Long $\mathbb{C}^2$'s without holomorphic functions

Franc Forstnerič

University of Ljubljana
Institute of Mathematics, Physics and Mechanics
The Slovenian Academy of Sciences and Arts

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What is a long $\mathbb{C}^n$?

A complex manifold X of dimension n is said to be a **long** $\mathbb{C}^n$ if it is the union of an increasing sequence of domains

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Wermer 1959 described this phenomenon with K a polydisk in $\mathbb{C}^3$.

History continued

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- Do there exist several non-equivalent (non-Stein) long $\mathbb{C}^2$'s?
- Does there exist a non-Stein long $\mathbb{C}^2$ with a nontrivial algebra of holomorphic functions?
- Does there exist a Stein long $\mathbb{C}^2$ which is not a $\mathbb{C}^2$?

Uncountably many long $\mathbb{C}^2$'s

Here we give a positive answer to the first question. The other two remain open for the time being.

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Let $n > 1$. To every open set $U \subset \mathbb{C}^n$ one can associate a long $\mathbb{C}^n$, $X(U)$ (containing U in the initial copy of $\mathbb{C}^n \subset X$) such that any biholomorphism $\Phi\colon X(U) \to X(V)$ maps U onto V.

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stable core **strongly stable core**.

The stable hull property

Definition

Let X be complex manifold. A compact set B in X is said to have the **stable hull property (SHP)** if there exists an exhaustion

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by compact sets such that $B \subset K_1$, $K_j \subset \mathring{K}_{j+1}$ for every $j \in \mathbb{N}$, and the increasing sequence of hulls $\widehat{B}_{\mathscr{O}(K_j)}$ stabilizes, i.e., there is a $j_0 \in \mathbb{N}$ such that

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Proof.

An exercise. □

The proof (just in case...)

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It follows that $\widehat{B}_{\mathscr{O}(L_{\ell_j})} = C$ for all $j \in \mathbb{N}$. Since the sequence of hulls $\widehat{B}_{\mathscr{O}(L_\ell)}$ is increasing with ℓ, we conclude that

$$\widehat{B}_{\mathscr{O}(L_\ell)} = C \quad \text{for all } \ell \geq \ell_1.$$

Hence, B has SHP with respect to the exhaustion $(L_\ell)_{\ell \in \mathbb{N}}$.

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(ii) A compact set $B \subset X$ is called the **strongly stable core** of X, denoted $SSC(X)$, if B has the stable hull property, but any compact set $L \subset X$ with $\mathring{L} \setminus B \neq \emptyset$ fails to have the stable hull property.

The second main theorem

Theorem

Let $n > 1$.

(a) *For every compact polynomially convex set* $B \subset \mathbb{C}^n$ *there exists a long* $\mathbb{C}^n$, $X(B)$, *whose strongly stable core equals* B*:*

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(b) *For every open set* $U \subset \mathbb{C}^n$ *there exists a long* $\mathbb{C}^n$, X, *such that*

$$SC(X) \subset U \quad \text{and} \quad \overline{U} = \overline{SC(X)}.$$

Connection to CR geometry

By part (a) of the theorem, every compact polynomially convex domain $B \subset \mathbb{C}^n$ $(n > 1)$ is the strongly stable core of a long $\mathbb{C}^n$.

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Corollary

There exist uncountably many biholomorphically inequivalent long $\mathbb{C}^2$*'s.*

The main tool: Andersén-Lempert theory

Varolin 1997 A complex manifold X enjoys the **holomorphic density property** if every holomorphic vector field on X can be approximated, uniformly on compacts, by Lie combinations (sums and commutators) of $\mathbb{C}$-complete holomorphic vector fields on X.

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Theorem (Andersén & Lempert 1992, Rosay & F. 1993, Varolin 2000)

Let X be a Stein manifold with the density property, and let

$$\Phi_t \colon \Omega_0 \longrightarrow \Omega_t = \Phi_t(\Omega_0) \subset X, \quad t \in [0,1]$$

be a smooth isotopy of biholomorphic maps of Ω_0 onto pseudoconvex Runge domains $\Omega_t \subset X$ such that $\Phi_0 = \mathrm{Id}_{\Omega_0}$.

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More generally, if $L \subset \mathbb{C}^n$ is a compact holomorphically contractible set disjoint from K such that $K \cup L$ is polynomially convex, then there exists an injective holomorphic map $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

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We follow **E.F. Wold's construction** (2008, 2010) up to a certain point, adding a new twist at the end. We consider the case $n = 2$.

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- $\psi_2 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ approximates the identity map on K' and $\psi_2(\psi_1(\theta(L))) \subset B$. Hence $\phi(L) \subset \widehat{\phi(K)} \not\subset \phi(\mathbb{C}^2)$.

Proof of the first main theorem

Pick a compact set $K \subset \mathbb{C}^n$ with nonempty interior and a countable dense sequence $\{a_j\}_{j\in\mathbb{N}}$ in $\mathbb{C}^n$. Set $K_1 = \widehat{K}$.

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From the first step, we also have $\phi_1(a_1) \in K_2$, and hence

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Continuing inductively, we get a sequence of injective holomorphic maps $\phi_j \colon \mathbb{C}^n \to \mathbb{C}^n$ for $j = 1, 2, \ldots$ such that, setting

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Since the set $\{a_j\}_{j \in \mathbb{N}}$ is everywhere dense in $\mathbb{C}^n = X_1$, it follows that every holomorphic function on X is bounded on $X_1 = \mathbb{C}^n$ and hence constant. By the identity principle it is constant on all of X.

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Lemma (2)

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Proof: Let $n = 2$. Choose a closed ball $\mathcal{B}$ containing B in its interior. By a suitable choice of coordinates we may assume that $\mathcal{B} \subset \mathbb{C}^* \times \mathbb{C}$.

Proof of Lemma 2

Choose an FB map $\theta_1 \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ whose image is Runge in $\mathbb{C}^2$. Hence, the set $\theta_1(\mathcal{B})$ is polynomially convex.

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The AL Theorem furnishes $\theta_2 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which approximates θ_1^{-1} on the set $\theta_1(\mathcal{B})$. The composition $\theta = \theta_2 \circ \theta_1 \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is then a FB map which is close to the identity on $\mathcal{B}$. The set $B' := \theta(B)$ is a small smooth perturbation of B, hence polynomially convex. Set

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Choose m pairwise disjoint copies $M_1, \ldots, M_m \subset (\mathbb{C}^* \times \mathbb{C}) \setminus B'$ of Stolzenberg's compact set M. By placing the sets M_j far apart and away from B', we may assume that

$$B' \cup (\cup_{j=1}^m M_j) \text{ is } \mathscr{O}(\mathbb{C}^* \times \mathbb{C})\text{-convex.}$$

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$$\phi := \psi \circ \theta \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$$

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If the approximation is close enough, the set $\phi(B) = \psi(B')$ is still polynomially convex. Clearly, ϕ satisfies Lemma 2.

Proof of the second main theorem

Recall:

Theorem (2)

(a) *For every compact polynomially convex set $B \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$, $X(B)$, whose strongly stable core equals B:*

$$SSC(X(B)) = B.$$

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The proof amount to a recursive application of Lemma 2. Assume that $n = 2$. We consider part (a); part (b) is similar.

Choose a decreasing sequence of compact strongly pseudoconvex polynomially convex domains in $\mathbb{C}^2$ whose intersection equals B:

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Step 1: Pick a countable dense set

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Let $r_1 > 0$ be small enough such that $\overline{\mathbb{B}}(a_{1,1}, r_1) \cap B_1 = \emptyset$ and $\overline{\mathbb{B}}(a_{1,1}, r_1) \cup B_1$ is polynomially convex.

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Choose the first FB map $\phi_1 : \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ such that

$$\phi_1 \text{ is close to the identity on } B_1$$

and the set

$$C^1_{1,1} = \phi_1(\overline{\mathbb{B}}(a_{1,1}, r_1))$$

satisfies

$$\widehat{C^1_{1,1}} \setminus \phi_1(\mathbb{C}^2) \neq \varnothing.$$

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The k-th step: We have found an FB map

$$\Phi_k = \phi_k \circ \cdots \circ \phi_1 \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$$

which is close to Id on B_k and ϕ_k performs the Wold process on the first $k(k+1)/2$ points of a countable dense set $A_k \subset \mathbb{C}^2 \setminus \Phi_{k-1}(B_k)$.

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***** THANK YOU FOR YOUR ATTENTION *****