Recent advances in elliptic complex geometry

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- S has a strictly plurisubharmonic exhaustion function.

What should the dual notion be?

Some important complex manifolds X admit *many* holomorphic maps $\mathbb{C} \to X$, and $\mathbb{C}^k \to X$; for example $\mathbb{C}^n$, $\mathbb{P}^n$, complex Lie groups and their homogeneous spaces.

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Runge Theorem. Let $K \subset \mathbb{C}$ be compact with no holes. Every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by entire functions.

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Formulate them as properties of an arbitrary target X, with source any Stein manifold (or Stein space).

Oka properties of a complex manifold

A property that a complex manifold X might or might not have:

Basic Oka Property (BOP): For every Stein inclusion $T \hookrightarrow S$ and every compact $\mathcal{O}(S)$-convex set $K \subset S$, a continuous map $f\colon S \to X$ that is holomorphic on $K \cup T$ can be deformed to a holomorphic map $F\colon S \to X$.
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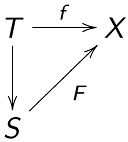

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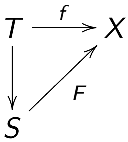

By Oka-Weil and Cartan, $\mathbb{C}$, and hence $\mathbb{C}^n$, satisfy BOP.

No hyperbolic (or volume hyperbolic) X has BOP.

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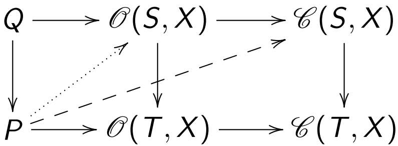

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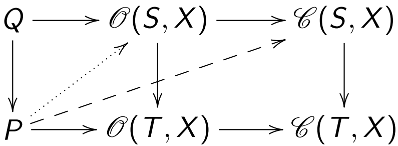

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Applying POP with parameter pairs $\emptyset \subset S^k$ and $S^k \subset B^{k+1}$ shows that for any POP manifold X and for any Stein manifold S:

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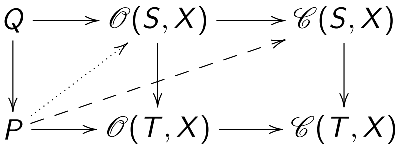

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The isomorphisms of homotopy groups are induced by the inclusion $\mathcal{O}(S,X) \hookrightarrow \mathcal{O}(S,X)$.

The Oka-Grauert Principle

Theorem (Grauert, 1957-58): Every complex homogeneous manifold enjoys POP for all pairs of finite polyhedra $Q \subset P$. The analogous result holds for sections of holomorphic G-bundles (G a complex Lie group) over a Stein space.

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Kiyoshi Oka proved this result for line bundles in 1939: A Cousin-II problem is solvable by holomorphic functions if it is solvable by continuous functions.

Elliptic manifolds

Gromov, 1989: A complex manifold X is elliptic if it admits a dominating spray:
A holomorphic map $s\colon E \to X$ defined on the total space of a holomorphic vector bundle E over X such that $s(0_x) = x$ and $s|E_x \to X$ is a submersion at 0_x for all $x \in X$.

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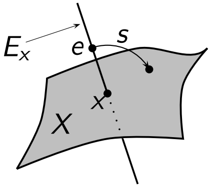

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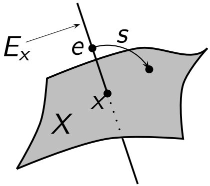

Subelliptic manifold (F. 2002): There exist finitely many sprays $s_j\colon E_j \to X$ such that

$$\sum_j ds_j(E_{j,x}) = T_x X, \quad \forall x \in X.$$

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• A spray of this type exists on $X = \mathbb{C}^n \backslash A$ where A is algebraic subvariety with $\dim A \leq n - 2$.
Use shear vector fields $f(\pi(z))v$ ($v \in \mathbb{C}^n$, $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ linear projection, $\pi(v) = 0$) that vanish on A: $f = 0$ on $\pi(A) \subset \mathbb{C}^{n-1}$.

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Problem: Lack of known functorial properties of (sub)ellipticity

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Theorem (Gromov 1989). POP holds in the following cases:

1. Maps $S \to X$ from a Stein manifold S to an elliptic manifold X.
2. Sections of a holomorphic fiber bundle $Z \to S$ with elliptic fiber over a Stein manifold S.
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$S = S_0 \supset S_1 \supset \cdots \supset S_m = \emptyset$, $M_k = S_k \backslash S_{k+1}$ smooth, the restriction of $Z|M_k$ a subelliptic submersion.

Sections avoiding subvarieties

Example: Let $E \to S$ be a holo. vector bundle with fiber $E_x \cong \mathbb{C}^k$, and let $\Sigma \subset E$ be a tame complex subvariety with fibers $\Sigma_x \subset E_x$ of codimension ≥ 2. Then $E \backslash \Sigma \to S$ is an elliptic submersion. Hence sections $S \to E$ avoiding Σ satisfy POP.

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Tameness: The closure of Σ in the associated bundle $\widehat{E} \to S$ with fibers $\widehat{E}_x \cong \mathbb{P}^k$ does not contain the hyperplane at infinity $\widehat{E}_x \backslash E_x \cong \mathbb{P}^{k-1}$ over any point $x \in X$.

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Existence of proper holo. embeddings $S^n \hookrightarrow \mathbb{C}^N$, $N = \left[\frac{3n}{2}\right] + 1$, when S^n is Stein and $n > 1$.

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F. Lárusson: What is an Oka manifold?
Notices Amer. Math. Soc. 57 (2010), no. 1, 50–52.

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Question: Is every Oka manifold also elliptic? (Gromov: Every Stein Oka manifold is elliptic.)

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- Split

$$\gamma = \beta \circ \alpha^{-1}, \quad \alpha, \beta \approx \mathrm{Id}.$$

Then $F \circ \alpha = G \circ \beta\colon (A \cup B) \times r\mathbb{B}^k \to X$ solves the problem.

Passing a critical point

Passing a critical point p_0 of an exhaustion function ρ on S

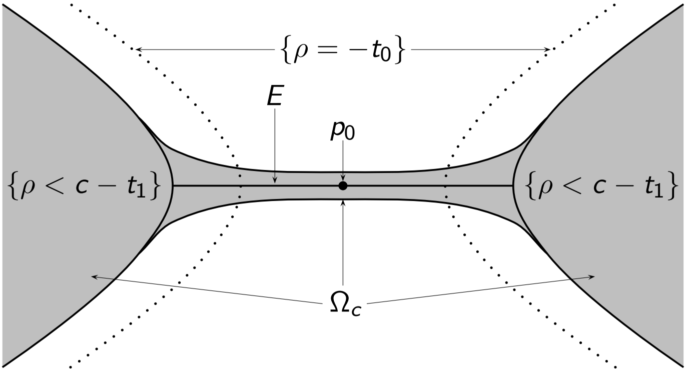

Figure: The set $\Omega_c = \{\tau < c\}$, $c > 0$.

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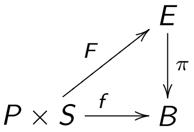

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Theorem (F. 2010) Let $\pi\colon E \to B$ a stratified holo. submersion.
(a) BOP $\Longrightarrow$ POP, and these are local properties.
(b) A stratified holo. fiber bundle with Oka fibers enjoys POP.
(c) A stratified subelliptic submersion enjoys POP.

Recent application to Gromov-Vaserstein Problem

Theorem (Ivarsson and Kutzschebauch, 2009)
Let S be a Stein manifold and $f \colon S \to SL_m(\mathbb{C})$ a null-homotopic holomorphic mapping. There exist $k \in \mathbb{N}$ and holomorphic mappings $G_1, \ldots, G_k \colon S \to \mathbb{C}^{m(m-1)/2}$ such that

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Vaserstein (1988): Factorization of continuous maps.

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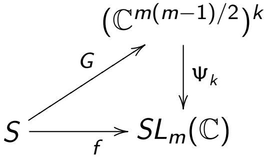

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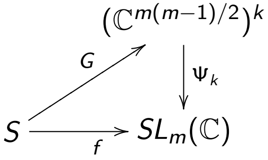

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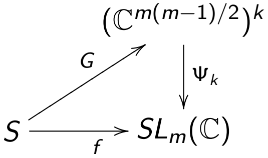

Vaserstein's result gives a *continuous* lifting of f.
We deform this continuous lifting to a holomorphic lifting by applying the Oka principle to certain auxiliary submersions (row projections $SL_m(\mathbb{C}) \to \mathbb{C}^n$) that are *stratified elliptic*.

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Assume that $\pi\colon E_g = \mathbb{D} \times \mathbb{C} \backslash \Gamma_g \to \mathbb{D}$ is Oka. Then there is a holomorphic map $F\colon \mathbb{D} \times \mathbb{C}^* \to E_g$ such that the dgm. commutes:

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Question: Must g be holomorphic?

Link with homotopy theory

Lárusson 2004: POP is a homotopy-theoretic property.

The category of complex manifolds can be embedded into a model category such that:
• a holomorphic map is acyclic iff it is topologically acyclic.
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Theorem (Lárusson 2004–5). In this model structure, a complex manifold is:

• cofibrant iff it is Stein.

• fibrant iff it has POP.