $F \colon K \to \mathbb{C}^{2n+1}$ can be approximated uniformly on K by proper holomorphic Legendrian embeddings

$$\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_{2n+1}) : M \hookrightarrow \mathbb{C}^{2n+1}.$$

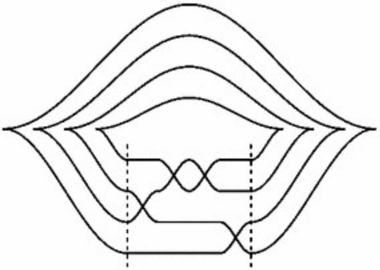
Furthermore, given a pair of indices $\{i, j\} \subset \{1, 2, \ldots, 2n+1\}$ with $i \neq j$, we may choose $\widetilde{F}$ such that

$$(\widetilde{F}_i, \widetilde{F}_j) \colon M \to \mathbb{C}^2 \text{ is a proper map.}$$

Proof: The front projection

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha_0 = dz + xdy$.

If $(x, y, z)\colon M \to \mathbb{C}^3$ is Legendrian, then its **front projection** $(y, z)\colon M \to \mathbb{C}^2$ is a zig-zag diagram, with $x = -dz/dy$.



We find proper holomorphic maps $(y, z)\colon M \to \mathbb{C}^2$ such that any critical point of y is also a critical point of z of order one more.

Proof: The Lagrangain projection

Consider $(\mathbb{C}^3, \alpha_0)$ with $\alpha_0 = dz + xdy$, $d\alpha_0 = dx \wedge dy$.

A holomorphic map $(x, y, z) \colon M \to \mathbb{C}^3$ is Legendrian if and only if xdy **is an exact 1-form** (i.e., with vanishing periods over all closed curves in M), and

$$z = -\int^{\cdot} xdy.$$

The construction of **proper exact immersions** $(x, y) \colon M \to \mathbb{C}^2$ proceeds by inductively enlarging their domain. Let $\rho \colon M \to \mathbb{R}_+$ be a strongly subharmonic Morse exhaustion function. We must consider two cases:

The noncritical case: Let $D \subset D'$ be Runge domains in M of the form

$$D = \{\rho < c\}, \quad D' = \{\rho < c'\}, \quad d\rho \neq 0 \text{ on } \overline{D}' \setminus D.$$

The critical case: ρ has a single critical point $p \in D' \setminus \overline{D}$.
The (only) nontrivial case is when the Morse index of p is equals one (critical points of Morse index zero are local minima of ρ).

Proof, 2: The period map

The noncritical case: Let $C_1, \ldots, C_\ell \subset D$ be closed curves forming a basis of the homology group $H_1(D;\mathbb{Z}) \cong H_1(D';\mathbb{Z}) = \mathbb{Z}^\ell$ such that $\bigcup_{j=1}^{\ell} C_j$ is Runge in M. Consider the **period map**

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) : \mathscr{A}^1(D)^2 \to \mathbb{C}^\ell$$

$$\mathcal{P}_j(x, y) = \int_{C_j} x\, dy, \qquad x, y \in \mathscr{A}^1(D),\ j = 1, \ldots, \ell.$$

We may assume that $y \in \mathscr{A}^1(D)$ is nonconstant. We find a holomorphic spray $X(\cdot\,, \zeta) : \overline{D} \to \mathbb{C}$ $(\zeta \in \mathbb{C}^\ell)$ of class $\mathscr{A}^1(D)$ and of the form

$$X(u, \zeta) = x(u) + \sum_{k=1}^{\ell} \zeta_k\, g_k(u), \quad u \in \overline{D},\ \zeta \in \mathbb{C}^\ell.$$

such that $X(\cdot\,, 0) = x$ and

$$\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=0} \mathcal{P}(X(\cdot\,, \zeta), y) : \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell \ \text{ is an isomorphism.}$$

Proof, 3: Sprays and Runge's theorem

By Runge's theorem we can find holomorphic maps

$$\widetilde{x}(\cdot\,,\cdot)\colon M\times\mathbb{C}^\ell\to\mathbb{C},\quad \tilde{y}\colon M\to\mathbb{C}$$

approximating X, y (respectively) in $\mathscr{C}^1(\overline{D})$.

Since $\mathcal{P}(X(\cdot\,,0),y)=0$, the period domination condition implies (by the implicit function theorem) that there is $\zeta_0\in\mathbb{C}^\ell$ close to 0 such that

$$\mathcal{P}(\widetilde{x}(\cdot\,,\zeta_0),\tilde{y})=0.$$

Hence, the 1-form $\widetilde{x}(\cdot\,,\zeta_0)d\tilde{y}$ is **exact** on $\overline{D}'$. Fix a point $p_0\in D$ and set

$$\widetilde{z}(p)=z(p_0)-\int_{p_0}^{p}\widetilde{x}(\cdot\,,\zeta_0)d\tilde{y},\quad p\in D'.$$

The Legendrian curve

$$(\widetilde{x}(\cdot\,,\zeta_0),\widetilde{y},\widetilde{z}):\overline{D}'\to\mathbb{C}^3$$

approximates (x,y,z) in $\mathscr{C}^1(\overline{D})$. This establishes the noncritical case.

Proof, 4: The critical case

The critical case: This amounts to a change of topology of the sublevel set. The new bigger domain $D' \subset M$ deformation retracts onto $\overline{D} \cup E$, where E is a smooth arc attached to $\overline{D}$ with its endpoints $a, b \in bD$.

Let $(x, y, z) \colon \overline{D} \to \mathbb{C}^3$ be a Legendrian curve. We extend the functions x, y to smooth functions $\tilde{x}, \tilde{y} \colon \overline{D} \cup E \to \mathbb{C}$ such that

$$\int_E \tilde{x} d\tilde{y} = z(b) - z(a).$$

This ensures that the extended function

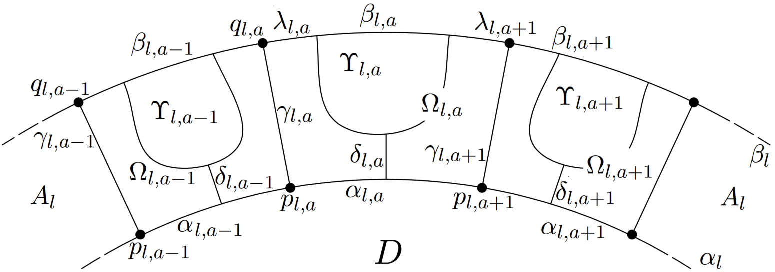
$$\tilde{z}(p) = z(a) + \int_a^p \tilde{x} d\tilde{y}, \quad p \in \overline{D} \cup E \subset M$$

is well defined and matches the function z on $\overline{D}$.

Hence, $(\tilde{x}, \tilde{y}, \tilde{z}) \colon \overline{D} \cup E \to \mathbb{C}^3$ is a **generalized Legendrian curve**.

Now, use period dominating sprays and Mergelyan approximation theorem to conclude the proof similarly as before.

Proof, 5: How to ensure properness of $(x, y)\colon M \to \mathbb{C}^2$



Assume $\max\{|x|, |y|\} > m$ on bD. Subdivide bD into arcs $\alpha_{l,a}$ such that on each of them, one of the functions $|x|, |y|$ is $> m$. Assume that $|x| > m$ on $\alpha_{l,a}$. Extend x smoothly to the arcs $\gamma_{l,a}$ and $\gamma_{l,a+1}$ such that $|x| > m$, and $|x| > m + 1$ at the outer endpoints of these two arcs. Apply Mergelyan to approximate x on $\overline{D} \cup \gamma_{l,a} \cup \gamma_{l,a+1}$ by $\tilde{x} \in \mathscr{O}(M)$. Choose a disc $\Upsilon_{l,a} \subset \Omega_{l,a}$ such that $|\tilde{x}| > m$ on $\overline{\Omega_{l,a} \setminus \Upsilon_{l,a}}$. Use Mergelyan to approximate y on $\overline{D}$ by $\tilde{y} \in \mathscr{O}(M)$ such that $|\tilde{y}| > m + 1$ on $\Upsilon_{l,a}$. $\Omega_{l,a}$. Then, $\max\{|\tilde{x}|, |\tilde{y}|\} > m + 1$ on bD' and $> m$ on $\overline{D}' \setminus D$.

Rough shape of the space of Legendrian curves in $\mathbb{C}^{2n+1}$

Let M be an open Riemann surface. Denote by $\mathscr{L}(M, \mathbb{C}^{2n+1})$ the space of all holomorphic Legendrian immersions $M \to (\mathbb{C}^{2n+1}, \xi_0)$. Let $pr\colon \mathbb{C}^{2n+1} \to \mathbb{C}^{2n}$ be the Lagrangian projection $(x, y, z) \mapsto (x, y)$.

Consider the sequence of maps

$$\mathscr{L}(M, \mathbb{C}^{2n+1}) \xrightarrow{pr} \mathfrak{I}_*(M, \mathbb{C}^{2n}) \overset{\iota}{\hookrightarrow} \mathfrak{I}(M, \mathbb{C}^{2n}) \xrightarrow{\phi} \mathscr{O}(M, \mathbb{C}^{2n}_*) \xrightarrow{\psi} \mathscr{C}(M, S^{4n-1}),$$

where $\mathfrak{I}_*(M, \mathbb{C}^{2n})$ is the space of exact holomorphic immersions $M \to \mathbb{C}^{2n}$, $\mathfrak{I}(M, \mathbb{C}^{2n})$ is the set of all holomorphic immersions,

$$\phi(x, y) = (dx/\theta, dy/\theta)\colon M \to \mathbb{C}^{2n}_*, \quad (x, y) \in \mathfrak{I}(M, \mathbb{C}^{2n}),$$

and ψ is induced by the retraction $\mathbb{C}^{2n}_* \to S^{4n-1}$.

Theorem (Lárusson and F., CAG and Math. Z. 2017)

All maps in the above sequence are weak homotopy equivalences, and homotopy equivalence when M has finite topological type.

Homotopy groups of $\mathscr{L}(M, \mathbb{C}^{2n+1})$

The proof combines methods explained above and the parametric version of **Gromov's convex integration lemma**.

Since M has the homotopy type of a bouquet of ℓ circles, where $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$, we get

Corollary (Lárusson and F., Math. Z. 2017)

The space $\mathscr{L}(M, \mathbb{C}^{2n+1})$ is weakly homotopy equivalent to the free ℓ-loop space $\mathcal{L}_\ell S^{4n-1}$, and is homotopy equivalent to it if $\ell < \infty$.

It follows that

$$\pi_k(\mathscr{L}(M, \mathbb{C}^{2n+1})) = \pi_k(S^{4n-1}) \times \pi_{k+1}(S^{4n-1})^\ell.$$

In particular, $\mathscr{L}(M, \mathbb{C}^{2n+1})$ is $(4n-3)$-connected.

A hyperbolic contact structure on $\mathbb{C}^{2n+1}$

The situation may be radically different for nonstandard contact structures on $\mathbb{C}^{2n+1}$. The **Kobayashi pseudometric** associated to a contact structure is defined by using **holomorphic Legendrian discs**.

Theorem (F., J. Geom. Anal. 2017)

For any $n \geq 1$ there exists a holomorphic contact structure ξ on $\mathbb{C}^{2n+1}$ which is **Kobayashi hyperbolic** *and isotopic to ξ_0. In particular, every holomorphic Legendrian curve from $\mathbb{C}$ or $\mathbb{C}^*$ to $(\mathbb{C}^{2n+1}, \xi)$ is constant.*

Idea of proof: We take $\alpha = \Phi^*\alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$ and $\Phi\colon \mathbb{C}^{2n+1} \longrightarrow \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image Ω avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^{2n}_{(x,y)} \times C_N \overline{\mathbb{D}}_z.$$

Assuming that $C_N \geq n2^{3N+1}$ for all $N \in \mathbb{N}$,

$\mathbb{C}^{2n+1} \setminus K$ is α_0-hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^*\alpha_0)$ is hyperbolic.

Complete bounded Legendrian curves in $(\mathbb{C}^{2n+1}, \xi_0)$

Martín-Umehara-Yamada 2014 Do there exists complete bounded holomorphic Legendrian curves in $\mathbb{C}^3$? Can they have Jordan boundaries? (Analogue of the **Calabi-Yau problem** in the theory of minimal surface.)

Theorem (Alarcón, F., López, Compositio Math.)

Let M be a compact bordered Riemann surface. Every holomorphic Legendrian curve $M \to \mathbb{C}^{2n+1}$ can be uniformly approximated by topological embeddings $F\colon M \to \mathbb{C}^{2n+1}$ such that $F|_{\mathring{M}}\colon \mathring{M} \to \mathbb{C}^{2n+1}$ is a complete holomorphic Legendrian embedding.

Besides the methods explained above, we use the following

Riemann-Hilbert lemma for Legendrian curves: given a Legendrian immersion $f\colon M \to \mathbb{C}^{2n+1}$ and a continuous family of Legendrian discs $F(u,\cdot)\colon \overline{\mathbb{D}} \to \mathbb{C}^{2n+1}$ with $F(u,0) = f(u)$ for all $u \in bM$, there is a Legendrian approximate solution $H\colon M \to \mathbb{C}^{2n+1}$ to the Riemann-Hilbert boundary value problem.

Darboux charts around isotropic Stein submanifolds

Theorem (Alarcón & F., preprint 2017)

Let (X, ξ) be a complex contact manifold of dimension $2n+1 \geq 3$.

Assume that M is an **open Riemann surface** *or a* **contractible Stein manifold** *of dimension $m \leq n$, $\theta_1, \ldots, \theta_m$ are holomorphic 1-forms on M providing a framing of T^*M, and $f \colon M \to X$ is a holomorphic isotropic immersion.*

Then there are a neighborhood $\Omega \subset M \times \mathbb{C}^{2n+1-m}$ of $M \times \{0\}$ and a holomorphic immersion $F \colon \Omega \to X$ such that $F|_{M\times\{0\}} = f$ and

$$F^*\xi = \ker\left(dz - \sum_{j=1}^{m} y_j \theta_j - \sum_{i=m+1}^{n} y_i dx_i\right),$$

where $(x_{m+1}, \ldots, x_n, y_1, \ldots, y_n, z)$ are complex coordinates on $\mathbb{C}^{2n+1-m}$.

A special case

Assume that there is a holomorphic immersion $(x_1, \ldots, x_m)\colon M \to \mathbb{C}^m$. (This always holds if M is an open Riemann surface.) Choosing $\theta_j = dx_j$ for $j = 1, \ldots, m$, we get the normal form

$$F^*\xi = \ker\left(dz - \sum_{j=1}^{n} y_j dx_j\right).$$

By using the shear automorphism

$$z \mapsto z + \sum_{j=1}^{n} x_j y_j,$$

we see that the above structure is contactomorphic to the one given by

$$\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.$$

This theorem is proved by standardizing the contact structure $F^*\xi$ along $R \times \{0\}^{2n}$ and applying **Moser's method**.

Moser's method for holomorphic contact forms

Lemma

Assume that (X, α) is complex contact manifold with a locally closed isotropic Stein submanifold $M \subset X$. Let α_t be a homotopy of holomorphic 1 forms on X, with $\alpha_0 = \alpha$, such that for all $t \in [0, 1]$,

$$\alpha = \alpha_t \text{ on } TX|_M \quad \text{and} \quad \alpha_t \text{ is contact near } M.$$

Then there exist a neighborhood $\Omega \subset X$ of M and a holomorphic flow $\phi_t \colon \Omega \to X$ $(t \in [0, 1])$, fixing M pointwise, such that

$$\phi_t^*(\alpha_t) = \alpha_0 \quad \text{for all } t \in [0, 1].$$

If in addition we have

$$d\alpha_t = d\alpha_0 \text{ on } TX|_M \quad \text{for all } t \in [0, 1],$$

then ϕ_t be chosen such that $T\phi_t = \mathrm{Id}$ on $TX|_M$ for every $t \in [0, 1]$.

Two consequences of the existence of Darboux charts

Corollary (Alarcón & F. 2017)

Let M be a compact bordered Riemann surface. Every holomorphic Legendrian immersion $M \to (X, \xi)$ can be uniformly approximated by topological embeddings $\tilde{f} : M \to X$ such that

$$\tilde{f}|_{\mathring{M}} : \mathring{M} \to X \text{ is a complete Legendrian embedding.}$$

Corollary (Deformation theory for Legendrian curves)

The space of all small Legendrian deformations of a Legendrian immersion $f : M \to (X, \xi)$ can be identified with an open set in a complex Banach space which can be explicitly described (as in the model contact space $(\mathbb{C}^{2n+1}, \xi_0)$).

A few open problems

1. How many contact structures are there on $\mathbb{C}^3$? On $\mathbb{C}^{2n+1}$? How could one distinguish them?
2. Does every Stein manifold X^{2n+1} satisfying LeBrun-Salamon condition (i.e., such that the canonical bundle K_X has $(n+1)$-st root) admit a contact structure?

   Note that a generic holomorphic 1-form on a Stein manifold is contact on the complement of a complex hypersurface.
3. Does the Runge approximation theorem hold for holomorphic contact structures? In particular, is it possible to approximate a holomorphic contact form on a convex set in $\mathbb{C}^{2n+1}$ by a contact form on all of $\mathbb{C}^{2n+1}$?
4. Does every Stein contact manifold (X, ξ) admit proper holomorphic Legendrian curves normalized by any bordered Riemann surface? We have a positive answer for pseudoconvex domains in the model contact space $(\mathbb{C}^{2n+1}, \alpha_0)$.