Minimal hulls of compact sets in $\mathbb{R}^3$

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Abstract

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- **We characterize the minimal surface hull of a compact set K in $\mathbb{R}^3$ by sequences of conformal minimal discs whose boundaries converge to K in the measure theoretic sense.** The analogous result is obtained for the **null hull** of a compact subset of $\mathbb{C}^3$. These are inspired by **Poletsky's characterization of the polynomial hull of a compact set in $\mathbb{C}^n$ by sequences of holomorphic discs**.

- We **describe the minimal hull by 2-dimensional minimal currents** which are limits of Green currents supported by conformal minimal discs, in the spirit of the **Duval-Sibony-Wold characterization of polynomial hulls by positive $(1, 1)$-currents**.

- We obtain a **polynomial hull version of Bochner's tube theorem.**

B. Drinovec Drnovšek; F. Forstnerič: Minimal hulls of compact sets in $\mathbb{R}^3$. Preprint (2014). arxiv.org/abs/1409.6906

Hulls of compact sets

When discussing hulls, one typically deals with dual sets of objects.

Given a set $\mathcal{P}$ of real functions on a manifold X, the **$\mathcal{P}$-hull** of a compact subset $K \subset X$ is defined by

$$\widehat{K}_{\mathcal{P}} = \{x \in X : f(x) \le \sup_K f \quad \forall f \in \mathcal{P}\}.$$

Suppose that $\mathcal{G}$ is a class of **geometric objects** in X (submanifolds, subvarieties, minimal surfaces,...) such that the restriction $f|_C$ satisfies the maximum principle for every $f \in \mathcal{P}$ and $C \in \mathcal{G}$. Then

$$C \subset \widehat{K}_{\mathcal{P}} \text{ for every } C \in \mathcal{G} \text{ with boundary } bC \subset K.$$

The Main question: How closely is the hull $\widehat{K}_{\mathcal{P}}$ described by the objects in $\mathcal{G}$ with boundaries in K (in a suitably general sense)?

Polynomial hulls and Poletsky's theorem

A classical example is the **polynomial hull**, $\widehat{K}$, of a compact set $K \subset \mathbb{C}^n$. This is the hull $\widehat{K}_{\mathcal{P}}$ for $\mathcal{P} = \{|f| : f \in \mathscr{O}(\mathbb{C}^n)\}$, or $\mathcal{P} = \mathrm{Psh}(\mathbb{C}^n)$ (plurisubharmonic functions).

A natural dual class $\mathcal{G}$ consists of **complex curves with boundaries**.

Wermer (1958), Stolzenberg (1966), Alexander (1971): If K is a compact sets of finite linear measure then $\widehat{K} \setminus K$ is a (possibly empty) one dimensional closed complex subvariety of $\mathbb{C}^n \setminus K$.
Stolzenberg (1963): in general one cannot describe $\widehat{K}$ by bounded complex curves with boundaries exactly in K.

Theorem (Poletsky 1991, 1993)

Let K be a compact set in $\mathbb{C}^n$, and let $B \subset \mathbb{C}^n$ be a ball containing K. A point $p \in B$ belongs to $\widehat{K}$ iff there exists a sequence of holomorphic discs $f_j \colon \overline{\mathbb{D}} \to B$ satisfying the following for every $j = 1, 2, \ldots$:

$$f_j(0) = p \ \ \textit{and} \ \ \left|\{t \in [0, 2\pi] : \mathrm{dist}(f(e^{\mathrm{i}t}), K) < 1/j\}\right| \geq 2\pi - 1/j.$$

Minimal surfaces in $\mathbb{R}^n$

Definition

A smoothly immersed surface $f\colon M \to \mathbb{R}^n$ is a **minimal surface** if its **mean curvature vector field $\mathbf{H}$**: $M \to \mathbb{R}^n$ is identically zero: $\mathbf{H} = 0$.

When $n = 3$, we have $\mathbf{H} = \mathrm{H} \cdot \mathbf{N}$ where $\mathbf{N}$ is the Gauss map and

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2} \ldots$$ **the mean curvature function**.

In local isothermal coordinates (x, y) on M we have:

$$\triangle f = 2\xi \mathbf{H}, \quad \xi = ||f_x||^2 = ||f_y||^2.$$

Lemma (Classical; proof in Appendix A)

The following are equivalent for a smooth **conformal** *immersion $f\colon M \to \mathbb{R}^n$ from an open Riemann surface M to $\mathbb{R}^n$:*

- *f is minimal (a stationary point of the area functional).*
- *f has vanishing mean curvature vector: $\mathbf{H} = 0$.*
- *f is harmonic: $\triangle f = 0$.*

Holomorphic null curves in $\mathbb{C}^n$

In the sequel M denotes an open or a bordered Riemann surface.

Definition

A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n)\colon M \to \mathbb{C}^n, \quad n \geq 3$$

is a **null curve** if the derivative $F' = (F_1', F_2', \ldots, F_n')$ with respect to any local holomorphic coordinate $\zeta = x + \imath y$ on M satisfies

$$(F_1')^2 + (F_2')^2 + \ldots + (F_n')^2 = 0.$$

The nullity condition is equivalent to $F'(\zeta) \in A_* = A \setminus \{0\}$ where

$$A = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0 \Big\} \ldots$$ **the null quadric**.

It is easily seen that A_* is **elliptic** in the sense of Gromov, and hence an **Oka manifold**.

The Oka principle for null curves

There exist plenty of null curves $M \to \mathbb{C}^n$ from any open Riemann surface; we have the Runge and the Mergelyan approximation theorem, and also the following Oka principle.

Theorem

Let ϑ be a nowhere vanishing holomorphic 1-form on an open Riemann surface M. Then every continuous map $h_0 \colon M \to A_*$ is homotopic to a holomorphic map $h \colon M \to A_*$ such that $h\vartheta$ has vanishing periods, and hence

$$F(x) = F(p) + \int_p^x h\vartheta, \qquad x \in M$$

is a null holomorphic immersion $M \to \mathbb{C}^n$. F can be chosen **proper**.
The same holds if $A \subset \mathbb{C}^n$ is an irreducible cone with the only singularity at $0 \in \mathbb{C}^n$ such that $A_* = A \setminus \{0\}$ is an Oka manifold.

[A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math. 196 (2014), 733–771]

Connection between null curves and minimal surfaces

- If $F = f + \imath g \colon M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$f = \Re F \colon M \to \mathbb{R}^n, \quad g = \Im F \colon M \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

Proof: Let $F = f + \imath g = (F^1, \ldots, F^n) : M \to \mathbb{C}^n$ be a holomorphic null curve and $\zeta = x + \imath y$ a local holomorphic coordinate on M. Then

$$0 = \sum_{j=1}^{n} (F_x^j)^2 = \sum_{j=1}^{n} \left(f_x^j + \imath g_x^j\right)^2 = \sum_{j=1}^{n} \left((f_x^j)^2 - (g_x^j)^2\right) + 2\imath \sum_{j=1}^{n} f_x^j g_x^j.$$

Since $g_x = -f_y$ by the Cauchy-Riemann equations, the above reads

$$0 = ||f_x||^2 - ||f_y||^2 - 2\imath\, f_x \cdot f_y \iff ||f_x|| = ||f_y||,\ f_x \cdot f_y = 0.$$

It follows that f is a conformal minimal immersion (CMI).

Conversely, ...

- A conformal minimal immersion (CMI) $f : \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{|\zeta| < 1\}$ is the real part of a null disc $F\colon \mathbb{D} \to \mathbb{C}^n$.

Proof: In any local holomorphic coordinate $\zeta = x + \imath y$ on M we have

$$||f_x|| = ||f_y|| > 0, \quad f_x \cdot f_y = 0 \qquad \text{(conformality)}$$

Equivalently, $f_x \pm \imath f_y \in A_*$ are null vectors. From $2\partial f = (f_x - \imath f_y) d\zeta$ we infer that f is conformal if and only if

$$(\partial f^1)^2 + (\partial f^2)^2 + \cdots + (\partial f^n)^2 = 0,$$

and is conformal harmonic (=conformal minimal) iff

$$\partial f = (\partial f^1, \ldots, \partial f^n) \text{ is a holomorphic 1-form with values in } A_*.$$

If g is any local harmonic conjugate of f, then CR equations imply $\partial(f + \imath g) = 2\partial f = 2\imath\,\partial g$, so $F = f + \imath g$ is a null holomorphic immersion.

Weierstrass representation of conformal minimal surfaces

Hence every conformal minimal immersion $f\colon M \to \mathbb{R}^n$ is of the form

$$f(x) = f(p) + \Re \int_p^x \phi \qquad (p,\ x \in M)$$

where $\phi = (\phi_1, \ldots, \phi_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M without zeros satisfying the nullity condition

$$\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = 0$$

such that $\Re\phi = (\Re\phi_1, \ldots, \Re\phi_n)$ has vanishing periods over all closed curves in M.

A. Alarcón, F. (2014): Every conformal minimal immersion $M \to \mathbb{R}^n$ is homotopic through conformal minimal immersions to the real part of a holomorphic null curve $M \to \mathbb{C}^n$. arxiv.org/abs/1408.5315

Null plurisubharmonic functions

Definition

An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ on a domain $\Omega \subset \mathbb{C}^n$ is **null plurisubharmonic** if the restriction of u to any affine complex line $L \subset \mathbb{C}^n$ directed by a null vector $\theta \in A_*$ is subharmonic on $L \cap \Omega$. The class of all such functions is denoted $\mathfrak{N}\mathrm{Psh}(\Omega)$.

Clearly $\mathrm{Psh}(\Omega) \subset \mathfrak{N}\mathrm{Psh}(\Omega)$. The inclusion is proper: the function

$$u(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 - |z_3|^2$$

is null plurisubharmonic on $\mathbb{C}^3$, but is not plurisubharmonic.

Null plurisubharmonic functions satisfy most of the standard properties of plurisubharmonic functions. The main point:

- If $F \colon M \to \Omega$ is a holomorphic null curve and $u \in \mathfrak{N}\mathrm{Psh}(\Omega)$ then $u \circ F$ is subharmonic on M.

Minimal plurisubharmonic functions

Definition (Harvey & Lawson; Drinovec Drnovšek & F.)

An upper semicontinuous function u on a domain $\omega \subset \mathbb{R}^n$ $(n \geq 3)$ is **minimal plurisubharmonic ($\mathfrak{M}$-psh)** if the restriction of u to any affine 2-dimensional plane $L \subset \mathbb{R}^n$ is subharmonic on $L \cap \omega$ (in isothermal coordinates). The set of all such functions is denoted $\mathfrak{M}\mathrm{Psh}(\omega)$.

Basic facts:

- Every $\mathfrak{M}\mathrm{Psh}$ function can be approximated by smooth $\mathfrak{M}\mathrm{Psh}$ functions.
- A $\mathscr{C}^2$ function u is $\mathfrak{M}\mathrm{Psh}$ if and only if the sum of the two smallest eigenvalues of $\mathrm{Hess}\, u$ is nonnegative at every point.
- The restriction of a $\mathfrak{M}\mathrm{Psh}$ function to an immersed minimal surface is subharmonic (in isothermal coordinates on the surface).

Connection between $\mathfrak{MPsh}(\omega)$ and $\mathfrak{NPsh}(\mathcal{T}_\omega)$

Lemma

Let $\omega \subset \mathbb{R}^n$ and let $\mathcal{T}_\omega = \omega \times \imath\mathbb{R}^n \subset \mathbb{C}^n$ be the tube over ω.

- *If $u \in \mathfrak{MPsh}(\omega)$ then the function $U(x + \imath y) = u(x)$ $(x + \imath y \in \mathcal{T}_\omega)$ is null plurisubharmonic on the tube $\mathcal{T}_\omega$.*
- *Conversely, if $U \in \mathfrak{NPsh}(\mathcal{T}_\omega)$ is independent of $y = \Im z$ then $u(x) = U(x + \imath 0)$ is minimal plurisubharmonic on ω.*

Proof.

Recall that

- the real and the imaginary part of a holomorphic null disc $f \in \mathfrak{N}(\mathbb{D}, \mathbb{C}^n)$ are conformal minimal discs in $\mathbb{R}^n$;
- conversely, every conformal minimal disc in $\mathbb{R}^n$ is the real part of a holomorphic null disc in $\mathbb{C}^n$.

Since $U \circ f = u \circ \Re f$ for all $f \in \mathfrak{N}(\mathbb{D}, \mathcal{T}_\omega)$, the lemma follows. □

Disc formulas for biggest $\mathfrak{M}$Psh and $\mathfrak{N}$Psh minorants

The following is our main result. It is analogous to **Poletsky's theorem** on plurisubharmonic minorants (with generalisations by Lárusson and Sigurdsson, Rosay, B.D.-F., Kuzman,...).
$\mathrm{CMI}(\overline{\mathbb{D}},\omega)$ ...conformal minimal immersions $\overline{\mathbb{D}} \to \omega \subset \mathbb{R}^3$.

Theorem (B. Drinovec Drnovšek & F., 2014)

(a) *If ω is a domain in $\mathbb{R}^3$ and $\phi\colon \omega \to \mathbb{R}\cup\{-\infty\}$ is an upper semicontinuous function, then the function $u\colon \omega \to \mathbb{R}\cup\{-\infty\}$,*

$$u(x) = \inf\left\{ \int_0^{2\pi} \phi(f(e^{it}))\,\frac{dt}{2\pi} \colon\ f \in \mathrm{CMI}(\overline{\mathbb{D}},\omega),\ f(0)=x \right\} \quad (1)$$

is minimal plurisubharmonic or identically $-\infty$; moreover, u is the supremum of functions in $\mathfrak{M}\mathrm{Psh}(\omega)$ which are bounded above by ϕ.

(b) *If Ω is a domain in $\mathbb{C}^3$, $\phi\colon \Omega \to \mathbb{R}\cup\{-\infty\}$ is an upper semicontinuous function, and we use null holomorphic discs $f\colon \overline{\mathbb{D}} \to \Omega$ in (1), then the resulting function u is the biggest null plurisubharmonic function on Ω which is bounded above by ϕ.*

A lemma of Edgar and Bu-Schachermayer

Part (a) follows from part (b) in view of the previous lemma.
The proof of (a) is based on the following **EBS-lemma** (**Edgar 1985; Bu-Schachermayer 1992**) adapted to null plurisubharmonic functions.

Lemma

Let Ω be a domain in $\mathbb{C}^n$ $(n \geq 3)$ and $\phi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Define $u_1 = \phi$ and for $j > 1$

$$u_j(z) = \inf \left\{ \int_0^{2\pi} u_{j-1}\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big)\, \frac{dt}{2\pi} \right\}, \quad z \in \Omega,$$

where the infimum is taken over all null vectors $\theta \in A_*$ such that

$$\{z + \zeta\theta : |\zeta| \leq 1\} \subset \Omega.$$

Then the functions u_j are upper semicontinuous and decrease pointwise to the largest null plurisubharmonic function u_ϕ on Ω with $u_\phi \leq \phi$.

Proof of EBS lemma

Proof: We first show by induction that the functions u_j are upper semicontinuous (u.s.c.); then u_ϕ is also u.s.c.

Assume that u_{j-1} is u.s.c.; this holds when $j=2$. Given $z_0 \in \Omega$ and $\epsilon > 0$, there exists a null vector $\theta \in A^*$ such that

$$u_j(z_0) \leq \int_0^{2\pi} u_{j-1}\big(z_0 + \mathrm{e}^{\mathrm{i}t}\theta\big)\,\frac{dt}{2\pi} < u_j(z_0) + \epsilon.$$

For z close enough to z_0 we have $u_{j-1}\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big) < u_{j-1}\big(z_0 + \mathrm{e}^{\mathrm{i}t}\theta\big) + \epsilon$ for all t (since u_{j-1} is u.s.c.). Hence the average increases by at most ϵ, so we get $u_j(z) < u_j(z_0) + 2\epsilon$.

Next we show that $u_\phi \in \mathfrak{N}\mathrm{Psh}(\Omega)$. Pick a point $z \in \Omega$ and a null vector $\theta \in A^*$ such that $z + \overline{\mathbb{D}}\,\theta \subset \Omega$. By the monotone convergence theorem:

$$u_\phi(z) = \lim_{j\to\infty} u_j(z) \leq \lim_{j\to\infty} \int_0^{2\pi} u_{j-1}\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big)\,\frac{dt}{2\pi} = \int_0^{2\pi} u_\phi\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big)\,\frac{dt}{2\pi}.$$

Proof of null plurisubharmonicity

It remains to prove that u_ϕ is the largest null plurisubharmonic function dominated by ϕ.

Choose a null plurisubharmonic function $v \leq \phi = u_1$. We show by induction that $v \leq u_j$ for every $j \in \mathbb{N}$; clearly this will imply that $v \leq u_\phi$.

Suppose that $v \leq u_{j-1}$ for some j. Then for every $z \in \Omega$ and for every null vector $\theta \in A^*$ such that $z + \overline{\mathbb{D}}\,\theta \subset \Omega$ we have

$$v(z) \leq \int_0^{2\pi} v\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big)\,\frac{dt}{2\pi} \leq \int_0^{2\pi} u_{j-1}\big(z + \mathrm{e}^{\mathrm{i}t}\theta\big)\,\frac{dt}{2\pi}.$$

Taking infimum over all θ we get $v(z) \leq u_j(z)$ which concludes the inductive step.

This proves the EBS lemma.

Riemann-Hilbert problem for null discs

We also use the following RH lemma. **(Proof in Appendix B.)**

Lemma (Alarcón, F., Math. Ann., to appear)

Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ be a null holomorphic immersion, let $\theta \in A_*$ be a null vector, and let $\mu\colon \mathbb{T} = b\mathbb{D} \to [0,\infty)$ be a continuous function. Set

$$g\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3, \quad g(\zeta, z) = f(\zeta) + \mu(\zeta) z\theta.$$

Given $\epsilon > 0$ and $r \in (0,1)$, there exist $r' \in [r,1)$ and a null holomorphic disc $h\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ with $h(0) = f(0)$ satisfying the following properties:

(i) $\mathrm{dist}(h(\zeta), g(\zeta,\mathbb{T})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$.

(ii) $\mathrm{dist}(h(\rho\zeta), g(\zeta,\overline{\mathbb{D}})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$ *and* $\rho \in [r',1)$.

(iii) *h is ϵ-close to f on* $\{\zeta \in \mathbb{C}\colon |\zeta| \le r'\}$.

Furthermore, given an upper semicontinuous function $u\colon \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ and an arc $I \subset \mathbb{T}$, we may achieve that

$$\int_I u(h(\mathrm{e}^{\mathrm{i}s}))\,\frac{ds}{2\pi} \le \int_0^{2\pi}\int_{s\in I} u(g(\mathrm{e}^{\mathrm{i}s},\mathrm{e}^{\mathrm{i}t}))\,\frac{ds}{2\pi}\frac{dt}{2\pi} + \epsilon.$$

Proof of the disc formula for the $\mathfrak{N}\mathrm{Psh}$ minorant

The EBS-lemma furnishes a decreasing sequence of upper semicontinuous functions u_n on Ω which converges pointwise to the largest null plurisubharmonic function u_ϕ on Ω with $u_\phi \leq \phi$.

Let $\mathfrak{N}(\mathbb{D}, \Omega, z)$ denote the set of null discs $f \colon \overline{\mathbb{D}} \to \Omega$ with $f(0) = z$. To conclude the proof, we need to show that for every $z \in \Omega$

$$u_\phi(z) = u(z) := \inf \Big\{ \int_0^{2\pi} \phi(f(\mathrm{e}^{\mathrm{i}t}))\, \frac{dt}{2\pi} \colon \; f \in \mathfrak{N}(\mathbb{D}, \Omega, z) \Big\}.$$

Since $u_\phi \in \mathfrak{N}\mathrm{Psh}(\Omega)$, $u_\phi \circ f$ is subharmonic for any $f \in \mathfrak{N}(\mathbb{D}, \Omega, z)$, and hence we get by the submeanvalue property (using also $u_\phi \leq \phi$) that

$$u_\phi(z) = u_\phi(f(0)) \leq \int_0^{2\pi} u_\phi(f(\mathrm{e}^{\mathrm{i}t}))\, \frac{dt}{2\pi} \leq \int_0^{2\pi} \phi(f(\mathrm{e}^{\mathrm{i}t}))\, \frac{dt}{2\pi}.$$

Taking the infimum over all such f we obtain $u_\phi \leq u$ on Ω.

Proof of the disc formula - page 2

Proof of the reverse inequality $u \leq u_\phi$:

Fix a point $z \in \Omega$ and a number $\epsilon > 0$. Pick $n \in \mathbb{N}$ such that

$$u_\phi(z) \leq u_n(z) < u_\phi(z) + \epsilon. \tag{2}$$

By the definition of u_n there exists a null vector $\theta \in A_*$ such that the linear null disc

$$\overline{\mathbb{D}} \ni \zeta \mapsto f_{n-1}(\zeta) := z + \zeta\theta \in \Omega$$

satisfies

$$u_n(z) \leq \int_0^{2\pi} u_{n-1}\left(f_{n-1}(\mathrm{e}^{\mathrm{i}s})\right) \frac{ds}{2\pi} < u_n(z) + \frac{\epsilon}{2n}. \tag{3}$$

Fix a point $\mathrm{e}^{\mathrm{i}s_0} \in \mathbb{T}$. By the definition of u_{n-1} there exists a null vector $\theta_{s_0} \in A_*$ such that the null disc $\overline{\mathbb{D}} \ni \zeta \mapsto f_{n-1}(\mathrm{e}^{\mathrm{i}s_0}) + \zeta\theta_{s_0} \in \Omega$ satisfies

$$\int_0^{2\pi} u_{n-2}\left(f_{n-1}(\mathrm{e}^{\mathrm{i}s_0}) + \mathrm{e}^{\mathrm{i}t}\theta_{s_0}\right) \frac{dt}{2\pi} \leq u_{n-1}(f_{n-1}(\mathrm{e}^{\mathrm{i}s_0})) + \frac{\epsilon}{4n}. \tag{4}$$

Proof of the disc formula - page 3

Setting

$$g_{n-1}(\mathrm{e}^{\mathrm{i}s}, \zeta) = f_{n-1}(\mathrm{e}^{\mathrm{i}s}) + \zeta\theta_{t_0},$$

it follows from (4) that there is an arc $I \subset \mathbb{T}$ around $\mathrm{e}^{\mathrm{i}s_0}$ such that

$$\int_0^{2\pi}\int_{s\in I} u_{n-2}\big(g_{n-1}(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}t})\big)\,\frac{ds}{2\pi}\frac{dt}{2\pi} \le \int_I u_{n-1}\big(f_{n-1}(\mathrm{e}^{\mathrm{i}s})\big)\,\frac{ds}{2\pi} + \frac{|I|}{2\pi}\frac{\epsilon}{3n}.$$

We may work in a compact subset of Ω, hence $u_{n-2} \le M$ for some $M \in \mathbb{R}$. Repeating this construction at other points of $\mathbb{T}$ we find pairwise disjoint arcs $I_1, \ldots, I_\ell \subset \mathbb{T}$ such that the set $E = \mathbb{T} \setminus \bigcup_{j=1}^{\ell} I_j$ has measure

$$|E| < \frac{\epsilon}{3nM}$$

and for each $j = 1, \ldots, \ell$ we have

$$\int_0^{2\pi}\int_{s\in I_j} u_{n-2}\big(g_{n-1}(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}t})\big)\,\frac{ds}{2\pi}\frac{dt}{2\pi} \le \int_{I_j} u_{n-1}\big(f_{n-1}(\mathrm{e}^{\mathrm{i}s})\big)\,\frac{ds}{2\pi} + \frac{|I_j|}{2\pi}\frac{\epsilon}{3n}.$$

Proof of the disc formula - page 4

Let $\chi\colon \mathbb{T} \to [0,1]$ be a smooth function such that $\chi \equiv 1$ on $\bigcup_{j=1}^{\ell} I_j$ and $\chi \equiv 0$ on $\mathbb{T} \setminus \bigcup_{j=1}^{\ell} I_j'$, where $I_j' \supset I_j$ are bigger pairwise disjoint arcs. Define the map $h_{n-1}\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3$ by

$$h_{n-1}(\zeta,\xi) = g_{n-1}(\zeta,\chi(\zeta)\xi), \quad (\zeta,\xi) \in \mathbb{T} \times \overline{\mathbb{D}}.$$

By the RH lemma for null discs we get $f_{n-2} \in \mathfrak{N}(\mathbb{D},\Omega,z)$ satisfying

$$\begin{aligned}
\int_0^{2\pi} u_{n-2}\left(f_{n-2}(\mathrm{e}^{\mathrm{i}s})\right) \frac{ds}{2\pi} &\leq \int_0^{2\pi}\int_0^{2\pi} u_{n-2}\big(h_{n-1}(\mathrm{e}^{\mathrm{i}s},\mathrm{e}^{\mathrm{i}t})\big) \frac{ds}{2\pi}\frac{dt}{2\pi} + \frac{\epsilon}{3n} \\
&\leq \int_0^{2\pi} u_{n-1}\big(f_{n-1}(\mathrm{e}^{\mathrm{i}t})\big) \frac{dt}{2\pi} + \frac{\epsilon}{n} \\
&\leq u_n(z) + \frac{2\epsilon}{n}.
\end{aligned}$$

(We apply the RH-lemma ℓ times, once for each of the segments $I_1,\dots,I_\ell$.)

Proof of the disc formula - page 5

Repeating this procedure we get null discs $f_1, f_2, \ldots, f_{n-1} \in \mathfrak{N}(\mathbb{D}, \Omega, z)$ such that for every $j = 1, \ldots, n-2$ we have that

$$\int_0^{2\pi} u_j\left(f_j(\mathrm{e}^{\mathrm{i}s})\right) \frac{ds}{2\pi} \leq \int_0^{2\pi} u_{j+1}(f_{j+1}(\mathrm{e}^{\mathrm{i}s})) \frac{ds}{2\pi} + \frac{\epsilon}{n}.$$

Since $u_1 = \phi$, we get by (2) and (3) that

$$\begin{aligned} u(z) \leq \int_0^{2\pi} \phi\left(f_1(\mathrm{e}^{\mathrm{i}s})\right) \frac{ds}{2\pi} &\leq \int_0^{2\pi} u_{n-1}(f_{n-1}(\mathrm{e}^{\mathrm{i}s})) \frac{ds}{2\pi} + \frac{(n-2)\epsilon}{n} \\ &< u_n(z) + \epsilon < u_\phi(z) + 2\epsilon. \end{aligned}$$

Since this holds for any $\epsilon > 0$, we get $u(z) \leq u_\phi(z)$ as claimed.

This completes the proof of the disc formula for $\mathfrak{N}\mathrm{Psh}$ minorants (part (b) of our main theorem). As said before, part (a) follows from (b).

Minimal hulls and null hulls

Definition

- The **minimal hull** of a compact set $K \subset \mathbb{R}^n$ $(n \geq 3)$ is the set

$$\widehat{K}_{\mathfrak{M}} = \{x \in \mathbb{R}^n \colon u(x) \leq \sup_K u \quad \forall u \in \mathfrak{M}\mathrm{Psh}(\mathbb{R}^n)\}.$$

- The **null hull** of a compact set $K \subset \mathbb{C}^n$ $(n \geq 3)$ is the set

$$\widehat{K}_{\mathfrak{N}} = \{z \in \mathbb{C}^n \colon u(z) \leq \sup_K u \quad \forall u \in \mathfrak{N}\mathrm{Psh}(\mathbb{C}^n)\}.$$

- By the maximum principle for subharmonic functions, every bounded minimal surface $M \subset \mathbb{R}^n$ with boundary $bM \subset K$ lies in $\widehat{K}_{\mathfrak{M}}$.
- If K is a compact set in $\mathbb{C}^n$ then every bounded holomorphic null curve $M \subset \mathbb{C}^n$ with $bM \subset K$ lies in the null hull $\widehat{K}_{\mathfrak{N}}$ of K.
- Clearly $K \subset \widehat{K}_{\mathfrak{M}} \subset \mathrm{Co}(K) =$ the convex hull of K, and $K \subset \widehat{K}_{\mathfrak{N}} \subset \widehat{K}$. All inclusions are proper in general.

Relationship between minimal hulls and null hulls

Let $\pi \colon \mathbb{C}^n \to \mathbb{R}^n$ be the projection $\pi(x + \imath y) = x$.

Recall that a minimal plurisubharmonic function u on $\mathbb{R}^n$ lifts to a null plurisubharmonic function $u \circ \pi$ on $\mathbb{C}^n$. This implies that for any compact set $L \subset \mathbb{C}^n$ we have the inclusion

$$\pi\big(\widehat{L}_{\mathfrak{N}}\big) \subset \widehat{\pi(L)}_{\mathfrak{M}}.$$

This inclusion may be strict:

Take $L \subset \mathbb{C}^3$ to be a smooth embedded Jordan curve such that $K = \pi(L) \subset \mathbb{R}^3$ is also a smooth Jordan curve. By the solution of the **Plateau problem** such K bounds a minimal surface M which is therefore contained in $\widehat{K}_{\mathfrak{M}}$. However, we have $L = \widehat{L}_{\mathfrak{N}} = \widehat{L}$ for most curves L.

Characterization of minimal hull and null hull

The following characterisation of the minimal hull and the null hull is a corollary to our disc formula for $\mathfrak{M}\mathrm{Psh}$ and $\mathfrak{N}\mathrm{Psh}$ functions.

Corollary

(a) *Let K be a compact set in $\mathbb{R}^3$ and let $\omega \Subset \mathbb{R}^3$ be a bounded open convex set containing K. A point $p \in \omega$ belongs to the minimal hull $\widehat{K}_{\mathfrak{M}}$ if and only if there exists a sequence of conformal minimal discs $f_j \colon \overline{\mathbb{D}} \to \omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and*

$$\left|\{t \in [0, 2\pi] : \mathrm{dist}(f_j(e^{\mathrm{i}t}), K) < 1/j\}\right| \geq 2\pi - 1/j. \qquad (5)$$

(b) *Let K be a compact set in $\mathbb{C}^3$, and let $\Omega \subset \mathbb{C}^3$ be a bounded pseudoconvex Runge domain containing K (hence $\widehat{K}_{\mathfrak{N}} \subset \widehat{K} \subset \Omega$). A point $p \in \Omega$ belongs to the null hull $\widehat{K}_{\mathfrak{N}}$ of K if and only if there exists a sequence of null holomorphic discs $f_j \colon \overline{\mathbb{D}} \to \Omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and the estimate (5).*

Proof

We prove (b); (a) is similar.

Assume that for some $p \in \Omega$ there exists a sequence $f_j \in \mathfrak{N}(\mathbb{D}, \Omega, p)$ satisfying (5). Pick $u \in \mathfrak{N}\mathrm{Psh}(\mathbb{C}^3)$.

Let $U_j = \{z \in \mathbb{C}^3 : \mathrm{dist}(z, K) < 1/j\}$, $M_j = \sup_{U_j} u$, $M = \sup_\Omega u$, and $E_j = \{t \in [0, 2\pi] \colon f_j(\mathrm{e}^{\mathrm{i}t}) \notin U_j\}$.

Then $|E_j| \leq 1/j$ by (5) and

$$u(p) = u(f_j(0)) \leq \int_{E_j} + \int_{[0,2\pi]\setminus E_j} u(f_j(\mathrm{e}^{\mathrm{i}t}))\,\frac{dt}{2\pi} \leq M/j + M_j.$$

Passing to the limit gives $u(p) \leq \sup_K u$; hence $p \in \widehat{K}_{\mathfrak{N}}$.

Proof

To prove the converse, pick an open set U in $\mathbb{C}^3$ with $K \subset U \Subset \Omega$.

The function $\phi\colon \Omega \to \mathbb{R}$, which equals -1 on U and 0 on $\Omega \setminus U$, is upper semicontinuous.

Let $u \in \mathfrak{N}\mathrm{Psh}(\Omega)$ be the associated extremal function. Then $-1 \leq u \leq 0$ on Ω and $u = -1$ on K, whence $u = -1$ on $\widehat{K}_{\mathfrak{N}}$.

Fix a point $p \in \widehat{K}_{\mathfrak{N}}$ and a number $\epsilon > 0$. By the disc formula there is a null disc $f \in \mathfrak{N}(\mathbb{D}, \Omega, p)$ such that

$$\int_0^{2\pi} \phi(f(\mathrm{e}^{\mathrm{i}t}))\, \frac{dt}{2\pi} < -1 + \epsilon/2\pi.$$

This implies $|\{t \in [0, 2\pi]\colon f(\mathrm{e}^{\mathrm{i}t}) \in U\}| \geq 2\pi - \epsilon$. Apply this with the sequences $U_j = \{z \in \mathbb{C}^3 : \mathrm{dist}(z, K) < 1/j\}$ and $\epsilon_j = 1/j$.

Null positive forms and currents

We also characterize the null hull and the minimal hull by currents.

Definition

- A 2-form α on a domain $\Omega \subset \mathbb{C}^n$ is *null positive* if for every point $z \in \Omega$ and null vector $\nu \in A_*$ we have

$$\langle \alpha(z), \nu \wedge J\nu \rangle \geq 0.$$

  (We identify ν with a tangent vector in $T_z\mathbb{C}^n$, and J denotes the complex structure operator.)

- A current T on $\mathbb{C}^n$ of bidimension $(1,1)$ is null positive if $T(\alpha) \geq 0$ for every null positive 2-form α with compact support.

Note that a $\mathscr{C}^2$ function u is null plurisubharmonic if and only if $dd^c u$ is null positive.

Null hulls in terms of null positive currents

The following result is analogous to the characterization of the polynomial hull due to **Duval and Sibony (1995)** and **Wold (2011)**.

Theorem

Let K be a compact set in $\mathbb{C}^3$. A point $p \in \mathbb{C}^3$ belongs to the null hull $\widehat{K}_{\mathfrak{N}}$ of K if and only if there exists a null positive $(1,1)$-current T with compact support on $\mathbb{C}^3$ satisfying

$$dd^c T = \mu - \delta_p,$$

where μ is a probability measure on K such that

$$u(p) \le \int_K u\, d\mu \qquad \forall u \in \mathfrak{N}\mathrm{Psh}(\mathbb{C}^3).$$

The support of any such current T is contained in the null hull $\widehat{K}_{\mathfrak{N}}$. (Here δ_p denotes the point evaluation at p.)

Proof

Let $\zeta = x + \imath y$ be the coordinate on $\mathbb{C} \cong \mathbb{R}^2$. The *Green current* on the disc $\overline{\mathbb{D}}$ is defined on any 2-form $\alpha = adx \wedge dy$ with $a \in \mathscr{C}(\overline{\mathbb{D}})$ by

$$G(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{D}} \log |\zeta| \cdot a(\zeta) dx \wedge dy.$$

G is a positive current of bidimension $(1,1)$ satisfying $dd^c G = \sigma - \delta_0$, where σ is the normalized Lebesgue measure on $b\mathbb{D} = \mathbb{T}$.

If $f\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ is a (null) holomorphic map then $f_* G$ is a (null) positive $(1,1)$ current on $\mathbb{C}^n$ satisfying $dd^c(f_* G) = f_*\sigma - \delta_{f(0)}$. Its mass equals

$$\mathrm{M}(f_* G) = \frac{1}{4} \left(\int_{\mathbb{T}} |f|^2 d\sigma - |f(0)|^2 \right).$$ **(Proof in Appendix C)**

Assume now that $p \in \widehat{K}_{\mathfrak{N}}$ and f_j is a bounded sequence of null discs in $\mathbb{C}^3$ with centers at p whose boundaries converge to K. The Green currents $T_j = (f_j)_* G$ have bounded masses, and hence there is a weakly convergent subsequence $(T_{j_k})_{k \in \mathbb{N}}$. The limit $T = \lim_{k \to \infty} T_{j_k}$ is a null positive current satisfying the conclusion of the theorem.

Bochner's tube theorem for polynomial hulls

Let $\pi\colon \mathbb{C}^n \to \mathbb{R}^n$ denote the projection $\pi(x+\mathrm{i}y) = x$.

Theorem

Let K be a connected compact set in $\mathbb{R}^n$. For every point $z_0 = p + \mathrm{i}q \in \mathbb{C}^n$ with $p \in \mathrm{Co}(K)$ there exists a positive $(1,1)$ current T on $\mathbb{C}^n$ with finite mass satisfying $\mathrm{supp}\, T \subset \mathrm{Co}(K) \times \mathrm{i}\mathbb{R}^n$ and $dd^c T = \mu - \delta_{z_0}$, where μ is a probability measure on $\mathcal{T}_K$.

Conversely, let T be a positive $(1,1)$ current on $\mathbb{C}^n$ with finite mass such that $\pi(\mathrm{supp}\, T)$ is a bounded set in $\mathbb{R}^n$. If $dd^c T \le 0$ on $\mathbb{C}^n \setminus \mathcal{T}_K$, then $\mathrm{supp}\, T \subset \mathrm{Co}(K) \times \mathrm{i}\mathbb{R}^n$.

Proof: Let $p \in \mathrm{Co}(K)$. It is classical that there exists a sequence of holomorphic discs $f_j = g_j + \mathrm{i}h_j\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ with $f_j(0) = p$ such that the sequence g_j is uniformly bounded and the boundaries $g_j(\mathbb{T})$ converge to K. It follows that the L^2-norms of g_j, and hence also of f_j, are uniformly bounded. Hence the Green currents $T_j = (f_j)_* G$ have uniformly bounded masses. A subsequence converges weakly to a current T satisfying the conclusion of the theorem. The converse is standard.

Minimal hulls in terms of currents

The same proof, together with the connection between conformal minimal discs in a domain $\omega \subset \mathbb{R}^3$ and null discs in the tube $\mathcal{T}_\omega = \omega \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over ω, gives the following analogous result, characterizing minimal hulls of compact sets in $\mathbb{R}^3$ by projections of supports of certain null currents in $\mathbb{C}^3$.

Theorem

Let K be a compact set in $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$ belongs to the minimal hull $\widehat{K}_{\mathfrak{M}}$ if and only if there exists a null positive current T on $\mathbb{C}^3$ of finite mass such that $\pi(\mathrm{supp}\ T) \subset \mathbb{R}^3$ is a bounded set and

$$dd^c T = \mu - \delta_p,$$

where μ is a probability measure on the tube $\mathcal{T}_K = K \times i\mathbb{R}^3$.

THANK YOU!

Appendix A: Basics on minimal surfaces in $\mathbb{R}^n$

Assume that D is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbf{x} = (x_1, \ldots, x_n)\colon D \to \mathbb{R}^n$ is a $\mathscr{C}^2$ embedding. Let $S = \mathbf{x}(D) \subset \mathbb{R}^n$, a parametrized surface in $\mathbb{R}^n$.

Every smooth embedded curve in S is of the form

$$\lambda(t) = \mathbf{x}(u_1(t), u_2(t)) \in S$$

where $t \mapsto (u_1(t), u_2(t))$ is a smooth embedded curve in D.

Let $s = s(t)$ denote the arc length on λ. The number

$$\kappa(\mathbf{T}, \mathbf{N}) := \frac{d^2\lambda}{ds^2} \cdot \mathbf{N} = \sum_{i,j=1}^{2} \left(\mathbf{x}_{u_i u_j} \cdot \mathbf{N} \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the **normal curvature** of S at $p = \lambda(t) \in S$ in the tangent direction $\mathbf{T} = \lambda'(s) \in T_p S$ with respect to the normal vector $\mathbf{N} \in N_p S$.
(It only depends on $\mathbf{T}$ and $\mathbf{N}$.)

A2: Curvature in terms of fundamental forms

In terms of t-derivatives we get

$$\kappa(\mathbf{T},\mathbf{N}) = \frac{\sum_{i,j=1}^{2}\left(\mathbf{x}_{u_i u_j}\cdot\mathbf{N}\right)\dot{u}_i\dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j}\,\dot{u}_i\dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}$$

Fix a normal vector $\mathbf{N} \in N_pS$ and vary the unit tangent vector $\mathbf{T} \in T_pS$. The **principal curvatures** of S at p in direction $\mathbf{N}$ are the numbers

$$\kappa_1(\mathbf{N}) = \max_{\mathbf{T}} \kappa(\mathbf{T},\mathbf{N}), \qquad \kappa_2(\mathbf{N}) = \min_{\mathbf{T}} \kappa(\mathbf{T},\mathbf{N}).$$

Their average

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1(\mathbf{N}) + \kappa_2(\mathbf{N})}{2} \in \mathbb{R}$$

is the **mean curvature** of S at p in the normal direction $\mathbf{N} \in N_pS$.

A3: The mean curvature vector

Let $G = (g_{i,j})$ and $h(\mathbf{N}) = (h_{i,j}(\mathbf{N})) = (\mathbf{x}_{u_i u_j} \cdot \mathbf{N})$ denote the matrices of the 1st and the 2nd fundamental form, respectively.

The extremal values of $\kappa(\mathbf{T}, \mathbf{N})$ are roots of the equation

$$\det\big(h(\mathbf{N}) - \mu G\big) = 0$$

$$\det G \cdot \mu^2 - \big(g_{2,2} h_{1,1}(\mathbf{N}) + g_{1,1} h_{2,2}(\mathbf{N}) - 2 g_{1,2} h_{1,2}(\mathbf{N})\big)\mu + \det h(\mathbf{N}) = 0.$$

The Vieta formula gives

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2}\mathbf{x}_{u_1 u_1} + g_{1,1}\mathbf{x}_{u_2 u_2} - 2 g_{1,2}\mathbf{x}_{u_1 u_2}}{2 \det G} \cdot \mathbf{N}.$$

There is a unique normal vector $\mathbf{H} \in N_p S$ such that

$$\mathrm{H}(\mathbf{N}) = \mathbf{H} \cdot \mathbf{N} \quad \text{for all } \mathbf{N} \in N_p S.$$

This $\mathbf{H}$ is the **mean curvature vector** of the surface S at p.

A4: The mean curvature in isothermal coordinates

The formulas simplify in **isothermal coordinates**:

$$G = (g_{i,j}) = \xi I,\ \det G = \xi^2; \quad \xi = ||\mathbf{x}_{u_1}||^2 = ||\mathbf{x}_{u_2}||^2,\ \mathbf{x}_{u_1} \cdot \mathbf{x}_{u_2} = 0$$

$$\mathrm{H}(\mathbf{N}) = \frac{\mathbf{x}_{u_1 u_1} + \mathbf{x}_{u_2 u_2}}{2\xi} \cdot \mathbf{N} = \frac{\triangle \mathbf{x}}{2\xi} \cdot \mathbf{N}.$$

Lemma

Assume that D is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbf{x}\colon D \to \mathbb{R}^n$ is a conformal immersion *of class $\mathscr{C}^2$ (i.e., $u = (u_1, u_2)$ are isothermal for $\mathbf{x}$.) Then the Laplacian $\triangle \mathbf{x} = \mathbf{x}_{u_1 u_1} + \mathbf{x}_{u_2 u_2}$ is orthogonal to $S = \mathbf{x}(D)$ and satisfies*

$$\triangle \mathbf{x} = 2\xi \mathbf{H}$$

where $\mathbf{H}$ is the mean curvature vector and $\xi = ||\mathbf{x}_{u_1}||^2 = ||\mathbf{x}_{u_2}||^2$.

A5: Proof of the lemma

Proof.

It suffices to show that the vector $\triangle \mathbf{x}(u)$ is orthogonal to the surface S at the point $\mathbf{x}(u)$ for every $u \in D$. If this holds, it follows from the preceding formula that the normal vector $(2\xi)^{-1}\triangle \mathbf{x}(u) \in N_{\mathbf{x}(u)}S$ fits the definition of the mean curvature vector $\mathbf{H}$, so it equals $\mathbf{H}$.

Conformality of the immersion $\mathbf{x}$ can be written as follows:

$$\mathbf{x}_{u_1} \cdot \mathbf{x}_{u_1} = \mathbf{x}_{u_2} \cdot \mathbf{x}_{u_2}, \qquad \mathbf{x}_{u_1} \cdot \mathbf{x}_{u_2} = 0.$$

Differentiating the first identity on u_1 and the second one on u_2 yields

$$\mathbf{x}_{u_1 u_1} \cdot \mathbf{x}_{u_1} = \mathbf{x}_{u_1 u_2} \cdot \mathbf{x}_{u_2} = -\mathbf{x}_{u_2 u_2} \cdot \mathbf{x}_{u_1},$$

whence $\triangle \mathbf{x} \cdot \mathbf{x}_{u_1} = 0$. Similarly we get $\triangle \mathbf{x} \cdot \mathbf{x}_{u_2} = 0$ by differentiating the first identity on u_2 and the second one on u_1. This proves the claim. □

A6: Lagrange's formula for the variation of area

The area of an immersed surface $\mathbf{x}\colon D \to \mathbb{R}^n$ with the 1st fundamental form $G = (g_{i,j})$ equals

$$\mathcal{A}(\mathbf{x}) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$

Let $\mathbf{N}\colon D \to \mathbb{R}^n$ be a *normal vector field* along $\mathbf{x}$ which vanishes on bD. Consider the 1-parameter family of maps $\mathbf{x}^t\colon D \to \mathbb{R}^n$:

$$\mathbf{x}^t(u) = \mathbf{x}(u) + t\,\mathbf{N}(u), \quad u \in D,\ t \in \mathbb{R}.$$

A calculation gives the formula for the first variation of area:

$$\delta\mathcal{A}(\mathbf{x})\mathbf{N} = \frac{d}{dt}\bigg|_{t=0} \mathcal{A}(\mathbf{x}^t) = -2\int_D \mathbf{H}\cdot \mathbf{N} \sqrt{\det G} \cdot du_1 du_2.$$

It follows that $\qquad \delta\mathcal{A}(\mathbf{x}) = 0 \Longleftrightarrow \mathbf{H} = 0.$

A7: Conformal minimal surfaces are harmonic

In view of the already established formula

$$\triangle \mathbf{x} = 2\xi\,\mathbf{H}$$

which holds for any conformal immersion $\mathbf{x}$ we get

Corollary

Let M be an open Riemann surface. The following are equivalent for a smooth conformal immersion $\mathbf{x}\colon M \to \mathbb{R}^n$:

- $\mathbf{x}$ *is minimal (a stationary point of the area functional).*
- $\mathbf{x}$ *has vanishing mean curvature vector:* $\mathbf{H} = 0$.
- $\mathbf{x}$ *is harmonic:* $\triangle \mathbf{x} = 0$.

In the sequel we shall always assume that $\mathbf{x}$ is conformal and hence

$$\delta\mathcal{A}(\mathbf{x}) \iff \mathbf{H} = 0 \iff \triangle \mathbf{x} = 0.$$

A8: Minimal surfaces versus area minimizing surfaces

We wish to emphasize the difference between

- **minimal surfaces:** these are stationary (critical) points of the area functional, and are (only) locally area minimizing; and
- **area-minimizing surfaces:** these are surfaces which globally minimize the area among all nearby surfaces with the same boundary.
- Minimal surfaces which are **graphs** are globally area minimizing.
- Recent work on conditions ensuring that a minimal surface is globally are minimizing was done by C. Arezzo.

Appendix B: Riemann-Hilbert problem for null discs

Lemma (A. Alarcón and F. F., The Calabi-Yau problem, null curves, and Bryant surfaces. arxiv.org/abs/1308.0903)

Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ *be a null holomorphic immersion, let* $\theta \in A_*$ *be a null vector, and let* $\mu\colon \mathbb{T} = b\mathbb{D} \to [0,\infty)$ *be a continuous function. Let*

$$g\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3, \quad g(\zeta, z) = f(\zeta) + \mu(\zeta) z\theta.$$

Given $\epsilon > 0$ *and* $0 < r < 1$, *there exist a number* $r' \in [r, 1)$ *and a null holomorphic disc* $h\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ *with* $h(0) = f(0)$, *satisfying the following:*

(i) $\mathrm{dist}(h(\zeta), g(\zeta, \mathbb{T})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$,

(ii) $\mathrm{dist}(h(\rho\zeta), g(\zeta, \overline{\mathbb{D}})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$ *and all* $\rho \in [r', 1)$, *and*

(iii) h *is* ϵ*-close to* f *on* $\{\zeta \in \mathbb{C}\colon |\zeta| \le r'\}$.

Furthermore, given an upper semicontinuous function $u\colon \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ *and an arc* $I \subset \mathbb{T}$, *we may achieve that*

$$\int_I u(h(\mathrm{e}^{\mathrm{i}t}))\,\frac{dt}{2\pi} \le \int_0^{2\pi}\!\!\int_I u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}s}))\,\frac{dt}{2\pi}\frac{ds}{2\pi} + \epsilon.$$

Appendix B: Proof - page 1

Consider the following unbranched two-sheeted holomorphic covering (the **spinor representation of the null quadric**):

$$\pi\colon \mathbb{C}^2 \setminus \{(0,0)\} \to A_*, \quad \pi(u,v) = \big(u^2 - v^2, \mathfrak{i}(u^2+v^2), 2uv\big).$$

Since $\overline{\mathbb{D}}$ is simply connected, the map $f'\colon \overline{\mathbb{D}} \to A_*$ lifts to a map $(u,v)\colon \overline{\mathbb{D}} \to \mathbb{C}^2 \setminus \{(0,0)\}$. Hence we have

$$\begin{aligned}
f' &= \pi(u,v) = \Big(u^2 - v^2, \mathfrak{i}(u^2+v^2), 2uv\Big) \in A_* \\
\vartheta &= \pi(a,b) = \Big(a^2 - b^2, \mathfrak{i}(a^2+b^2), 2ab\Big) \in A_* \\
\eta &= \sqrt{\mu}\colon b\mathbb{D} \to \mathbb{R}_+ \\
\eta(\zeta) &\approx \tilde\eta(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \ \ \text{(rational approximation)} \\
\mu(\zeta) &\approx \tilde\eta^2(\zeta) = \sum_{j=1}^{2N} B_j \zeta^{j-2m}.
\end{aligned}$$

Appendix B: Proof - page 2

For any integer $n \in \mathbb{N}$ we consider the following functions and maps on the closed disc $\overline{\mathbb{D}}$:

$$\begin{aligned} u_n(z) &= u(z) + \sqrt{2n+1}\,\tilde{\eta}(z) z^n a, \\ v_n(z) &= v(z) + \sqrt{2n+1}\,\tilde{\eta}(z) z^n b, \\ \Phi_n(z) &= \pi(u_n(z), v_n(z)) = \big(u_n^2 - v_n^2, \imath(u_n^2 - v_n^2), 2u_n v_n\big) \colon \overline{\mathbb{D}} \to A_*, \\ f_n(\zeta) &= f(0) + \int_0^\zeta \Phi_n(z)\, dz, \qquad \zeta \in \overline{\mathbb{D}}. \end{aligned}$$

Then $f_n \colon \overline{\mathbb{D}} \to \mathbb{C}^3$ is a null disc of the form

$$f_n(\zeta) = f(\zeta) + \mathbf{B}_n(\zeta)\,\vartheta + \mathbf{A}_n(\zeta).$$

Appendix B: Proof - page 3

The $\mathbb{C}$-valued term $\mathbf{B}_n$ equals

$$\begin{aligned}\mathbf{B}_n(\zeta) &= (2n+1)\sum_{j=1}^{2N}\int_0^{\zeta} B_j z^{2n+j-2m}\,dz \\ &= \sum_{j=1}^{2N}\frac{2n+1}{2n+1+j-2m}B_j\,\zeta^{2n+1+j-2m}.\end{aligned}$$

Since the coefficients $(2n+1)/(2n+1+j-2m)$ in the sum for $\mathbf{B}_n$ converge to 1 as $n \to +\infty$, we have that

$$\sup_{|\zeta|\leq 1}\left|\mathbf{B}_n(\zeta) - \zeta^{2n+1}\tilde{\eta}^2(\zeta)\right| \to 0 \quad \text{as } n \to \infty.$$

Appendix B: Proof - page 4

The remainder $\mathbb{C}^3$-valued term $\mathbf{A}_n(\zeta)$ equals

$$\begin{aligned}
\mathbf{A}_n(\zeta) &= 2\sqrt{2n+1}\int_0^\zeta \sum_{j=1}^N A_j\, z^{n+j-m}\big(u(z)(a,\mathrm{i}a,b)+v(z)(-b,\mathrm{i}b,a)\big)\,dz \\
|\mathbf{A}_n(\zeta)| &\leq 2\sqrt{2n+1}\,C_0\sum_{j=1}^N |A_j|\int_0^{|\zeta|}|z|^{n+j-m}\,d|z| \\
&\leq 2C_0\sum_{j=1}^N \frac{\sqrt{2n+1}}{n+1+j-m}|A_j|.
\end{aligned}$$

It follows that $|\mathbf{A}_n|\to 0$ uniformly on $\overline{\mathbb{D}}$ as $n\to+\infty$. Hence

$$f_n(\zeta)\approx f(\zeta)+\zeta^{2n+1}\tilde{\mu}(\zeta)\vartheta, \qquad \zeta\in\overline{\mathbb{D}},\ n>>0.$$

The map $h=f_n$ for large enough $n\in\mathbb{N}$ solves the problem.

Appendix C: Green currents

Recall that the *Green current* on the disc $\overline{\mathbb{D}}$ is defined on any 2-form $\alpha = a dx \wedge dy$ with $a \in \mathscr{C}(\overline{\mathbb{D}})$ by

$$G(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{D}} \log|\zeta| \cdot a(\zeta) dx \wedge dy.$$

Green's formula

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt + \frac{1}{2\pi} \int_{\mathbb{D}} \log|\zeta| \cdot \triangle u(\zeta)\, dx \wedge dy, \qquad (6)$$

which holds for any $u \in \mathscr{C}^2(\overline{\mathbb{D}})$, tells us that $dd^c G = \sigma - \delta_0$.

Given a smooth map $f = (f_1, \ldots, f_n) \colon \overline{\mathbb{D}} \to \mathbb{R}^n$ we denote by $f_* G$ the 2-dimensional current on $\mathbb{R}^n$ given on $\alpha = \sum_{i,j=1}^n a_{i,j} dx_i \wedge dx_j$ by

$$(f_* G)(\alpha) = G(f^* \alpha) = -\frac{1}{2\pi} \int_{\mathbb{D}} \log|\zeta| \cdot f^* \alpha. \qquad (7)$$

We call $f_* G$ the *Green current supported by* f.

Mass of Green currents

The following lemma is crucial in the proof of the characterisation of minimal hulls by Greens currents, and also in the proof of Bochner's tube theorem for polynomial hulls.

Lemma

If $f = (f_1, \ldots, f_n)\colon \overline{\mathbb{D}} \to \mathbb{R}^n$ is a conformal harmonic immersion of class $\mathscr{C}^2(\overline{\mathbb{D}})$, then the mass of the Green current $f_* G$ satisfies

$$\mathrm{M}(f_* G) \leq \frac{1}{4}\left(\int_{\mathbb{T}} |f|^2 d\sigma - |f(0)|^2\right). \qquad (8)$$

If f is injective outside of a closed set of measure zero in $\overline{\mathbb{D}}$, or if $f\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ is a holomorphic disc, then we have equality in (8).

The possible loss of mass for a non-holomorphic disc f may be caused by the cancellation of parts of the immersed surface $f(\overline{\mathbb{D}})$ (considered as a current) due to the reversal of the orientation.

Proof of the mass formula

Proof. Denote the partial derivatives of $f\colon \overline{\mathbb{D}} \to \mathbb{R}^n$ by f_x and f_y. Write

$$|f|^2 = \sum_{i=1}^{n} f_i^2, \qquad |\nabla f|^2 = \sum_{i=1}^{n} |\nabla f_i|^2 = \sum_{i=1}^{n} \left(f_{i,x}^2 + f_{i,y}^2 \right).$$

Since f is conformal, the vector fields f_x and f_y are orthogonal and satisfy $|f_x| = |f_y|$. Let

$$\vec{T} = \frac{f_x \wedge f_y}{|f_x| \cdot |f_y|} = \frac{f_x \wedge f_y}{|f_x|^2}.$$

Given a 2-form α on $\mathbb{R}^n$, we have

$$f^*\alpha = \langle \alpha \circ f, f_x \wedge f_y \rangle \, dx \wedge dy = \langle \alpha \circ f, \vec{T} \rangle \, |f_x|^2 \, dx \wedge dy. \tag{9}$$

The definition of $T = f_* G$ and the formula (9) imply

$$T(\alpha) = -\frac{1}{2\pi} \int_{\mathbb{D}} \log |\zeta| \cdot \langle \alpha \circ f, \vec{T} \rangle \cdot |f_x|^2 \, dx \wedge dy. \tag{10}$$

Conclusion of the proof

From the definition of the mass of a current and (10) it follows that

$$\mathrm{M}(T) = \sup\{T(\alpha) \colon |\langle \alpha, \vec{T} \rangle| \le 1\} \le -\frac{1}{2\pi} \int_{\mathbb{D}} \log|\zeta| \cdot |f_x|^2 \, dx \wedge dy; \tag{11}$$

equality holds for holomorphic discs. For any harmonic function $v \in \mathscr{C}^2(\overline{\mathbb{D}})$ we have

$$dd^c v^2 = d(2v\, d^c v) = 2dv \wedge d^c v = 2|\nabla v|^2 dx \wedge dy.$$

Applying this to each component f_i of f we get

$$|\nabla f|^2 \, dx \wedge dy = \sum_{i=1}^{n} |\nabla f_i|^2 \, dx \wedge dy = \frac{1}{2} \sum_{i=1}^{n} dd^c f_i^2 = \frac{1}{2} dd^c |f|^2.$$

Inserting the identity $|f_x|^2 \, dx \wedge dy = \frac{1}{2}|\nabla f|^2 \, dx \wedge dy = \frac{1}{4} dd^c |f|^2$ into (11) and applying Green's identity (6) gives (8) and proves the lemma.