Holomorphic Legendrian curves

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Complex contact manifolds

Kobayashi 1959 A complex contact manifold is a pair (X, ξ) where:

- X is a complex manifold of odd dimension $2n+1 \geq 3$,
- ξ is a holomorphic hyperplane subbundle of the tangent bundle TX which is maximally nonintegrable, in the sense that the O'Neill tensor

  $$O : \xi \times \xi \to TX/\xi = L, \quad (v, w) \mapsto [v, w] \mod \xi$$

  (also called the Frobenius obstruction) is nondegenerate.
- Equivalently, every point $p \in X$ admits an open neighborhood $U \subset X$ such that

  $$\xi|_U = \ker \alpha,$$

  where α is a holomorphic 1-form on U satisfying

  $$\alpha \wedge (d\alpha)^n \neq 0.$$

Such ξ is a holomorphic contact structure, and α is a contact form.

Darboux's theorem and stability results

Two complex contact manifolds (X, ξ) and (X', ξ') are said to be contactomorphic if there exists a biholomorphism $F\colon X \to X'$ satisfying

$$dF_x(\xi_x) = \xi'_{F(x)} \quad \text{for all } x \in X.$$

Example (Model complex contact space)

$$(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0), \quad \alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j.$$

Darboux 1882; Moser 1965 Every complex contact manifold (X^{2n+1}, ξ) is locally contactomorphic to $(\mathbb{C}^{2n+1}, \xi_0)$.

Gray 1959 If (X, ξ) is a compact contact manifold then any small contact perturbation ξ' of ξ is contactomorphic to ξ.

LeBrun & Salamon 1994 Any two complex contact structures on a simply connected compact complex manifold are contactomorphic.

The normal bundle of a contact structure

A holomorphic 1-form α with $\xi = \ker \alpha$ is determined up to a nowhere vanishing multiplier f; note that $f\alpha \wedge (d(f\alpha))^n = f^{n+1}\alpha \wedge (d\alpha)^n$.
Thus, (X, ξ) admits a complex contact atlas $\{(U_j, \alpha_j)\}$ with
$\alpha_i = f_{i,j}\alpha_j$ on $U_{i,j} = U_i \cap U_j$.

LeBrun & Salamon 1994 The collection (α_j) determines a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$ given by the tautological projection

$$TX \xrightarrow{\alpha} L := TX/\xi \quad \text{the normal bundle.}$$

From $d(f\alpha) = df \wedge \alpha + fd\alpha$ we see that

$$d\alpha \text{ is a section of } \Lambda^2(\xi^*) \otimes L.$$

Thus, letting $K_X = \Lambda^{2n+1}(T^*X)$ (the canonical bundle), we see that

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ is a section of } K_X \otimes L^{\otimes(n+1)}.$$

This provides a holomorphic line bundle isomorphism

$$L^{\otimes(n+1)} \cong K_X^{-1} = \Lambda^{2n+1}(TX).$$

The space of complex contact structures

Conversely, assume X^{2n+1} is a complex manifold with $H^1(X, \mathbb{Z}_{n+1}) = 0$ and $c_1(TX)$ divisible by $n+1$. Then there exists the line bundle

$$L = K_X^{-1/(n+1)}, \quad L^{\otimes(n+1)} \cong K_X^{-1}.$$

Given a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$, consider

$$\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \Gamma(X, \mathscr{O}).$$

If X is **compact** then $\Gamma(X, \mathscr{O}) = \mathbb{C}$. The map

$$\Gamma(X, \Omega^1(L)) \ni \alpha \mapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}$$

is homogeneous of degree $n+1$. Hence, if X admits a complex contact structure, then the set of all such structures is the complement of a degree $n+1$ hypersurface in $\mathbb{P}(\Gamma(X, \Omega^1(L)))$ (a complex manifold).

A contact structure on $\mathbb{CP}^{2n+1}$

Let $z_1, \ldots, z_{2n+2}$ be complex coordinates on $\mathbb{C}^{2n+2}$ and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$

Let θ_j $(j = 1, \ldots, 2n+2)$ be the pull-back of θ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots .$$

Then (H_j, θ_j) is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each j, and this collection defines a contact structure on $X = \mathbb{CP}^{2n+1}$. We have

$$K_X^{-1} = \mathscr{O}_X(2n+2), \quad L = K_X^{-1/(n+1)} = \mathscr{O}_X(2),$$

$$\alpha \in \Gamma(\mathbb{CP}^{2n+1}, \Omega^1(2)).$$

Contact hypersurfaces in complex symplectic manifolds

Let (Z, ω) be a holomorphic symplectic manifold, $\dim Z = 2n + 2 \geq 4$; ω a holomorphic 2-form on Z,

$$d\omega = 0, \quad \omega^{n+1} \neq 0.$$

A holomorphic vector field V on Z is a Liouville vector field (for ω) if

$$L_V \omega = \omega \iff d(i_V \omega) = \omega.$$

In this case, the holomorphic 1-form

$$\theta = i_V \omega = V \rfloor \omega$$

induces a holomorphic contact form $\alpha = \theta|_{TX}$ on any complex hypersurface $X \subset Z$ transverse to V. Indeed:

$$\theta \wedge (d\theta)^n = i_V \omega \wedge \omega^n = \frac{1}{n+1} i_V (\omega^{n+1})$$

which is a volume form on X provided that V is transverse to X.

The converse process is the symplectization of a contact manifold (X, α):

$$Z = \mathbb{C} \times X, \quad \omega = d(e^t \alpha) = e^t \left(dt \wedge \alpha + d\alpha \right), \quad i_{\partial_t} \omega|_{t=0} = \alpha.$$

Examples in $\mathbb{C}^{2n+2}$

Example: $\mathbb{C}^{2n+2}$, $\omega = \sum_{j=0}^n d\zeta_j \wedge dz_j$,

$$V = \sum_{j=0}^n \zeta_j \partial_{\zeta_j}, \quad \theta = i_V \omega = \sum_{j=0}^n \zeta_j dz_j, \quad d\theta = \omega;$$

$$W = \sum_{j=0}^n \zeta_j \partial_{\zeta_j} + z_j \partial_{z_j}, \quad \theta = i_W \omega = \sum_{j=0}^n \zeta_j dz_j - z_j d\zeta_j, \quad d\theta = 2\omega.$$

Any complex hypersurface $X \subset \mathbb{C}^{2n+2}$ transverse to V or W carries the complex contact structure $\ker \theta \cap TX$.

An example is the special complex linear group

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : z_{11} z_{22} - z_{21} z_{12} = 1 \right\} \subset \mathbb{C}^4$$

with the holomorphic contact form

$$\theta = z_{11} dz_{22} - z_{21} dz_{12}.$$

Contact structure on the projectivized cotangent bundle

Let Z^{n+1} be a complex manifold. The holomorphic cotangent bundle T^*Z carries the tautological 1-form θ defined by

$$\theta(u) = v(d\pi(u)), \quad u \in T_v(T^*Z),$$

where $\pi\colon T^*Z \to Z$ is the natural projection and $d\pi$ its differential.

In local coordinates $z_0, \ldots, z_n$ on Z and the induced fiber coordinates $\zeta_0, \ldots, \zeta_n$ on T^*Z, we have

$$\theta = \zeta_0 dz_0 + \ldots + \zeta_n dz_n \quad (= \mathbf{p}d\mathbf{q} \text{ in classical notation}).$$

Then, θ determines a contact structure ξ on the projectivised cotangent bundle $X = \mathbb{P}(T^*Z)$. On the affine chart $\{\zeta_j = 1\}$ we have

$$\xi = \ker\Big(dz_j + \sum_{i \neq j} \zeta_i dz_i\Big).$$

Note that $d\theta = d\zeta \wedge dz = \omega$ is the canonical symplectic form on T^*Z, and $\theta = i_V\omega$ where $V = \sum_{j=0}^n \zeta_j \partial_{\zeta_j}$ is the Euler vector field.

Isotropic and Legendrian submanifolds

A smooth map $F\colon M \to (X,\xi)$ is said to be isotropic if

$$dF_p(T_pM) \subset \xi_{F(p)}, \quad p \in M.$$

An isotropic immersion is Legendrian if $\dim_{\mathbb{R}} M = 2n$ is maximal.
If $\xi = \ker\alpha$ then $F\colon M \to X$ is isotropic iff $F^*\alpha = 0$.

Lemma

If $\dim X = 2n+1$ and F is an isotropic immersion, then $\dim_R M \leq 2n$; if $\dim_{\mathbb{R}} M = 2n$ then $F(M)$ is an immersed complex submanifold of X.

Proof.

Note that $\omega := d\alpha|_\xi$ is a holomorphic symplectic form on $\xi = \ker\alpha$. Isotropic subspaces $U \subset \xi_x$ $(x \in X)$ are characterized by the condition $U \subset U^\perp_\omega$. Note that $U^\perp_\omega$ is a complex subspace of ξ_x, so we also have $U^{\mathbb{C}} := \mathrm{Span}_{\mathbb{C}}(U) \subset U^\perp_\omega$. From $\dim_{\mathbb{C}} U^{\mathbb{C}} + \dim_{\mathbb{C}} U^\perp_\omega = 2n$ we get the result. Note that $\dim_{\mathbb{R}} U = 2n$ iff $U = U^{\mathbb{C}} = U^\perp_\omega$, so U is complex. Hence, if $\dim_{\mathbb{R}} M = 2n$ then $dF_p(T_pM) \subset \xi_{F(p)}$ is a complex subspace for every $p \in M$, so $F(M)$ is a complex submanifold. □

How many Legendrian submanifolds are there?

Problem

What can be said about the existence of (proper) complex isotropic and Legendrian submanifolds of a complex contact manifold (X, ξ)?

Example

Let $(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)$ with $\alpha_0 = dz + \sum_{j=1}^{n} x_j dy_j$. Given a holomorphic function $z = z(y_1, \ldots, y_n)$, the formula

$$dz - \sum_{j=1}^{n} \frac{\partial z}{\partial y_j} dy_j = 0$$

shows that $y \mapsto (-\partial z/\partial y, y, z(y))$ is a Legendrian submanifold.

Segre 1926, Bryant 1981 Every compact Riemann surface embeds as a complex Legendrian curve in $\mathbb{CP}^3$.

Merkulov 1994 Deformation theory of compact isotropic submanifolds in compact complex contact manifolds.

Proper Legendrian curves in $(\mathbb{C}^{2n+1}, \xi_0)$

In this talk, we mainly consider isotropic holomorphic curves and call them (holomorphic) Legendrian curves.

Theorem (Alarcón, F., López, Compositio Math., in press)

Let M be an open Riemann surface and $K \subset M$ be a compact set in M whose complement has no relatively compact connected components.

Then every holomorphic Legendrian curve $F\colon K \to \mathbb{C}^{2n+1}$ $(n \in \mathbb{N})$ on an open neighborhood of K can be approximated uniformly on K by proper holomorphic Legendrian embeddings $\widetilde{F}\colon M \hookrightarrow \mathbb{C}^{2n+1}$.

Furthermore, given a pair of indices $\{i, j\} \subset \{1, 2, \ldots, 2n+1\}$ with $i \neq j$, we may choose $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_{2n+1})$ as above such that $(\widetilde{F}_i, \widetilde{F}_j)\colon M \to \mathbb{C}^2$ is a proper map.

Example: If $(x, y)\colon \mathbb{C} \to \mathbb{C}^2$ is a proper holomorphic immersion and

$$z(\zeta) := z_0 - \int_0^\zeta x(t)dy(t), \quad \zeta \in \mathbb{C},$$

then $F = (x, y, z)\colon \mathbb{C} \to \mathbb{C}^3$ is a proper Legendrian immersion.

Proof, 1: The basic scheme

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha_0 = dz + xdy$.

A holomorphic map $(x, y, z)\colon M \to \mathbb{C}^3$ is Legendrian iff xdy is an exact 1-form and

$$z = -\int^{\cdot} xdy.$$

The construction proceeds by inductively enlarging the domain of the Legendrian curve. Let $\rho\colon M \to \mathbb{R}_+$ be a strongly subharmonic Morse exhaustion function. We must consider two cases:

The noncritical case: Let $D \subset D'$ be Runge domains in M of the form

$$D = \{\rho < c\}, \quad D' = \{\rho < c'\}, \quad d\rho \neq 0 \text{ on } \overline{D}' \setminus D.$$

The critical case: ρ has a single critical point $p \in D' \setminus \overline{D}$.
The (only) nontrivial case is when the Morse index of p is equals one (critical points of Morse index zero are local minima of ρ).

Proof, 2: The period map

The noncritical case: Let $C_1, \ldots, C_\ell \subset D$ be closed curves forming a basis of the homology group $H_1(D;\mathbb{Z}) \cong H_1(D';\mathbb{Z}) = \mathbb{Z}^\ell$ such that $\bigcup_{j=1}^{\ell} C_j$ is Runge in M. Consider the period map

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_\ell) : \mathscr{A}^1(D)^2 \to \mathbb{C}^\ell$$

$$\mathcal{P}_j(x,y) = \int_{C_j} x\, dy, \qquad x, y \in \mathscr{A}^1(D),\ j = 1, \ldots, \ell.$$

We may assume that $y \in \mathscr{A}^1(D)$ is nonconstant. We find a holomorphic spray $X(\cdot\,,\zeta) \colon \overline{D} \to \mathbb{C}$ $(\zeta \in \mathbb{C}^\ell)$ of class $\mathscr{A}^1(D)$ and of the form

$$X(u,\zeta) = x(u) + \sum_{k=1}^{\ell} \zeta_k\, g_k(u), \quad u \in \overline{D},\ \zeta \in \mathbb{C}^\ell.$$

such that $X(\cdot\,,0) = x$ and

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} \mathcal{P}(X(\cdot\,,\zeta), y) : \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell \ \text{ is an isomorphism.}$$

Proof, 3: Sprays and Runge's theorem

By Runge's theorem we can find holomorphic maps

$$\widetilde{x}(\cdot\,,\cdot)\colon M\times\mathbb{C}^\ell\to\mathbb{C},\quad \tilde{y}\colon M\to\mathbb{C}$$

approximating X, y (respectively) in $\mathscr{C}^1(\overline{D})$.

Since $\mathcal{P}(X(\cdot\,,0),y)=0$, the period domination condition implies (by the implicit function theorem) that there is $\zeta_0\in\mathbb{C}^\ell$ close to 0 such that

$$\mathcal{P}(\widetilde{x}(\cdot\,,\zeta_0),\tilde{y})=0.$$

Hence, the 1-form $\widetilde{x}(\cdot\,,\zeta_0)d\tilde{y}$ is exact on $\overline{D}'$. Fix a point $p_0\in D$ and set

$$\widetilde{z}(p)=z(p_0)-\int_{p_0}^{p}\widetilde{x}(\cdot\,,\zeta_0)d\tilde{y},\quad p\in D'.$$

The Legendrian curve

$$(\widetilde{x}(\cdot\,,\zeta_0),\widetilde{y},\widetilde{z}):\overline{D}'\to\mathbb{C}^3$$

approximates (x,y,z) in $\mathscr{C}^1(\overline{D})$. This establishes the noncritical case.

Proof, 4: The critical case

The critical case: This amounts to a change of topology of the sublevel set. The new bigger domain $D' \subset M$ deformation retracts onto $\overline{D} \cup E$, where E is a smooth arc attached to $\overline{D}$ with its endpoints $a, b \in bD$.

Let $(x, y, z) \colon \overline{D} \to \mathbb{C}^3$ be a Legendrian curve. We extend the functions x, y to smooth functions $\tilde{x}, \tilde{y} \colon \overline{D} \cup E \to \mathbb{C}$ such that

$$\int_E \tilde{x} d\tilde{y} = z(b) - z(a).$$

This ensures that the extended function

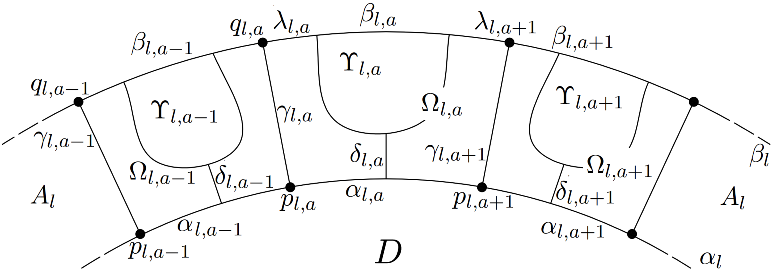
$$\tilde{z}(p) = z(a) + \int_a^p \tilde{x} d\tilde{y}, \quad p \in \overline{D} \cup E \subset M$$

is well defined and matches the function z on $\overline{D}$.

Hence, $(\tilde{x}, \tilde{y}, \tilde{z}) \colon \overline{D} \cup E \to \mathbb{C}^3$ is a generalized Legendrian curve.

Now, use period dominating sprays and Mergelyan approximation theorem to conclude the proof similarly as before.

Proof, 5: How to ensure properness of $(x, y)\colon M \to \mathbb{C}^2$



Assume $\max\{|x|, |y|\} > m$ on bD. Subdivide bD into arcs $\alpha_{l,a}$ such that on each of them, one of the functions $|x|, |y|$ is $> m$. Assume that $|x| > m$ on $\alpha_{l,a}$. Extend x smoothly to the arcs $\gamma_{l,a}$ and $\gamma_{l,a+1}$ such that $|x| > m$, and $|x| > m + 1$ at the outer endpoints of these two arcs. Apply Mergelyan to approximate x on $\overline{D} \cup \gamma_{l,a} \cup \gamma_{l,a+1}$ by $\tilde{x} \in \mathscr{O}(M)$. Choose the disc $\Upsilon_{l,a} \subset \Omega_{l,a}$ such that $|\tilde{x}| > m$ on $\overline{\Omega_{l,a} \setminus \Upsilon_{l,a}}$. Use Mergelyan to approximate y on $\overline{D}$ by $\tilde{y} \in \mathscr{O}(M)$ such that $|\tilde{y}| > m + 1$ on $\Upsilon_{l,a}$. Apply the analogous procedure on every $\Omega_{l,a}$. Then, $\max\{|\tilde{x}|, |\tilde{y}|\} > m + 1$ on bD' and $> m$ on $\overline{D}' \setminus D$.

A hyperbolic contact structure on $\mathbb{C}^{2n+1}$

The situation may be radically different for nonstandard contact structures on $\mathbb{C}^{2n+1}$. The Kobayashi pseudometric associated to a contact structure is defined by using holomorphic Legendrian discs.

Theorem (F., J. Geom. Anal. 2017)

For any $n \geq 1$ there exists a holomorphic contact structure ξ on $\mathbb{C}^{2n+1}$ which is **Kobayashi hyperbolic** *and isotopic to ξ_0. In particular, every holomorphic Legendrian curve $\mathbb{C} \to (\mathbb{C}^{2n+1}, \xi)$ is constant.*

Idea of proof: We take $\alpha = \Phi^*\alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$ and $\Phi\colon \mathbb{C}^{2n+1} \longrightarrow \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image Ω avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^{2n}_{(x,y)} \times C_N \overline{\mathbb{D}}_z.$$

Assuming that $C_N \geq n2^{3N+1}$ for all $N \in \mathbb{N}$,

$\mathbb{C}^{2n+1} \setminus K$ is α_0-hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^*\alpha_0)$ is hyperbolic.

On α_0-hyperbolicity of $\mathbb{C}^3 \setminus K$

Let $\alpha_0 = dz + xdy$ in $\mathbb{C}^3$.

Lemma

Assume that $C_N \geq 2^{3N+1}$ *for every* $N \in \mathbb{N}$ *and let*

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^2_{(x,y)} \times C_N \overline{\mathbb{D}}_z \subset \mathbb{C}^3$$

For every holomorphic α_0*-horizontal disk*

$$f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3 \setminus K, \quad \zeta \in \mathbb{D}$$

with $f(0) \in 2^{N_0}\mathbb{D}^3$ *for some* $N_0 \in \mathbb{N}$ *we have the estimates*

$$|x'(0)| < 2^{N_0+1}, \quad |y'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1}.$$

Oka principle for holomorphic Legendrian curves in $\mathbb{C}^{2n+1}$

Let M be an open Riemann surface. We can describe the rough shape of the space $\mathscr{L}(M, \mathbb{C}^{2n+1})$ of all holomorphic Legendrian immersions $M \to (\mathbb{C}^{2n+1}, \xi_0)$ into the model contact space.

Fix a nowhere vanishing holomorphic 1-form θ on M. To each holomorphic Legendrian immersion

$$f = (x, y, z) \colon M \to (\mathbb{C}^{2n+1}, \xi_0)$$

(not necessarily proper) we associate the map

$$\phi(f) = (dx/\theta, dy/\theta) \colon M \to \mathbb{C}^{2n}_* \to S^{4n-1}.$$

Theorem (Lárusson and F., Math. Z., in press)

The map $\phi : \mathscr{L}(M, \mathbb{C}^{2n+1}) \to \mathscr{C}(M, S^{4n-1})$ into the space of continuous maps $M \to S^{4n-1}$ is a weak homotopy equivalence, and is a homotopy equivalence when M has finite topological type.

Homotopy groups of $\mathscr{L}(M, \mathbb{C}^{2n+1})$

The proof combines methods explained above and the parametric version of Gromov's convex integration lemma. Since M has the homotppy type of a bouquet of ℓ circles, where $H_1(M;\mathbb{Z}) = \mathbb{Z}^\ell$, we get

Corollary (Lárusson and F., Math. Z., in press)

Let M be a connected open Riemann surface with $H_1(M;\mathbb{Z}) = \mathbb{Z}^\ell$, $\ell \in \mathbb{Z}_+$. For each $n \geq 1$, the space $\mathscr{L}(M, \mathbb{C}^{2n+1})$ is weakly homotopy equivalent to the free ℓ-loop space $\mathcal{L}_\ell S^{4n-1}$ of the sphere S^{4n-1}, and is homotopy equivalent to it of $\ell < \infty$.

It follows that $\mathscr{L}(M, \mathbb{C}^{2n+1})$ is path connected and simply connected, and for each $k \geq 2$ we have

$$\pi_k(\mathscr{L}(M, \mathbb{C}^{2n+1})) = \pi_k(S^{4n-1}) \times \pi_{k+1}(S^{4n-1})^\ell.$$

In particular, $\mathscr{L}(M, \mathbb{C}^{2n+1})$ is $(4n-3)$-connected.

Complete bounded Legendrian curves in $(\mathbb{C}^{2n+1}, \xi_0)$

Martín-Umehara-Yamada 2014 Do there exists complete bounded holomorphic Legendrian curves in $\mathbb{C}^3$? Can they have Jordan boundaries? (Analogue of the **Calabi-Yau problem** in the theory of minimal surface.)

Theorem (Alarcón, F., López, Compositio Math., in press)

Let M be a compact bordered Riemann surface. Every Legendrian curve $M \to \mathbb{C}^{2n+1}$ of class $\mathscr{A}^1(M)$ can be uniformly approximated by topological embeddings $F\colon M \to \mathbb{C}^{2n+1}$ such that $F|_{\mathring{M}}\colon \mathring{M} \to \mathbb{C}^{2n+1}$ is a complete Legendrian embedding.

Besides the methods explained above, we use the following

Riemann-Hilbert lemma for Legendrian curves: given a Legendrian immersion $f\colon M \to \mathbb{C}^{2n+1}$ and a continuous family of Legendrian discs $F(u,\cdot)\colon \overline{\mathbb{D}} \to \mathbb{C}^{2n+1}$ with $F(u,0) = f(u)$ for all $u \in bM$, there is a Legendrian approximate solution $H\colon M \to \mathbb{C}^{2n+1}$ to the Riemann-Hilbert boundary value problem.

The Riemann-Hilbert problem for Legendrian curves

Theorem (Alarcón, F., López, Compositio Math.)

Assume that M is a compact bordered Riemann surface, $I \subset bM$ is an arc which is not a boundary component of M, $f = (x, y, z)\colon M \to \mathbb{C}^{2n+1}$ is a Legendrian map of class $\mathscr{A}^1(M)$, and for every point $u \in bM$ the map

$$\overline{\mathbb{D}} \ni v \longmapsto F(u,v) = \big(X(u,v), Y(u,v), Z(u,v)\big) \in \mathbb{C}^{2n+1}$$

is a Legendrian disk of class $\mathscr{A}^1(\mathbb{D})$, depending continuously on $u \in bM$, such that $F(u,0) = f(u)$ for all $u \in bM$ and $F(u,v) = f(u)$ for all $u \in bM \setminus I$ and $v \in \overline{\mathbb{D}}$. Given a number $\epsilon > 0$ and a neighborhood $U \subset M$ of the arc I, there exist a holomorphic Legendrian map $h\colon M \to \mathbb{C}^{2n+1}$ and a neighborhood $V \Subset U$ of I with a smooth retraction $\rho\colon V \to V \cap bM$ such that the following conditions hold:

(i) $\sup\{|h(u) - f(u)| : u \in M \setminus V\} < \epsilon$,

(ii) $\mathrm{dist}(h(u), F(u, \mathbb{T})) < \epsilon$ *for all* $u \in bM$, *and*

(iii) $\mathrm{dist}(h(u), F(\rho(u), \overline{\mathbb{D}})) < \epsilon$ *for all* $u \in V$.

Outline of proof

1. The main point is to solve the problem on the disc $\overline{\mathbb{D}} = \{|\zeta| \leq 1\}$. Indeed, we then obtain a solution on a small closed disc $D \subset M$ containing the arc $I \subset bM$ in its boundary, and we glue it with the Legendrian immersion $f\colon M \to X$ by using period dominating sprays.

2. By the standard Riemann-Hilbert method we obtain a sequence of analytic (non-Legendrian) discs $h_N\colon \overline{\mathbb{D}} \to \mathbb{C}^{2n+1}$ $(N \in \mathbb{N})$ which satisfy the stated conditions for big enough $N \in \mathbb{N}$.

3. Write $h_N = (x_N, y_N, z_N)$. A calculation shows that

$$z_N(\zeta) + \int_0^\zeta x_N dy_N \to z_N(0) \text{ uniformly on } \zeta \in \overline{\mathbb{D}} \text{ as } N \to \infty.$$

Set

$$\tilde{z}_N(\zeta) = z_N(0) - \int_0^\zeta x_N dy_N, \quad \zeta \in \mathbb{D}.$$

The disc $\tilde{h}_N = (x_N, y_N, \tilde{z}_N)$ is then Legendrian and $|\tilde{z}_N - z_N| \to 0$ as $N \to \infty$. Hence, $h = \tilde{h}_N$ solves the problem for big enough N.

Darboux charts around immersed Legendrian curves

Theorem (Alarcón & F., Preprint 2017)

Assume that

- (X, ξ) *is a complex contact manifold,*
- R *is an open Riemann surface,*
- $x_1 \colon R \to \mathbb{C}$ *is a holomorphic immersion, and*
- $f \colon R \to (X, \xi)$ *is a holomorphic Legendrian immersion.*

Given a relatively compact domain $U \Subset R$ *there exists a holomorphic immersion* $F \colon U \times \mathbb{B}^{2n} \to X$ *such that* $F(\cdot\,, 0) = f$ *and*

$$F^*\xi = \ker\Big(dz + \sum_{j=1}^{n} x_j dy_j\Big) \quad \text{on } U \times \mathbb{B}^{2n},$$

where $x_2, \ldots, x_n, y_1, \ldots, y_n, z$ *are Euclidean coordinates on* $\mathbb{C}^{2n}$.

This is proved by standardizing the contact structure $\xi \subset TX$ along the Legendrian curve $f(R)$ and applying Moser's method.

Two consequences of the existence of Darboux charts

Corollary (Alarcón & F. 2017)

Assume that

- (X, ξ) *is a complex contact manifold,*
- R *is an open Riemann surface,*
- $f \colon R \to (X, \xi)$ *is a holomorphic Legendrian immersion, and*
- $M \subset R$ *is a smoothly bounded compact domain.*

Then $f|_M$ *can be uniformly approximated by topological embeddings* $\tilde{f} \colon M \to X$ *such that* $\tilde{f}|_{\mathring{M}} \colon \mathring{M} \to X$ *is a complete Legendrian embedding.*

Corollary (Deformation theory for Legendrian curves)

(Assumptions as above.) The space of all small Legendrian deformations of $f|_M : M \to (X, \xi)$ *can be identified with an open set in a complex Banach space which can be explicitly described (as in the standard case when* (X, ξ) *is the model contact space* $(\mathbb{C}^{2n+1}, \xi_0)$*).*

A few open problems

1. How many contact structures are there on $\mathbb{C}^3$? On $\mathbb{C}^{2n+1}$?
   How can one distinguish them?
   Is there an analogue of the tight/overtwisted phenomenon from smooth contact geometry (Eliashberg)?
2. Does every Stein manifold X^{2n+1} satisfying LeBrun-Salamon condition (the canonical bundle K_X has $(n+1)$-st root) admit a contact structure?

   Note that a generic holomorphic 1-form on a Stein manifold is contact on the complement of a complex hypersurface.
3. Does the Runge approximation theorem hold for holomorphic contact structures? In particular, is it possible to approximate a holomorphic contact form on a convex set in $\mathbb{C}^{2n+1}$ by a contact form on all of $\mathbb{C}^{2n+1}$?
4. Does every Stein contact manifold (X, ξ) admit proper holomorphic Legendrian curves normalized by any bordered Riemann surface? We have a positive answer for pseudoconvex domains in the model contact space $(\mathbb{C}^{2n+1}, \alpha_0)$.