Non-orientable minimal surfaces in $\mathbb{R}^n$

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Abstract

We will show how complex analytic methods can be used for constructions of **orientable** and also **non-orientable minimal surfaces** in $\mathbb{R}^n$ for any ≥ 3. In particular, we obtain

- the Runge-Mergelyan approximation theorem for conformal minimal surfaces in $\mathbb{R}^n$;
- general position and properness theorems;
- complete minimal surfaces in $\mathbb{R}^n$ bounded by Jordan curves;
- (complete) proper minimal surfaces in convex domains in $\mathbb{R}^n$ $(n \geq 3)$ and in minimally convex domains in $\mathbb{R}^3$.

Based on joint work with

- **Antonio Alarcón and Francisco J. López, University of Granada**
- **Barbara Drinovec Drnovšek, University of Ljubljana**

A brief history of minimal surface theory

1744 **Euler** The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

1760 **Lagrange**: A graph $z = f(x, y)$ is area minimizing if and only if

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

1776 **Meusnier** A smooth surface $S \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function $\mathbb{H}$ vanishes identically. The helicoid is a minimal surface.

1873 **Plateau** Minimal surfaces can be obtained as soap films.

1932 **Douglas, Radó** Every Jordan curve in $\mathbb{R}^3$ spans a minimal surface.

1965 **Calabi's Conjecture:** Every complete minimal surface in $\mathbb{R}^3$ is unbounded. (Complete: every divergent curve has infinite length.) This conjecture, which is wrong as stated, opened a major direction.

2000 **S.-T. Yau: The Calabi-Yau Problem**.

Conformal minimal = conformal harmonic

Theorem (Classical; see e.g. Osserman, A survey of minimal surfaces, Dover, New York,1986)

Let M be a surface endowed with a conformal structure. The following are equivalent for a **conformal** *immersion $\mathbf{X} : M \to \mathbb{R}^n$ $(n \geq 3)$:*

- $\mathbf{X}$ *is minimal (a stationary point of the area functional).*
- $\mathbf{X}$ *has identically vanishing mean curvature vector:* $\mathbf{H} = 0$.
- $\mathbf{X}$ *is harmonic:* $\triangle \mathbf{X} = 0$.

Indeed, direct calculations show that

$$\triangle \mathbf{X} = 2\xi \mathbf{H}$$

where

$$\xi = |\mathbf{X}_u|^2 = |\mathbf{X}_v|^2$$

and $\zeta = u + iv$ be a local holomorphic coordinate on M.

Weierstrass representation

Let M be an open Riemann surface and $\mathbf{X} = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ be a smooth immersion. Fix a nonvanishing holomorphic 1-form θ on M.

Conformality of $\mathbf{X}$ is equivalent to the **nullity condition**

$$(\partial X_1)^2 + (\partial X_2)^2 + \cdots + (\partial X_n)^2 = 0.$$

Hence $\partial \mathbf{X} = \mathbf{f}\theta$, where the map $\mathbf{f} \colon M \to \mathbb{C}^n$ assumes values in

$$\mathcal{A}_* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} \colon \sum_{j=1}^{n} z_j^2 = 0\} \quad \textbf{(null quadric)}.$$

Since $\bar\partial \partial \mathbf{X} = \bar\partial \mathbf{f} \wedge \theta$, $\mathbf{X}$ is harmonic iff $\mathbf{f} = \partial \mathbf{X}/\theta$ is holomorphic.

Conclusion: Every conformal minimal immersion $M \to \mathbb{R}^n$ is of the form

$$\mathbf{X}(p) = \mathbf{X}(p_0) + 2 \int_{p_0}^{p} \Re(\mathbf{f}\theta), \quad p_0, p \in M,$$

where $\mathbf{f} \colon M \to \mathcal{A}_*$ is holomorphic and the real periods of $\mathbf{f}\theta$ vanish.

Holomorphic null curves

The **flux homomorphism** $\mathrm{Flux}(\mathbf{X})\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$:

$$\mathrm{Flux}(\mathbf{X})(\gamma) = \int_\gamma d^c\mathbf{X} = 2\int_\gamma \Im(\mathbf{f}\theta), \quad [\gamma] \in H_1(M;\mathbb{Z}).$$

If $\mathrm{Flux}(\mathbf{X}) = 0$, then

$$\mathbf{Z}(p) = \int_\cdot^p \mathbf{f}\theta \in \mathbb{C}^n, \qquad p \in M$$

is a **holomorphic null curve** $\mathbf{Z} = (Z_1,\ldots,Z_n)\colon M \to \mathbb{C}^n$, i.e.,

$$(\partial Z_1)^2 + (\partial Z_2)^2 + \cdots + (\partial Z_n)^2 = 0.$$

The real and the imaginary part of a holomorphic null curve $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}\colon M \to \mathbb{C}^n$ are conformal minimal immersions $M \to \mathbb{R}^n$. The converse holds on the disk $\mathbb{D} = \{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$.

Runge-Mergelyan approximation theorem

Let M be an open Riemann surface.

Theorem (1)

If K is a compact Runge subset of M, then every conformal minimal immersion $K \to \mathbb{R}^n$ can be approximated by proper conformal minimal immersions $M \to \mathbb{R}^n$; embeddings if $n \geq 5$.

The analogous result holds for null holomorphic curves $M \to \mathbb{C}^n$, $n \geq 3$.

A. Alarcón, F. Forstnerič, Inventiones Math. 196 (2014)
A. Alarcón, F. Forstnerič, F.J. López, Embedded minimal surfaces in $\mathbb{R}^n$. Math. Z. 283(1) (2016)

$n = 3$: **A. Alarcón, F.J. López: J. Diff. Geom. 90 (2012)**

Open Problem: Does every Riemann surface admit a proper conformal minimal embedding in $\mathbb{R}^4$? Does it admit a proper holomorphic embedding in $\mathbb{C}^2$?

Complete minimal surfaces with Jordan boundaries

Theorem (2)

Assume that M is a compact bordered Riemann surface.
Every conformal minimal immersion $\mathbf{X}_0 \colon M \to \mathbb{R}^n$ $(n \geq 3)$ can be approximated, uniformly on M, by continuous maps $\mathbf{X} \colon M \to \mathbb{R}^n$ such that $\mathbf{X} \colon \mathring{M} \to \mathbb{R}^n$ is a **complete conformal minimal immersion** *and $\mathbf{X} \colon bM \to \mathbb{R}^n$ is a* **topological embedding**.

If $n \geq 5$ then $\mathbf{X} \colon M \to \mathbb{R}^n$ can be chosen a topological embedding.

A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Proc. London Math. Soc. (3) 111 (2015)

This result answers a long standing problem. Previous constructions are due to **Nadirashvili (1996)** (for the disk) and several other authors. However, earlier attempts to obtain **complete conformal minimal surfaces with Jordan boundaries** were inconclusive.

The main new ingredient used in our construction is a suitable version of the Riemann-Hilbert boundary value problem.

Proper minimal surfaces in (minimally) convex domains

Theorem (3)

Let M be a compact bordered Riemann surface and D be a convex domain in $\mathbb{R}^n$ for some $n \geq 3$. Then, every conformal minimal immersion $\mathbf{X}_0 \colon M \to D$ can be approximated, uniformly on compacts in $\mathring{M}$, by proper (and complete) conformal minimal immersions $\mathbf{X} \colon \mathring{M} \to D$.

If D is bounded with smooth strongly convex boundary, then $\mathbf{X}$ can be chosen continuous on M, hence mapping bM to bD.

The same result holds if D is **minimally convex***, i.e., if it admits a smooth exhaustion function $\rho \colon D \to \mathbb{R}$ such that for every point $\mathbf{x} \in D$, the sum of the smallest two eigenvalues of $\mathrm{Hess}_\rho(\mathbf{x})$ is positive.*

A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López:

Proc. London Math. Soc. (3) 111 (2015);

Minimal surfaces in minimally convex domains, arxiv:1510.04006

What about non-orientable minimal surfaces?

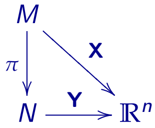
Assume that N is a non-orientable surface endowed with a conformal structure.

There is a 2-sheeted covering $\pi\colon M \to N$ by a Riemann surface M and a fixed-point-free antiholomorphic involution $\mathfrak{I}\colon M \to M$ (the deck transformation of π) such that $N = M/\mathfrak{I}$.

Every conformal minimal immersion $\mathbf{Y}\colon N \to \mathbb{R}^n$ lifts to a $\mathfrak{I}$-invariant conformal minimal immersion $\mathbf{X}\colon M \to \mathbb{R}^n$, i.e.,

$$\mathbf{X} = \mathbf{Y} \circ \pi \quad \text{and} \quad \mathbf{X} \circ \mathfrak{I} = \mathbf{X}.$$

Conversely, a $\mathfrak{I}$-invariant conformal minimal immersion $\mathbf{X}\colon M \to \mathbb{R}^n$ descends to a conformal minimal immersion $\mathbf{Y}\colon N \to \mathbb{R}^n$.



Main theorem for non-orientable minimal surfaces

Theorem (A. Alarcón, F. Forstnerič, F.J. López, 2016)

Let M be an open Riemann surface (or a bordered Riemann surface) with a fixed-point-free antiholomorphic involution $\mathfrak{I}$.

Then, Theorems 1–3 mentioned above hold also for $\mathfrak{I}$-invariant conformal minimal immersions $M \to \mathbb{R}^n$.

Hence, all mentioned results also hold for conformal minimal immersions $N \to \mathbb{R}^n$ from any non-orientable surface N endowed with a conformal structure, without having to change the conformal structure.

Non-orientable surfaces lie at the very origin of minimal surface theory. For instance, one can easily find a Möbius strip as solution to a Plateau problem; that is, non-orientable minimal surfaces do appear in nature.

Example: A properly embedded Möbius strip in $\mathbb{R}^4$

Let $\mathfrak{I}\colon \mathbb{C}_* \to \mathbb{C}_*$ be the fixed-point-free antiholomorphic involution on the punctured plane $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ given by

$$\mathfrak{I}(\zeta) = -\frac{1}{\bar{\zeta}}, \quad \zeta \in \mathbb{C}_*.$$

The harmonic map $\mathbf{X}\colon \mathbb{C}_* \to \mathbb{R}^4$ given by

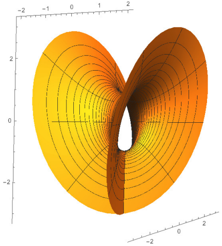
$$\mathbf{X}(\zeta) = \Re\left(\mathrm{i}\big(\zeta + \frac{1}{\zeta}\big), \zeta - \frac{1}{\zeta}, \frac{\mathrm{i}}{2}\big(\zeta^2 - \frac{1}{\zeta^2}\big), \frac{1}{2}\big(\zeta^2 + \frac{1}{\zeta^2}\big)\right)$$

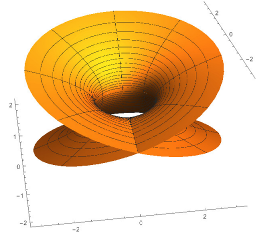
is an $\mathfrak{I}$-invariant proper conformal minimal immersion such that $\mathbf{X}(\zeta_1) = \mathbf{X}(\zeta_2)$ if and only if $\zeta_1 = \zeta_2$ or $\zeta_1 = \mathfrak{I}(\zeta_2)$.

Hence, $\mathbf{X}(\mathbb{C}_*) \subset \mathbb{R}^4$ is a properly embedded minimal Möbius strip in $\mathbb{R}^4$.

This seems to be the first known example of a properly embedded non-orientable minimal surface in $\mathbb{R}^4$. There is a well known example of a properly immersed minimal Möbius strip in $\mathbb{R}^3$ **(Meeks, 1981)**.

Example: A properly embedded Möbius strip in $\mathbb{R}^4$





Two views of the projection into $\mathbb{R}^3 = \{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4$ of the properly embedded minimal Möbius strip in $\mathbb{R}^4$ in the above example.

Topological structure of non-orientable surfaces

Every closed non-orientable surface N is the connected sum

$$N = \overbrace{\mathbb{P}^2 \sharp \cdots \sharp \mathbb{P}^2}^{g}$$

of $g \geq 1$ copies of the real projective plane $\mathbb{P}^2$. The number g is called the genus of N and equals the maximal number of pairwise disjoint closed curves in N which reverse the orientation.

Furthermore, $\mathbb{K} = \mathbb{P}^2 \sharp \mathbb{P}^2$ is the Klein bottle, and for any non-orientable surface N we have $N \sharp \mathbb{K} = N \sharp \mathbb{T}$ where $\mathbb{T}$ is the torus. This gives the following dichotomy according to whether the genus g is even or odd:

(I) $g = 1 + 2k \geq 1$ is odd. In this case, $N = \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$, and $k = 0$ corresponds to the projective plane $\mathbb{P}^2$.

(II) $g = 2 + 2k \geq 2$ is even. In this case, $N = \mathbb{P}^2 \sharp \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$.

Geometric model of a 2-sheeted oriented covering

Let $\iota\colon M \to N$ be a 2-sheeted covering by a compact orientable surface with involution $(M, \mathfrak{I})$. Then M has genus $g-1$, and hence it is a connected sum of $g-1$ copies of the torus $\mathbb{T}$.

We construct an explicit geometric model for $(M, \mathfrak{I})$ in $\mathbb{R}^3$.

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the involution $\tau(\mathbf{x}) = -\mathbf{x}$.

Case (I): $N = \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$. We take M to be an embedded surface

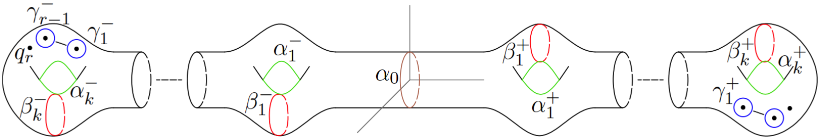
$$(\mathbb{T}_1^- \sharp \cdots \sharp \mathbb{T}_k^-) \sharp \mathbb{S}^2 \sharp (\mathbb{T}_1^+ \sharp \cdots \sharp \mathbb{T}_k^+)$$

of genus $g-1 = 2k$ in $\mathbb{R}^3$ which is invariant by the symmetry with respect to the origin (i.e., $\tau(M) = M$), where $\mathbb{T}_j^-$, $\mathbb{T}_j^+$ are embedded tori in $\mathbb{R}^3$ with $\tau(\mathbb{T}_j^-) = \mathbb{T}_j^+$ for all $j = 1, \ldots, k$. Set $\mathfrak{I} = \tau|_M\colon M \to M$.

Case I – illustration

If $k = 0$, the model is the round sphere $\mathbb{S}^2$ with the orientation reversing antipodal map $\mathfrak{I}$. Identifying $\mathbb{S}^2 \cong \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$ by the stereographic projection, we have $\mathfrak{I}(z) = -1/\bar{z}$.

If $k > 0$, we have $M = M^- \cup C \cup M^+$, where C is a $\mathfrak{I}$-invariant cylinder and M^-, M^+ are the closure of the two components of $M \setminus C$, both homeomorphic to the connected sum of k tori minus an open disk. Obviously $\mathfrak{I}(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$.



Geometric model, Case II

Case (II): $N = \mathbb{P}^2 \sharp \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k} = \mathbb{K} \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$.

Let $\mathbb{T}_0 \subset \mathbb{R}^3$ be the standard torus of revolution centered at the origin and invariant under the antipodal map $\tau(x) = -x$. In this case we let $M \subset \mathbb{R}^3$ be an embedded τ-invariant surface

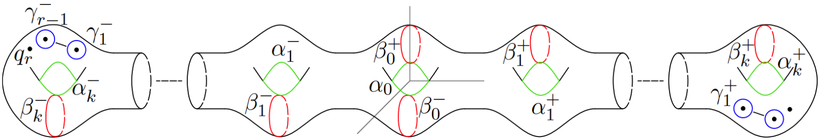
$$\left(\mathbb{T}_1^- \sharp \cdots \sharp \mathbb{T}_k^-\right) \sharp \mathbb{T}_0 \sharp \left(\mathbb{T}_1^+ \sharp \cdots \sharp \mathbb{T}_k^+\right),$$

where the tori $\mathbb{T}_j^\pm$ are as above, and set $\mathfrak{I} = \tau|_M$.

If $k = 0$, the model is the torus $\mathbb{T}_0$ with the involution $\mathfrak{I} = \tau|_{\mathbb{T}_0}$.

Case II – illustration

If $k > 0$, we have $M = M^- \cup K \cup M^+$, where $K \subset \mathbb{T}_0 \subset \mathbb{R}^3$ is a $\mathfrak{I}$-invariant torus minus two disjoint open disks, and M^- and M^+ are the closure of the two components of $M \setminus K$, both homeomorphic to the connected sum of k tori minus an open disk. Obviously, $\mathfrak{I}(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$.



Conclusion: In either case, there is a **homology basis** $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ for $H_1(M;\mathbb{Z})$ satisfying the following symmetry conditions:

$$\mathcal{B}^+ = \{\delta_1, \ldots, \delta_\ell\}, \quad \mathcal{B}^- = \{\mathfrak{I}(\delta_2), \ldots, \mathfrak{I}(\delta_\ell)\}, \quad \mathfrak{I}_*\delta_1 = \delta_1.$$

Analytic tools: $\mathfrak{I}$-invariant functions, 1-forms, and sprays

Definition

Let $(M, \mathfrak{I})$ be a Riemann surface with a fixed-point-free antiholomorphic involution. A holomorphic function $f \in \mathscr{O}(M)$ is $\mathfrak{I}$-invariant if

$$f \circ \mathfrak{I} = \bar{f}.$$

A holomorphic 1-form ϕ on M is $\mathfrak{I}$-invariant if

$$\mathfrak{I}^* \phi = \bar{\phi}.$$

Notation: $\mathscr{O}_{\mathfrak{I}}(M)$, $\Omega_{\mathfrak{I}}(M)$. Note that these are real algebras.
Clearly, a function $f = u + \imath v \colon M \to \mathbb{C}$ belongs to $\mathscr{O}_{\mathfrak{I}}(M)$ iff $u, v \colon M \to \mathbb{R}$ are conjugate harmonic functions satisfying

$$u \circ \mathfrak{I} = u, \quad v \circ \mathfrak{I} = -v.$$

For every $f \in \mathscr{O}(M)$ we have that $\overline{f \circ \mathfrak{I}} \in \mathscr{O}(M)$ and

$$f + \overline{f \circ \mathfrak{I}} \in \mathscr{O}_{\mathfrak{I}}(M), \quad f \cdot \overline{f \circ \mathfrak{I}} \in \mathscr{O}_{\mathfrak{I}}(M).$$

$\mathfrak{I}$-invariant sprays

Definition

Let $\mathbb{B}^N \subset \mathbb{C}^N$ be the unit ball for some $N \in \mathbb{N}$ and let $r > 0$. A holomorphic spray of maps $F\colon M \times r\mathbb{B}^N \to \mathbb{C}^n$ is $\mathfrak{I}$-invariant if

$$F(\mathfrak{I}p, \bar{z}) = \overline{F(p, z)}, \qquad p \in M,\ z \in r\mathbb{B}^N.$$

Note that $F(\cdot\,, z)\colon M \to \mathbb{C}^n$ is $\mathfrak{I}$-invariant if $z \in \mathbb{R}^N \subset \mathbb{C}^N$.

Example (Invariant sprays given by flows of vector fields)

Let $V_1, \ldots, V_N$ be holomorphic vector fields on $\mathbb{C}^n$ which are real on $\mathbb{R}^n$ (i.e., with real coefficients), and let ϕ_t^j denote the flow of V_j. Given a $\mathfrak{I}$-invariant holomorphic map $\mathbf{X}\colon M \to \mathbb{C}^n$, the map

$$F(p, t_1, \ldots, t_N) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_N}^N (\mathbf{X}(p))$$

is a $\mathfrak{I}$-invariant holomorphic spray of maps $M \to \mathbb{C}^n$.

$\mathfrak{I}$-invariant homology basis and period map

Lemma (0)

Let $(M, \mathfrak{I})$ be a bordered Riemann surface with a fixed-point-free involution $\mathfrak{I}\colon M \to M$. Then there exists a **Runge homology basis** *$\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ for $H_1(M;\mathbb{Z})$ satisfying*

$$\mathcal{B}^+ = \{\delta_1, \ldots, \delta_\ell\}, \quad \mathcal{B}^- = \{\mathfrak{I}(\delta_2), \ldots, \mathfrak{I}(\delta_\ell)\}, \quad \mathfrak{I}_*\delta_1 = \delta_1.$$

Denote by E the union of supports of the curves in $\mathcal{B}$. The Runge property means that $M \setminus E$ has no relatively compact connected components; this guarantees Mergelyan approximation on E.

Let $\mathcal{P}^+ = (\mathcal{P}^+_1, \ldots, \mathcal{P}^+_\ell)\colon \mathscr{O}(M) \to \mathbb{C}^\ell$ denote the **period map** given by

$$\mathcal{P}^+_j(f) = \int_{\delta_j} f\theta, \qquad f \in \mathscr{O}(M),\ j = 1, \ldots, \ell.$$

Similarly, we define $\mathcal{P}^+(\phi) = \left(\int_{\delta_j} \phi\right)_{j=1,\ldots,\ell}$ for a holomorphic 1-form ϕ.

Exactness of $\mathfrak{I}$-invariant 1-forms

Lemma (1)

Let ϕ be a $\mathfrak{I}$-invariant holomorphic 1-form on M. Then:

(a) *ϕ is exact if and only if $\mathcal{P}^+(\phi) = 0$.*

(b) *$\Re\phi$ is exact if and only if $\Re\mathcal{P}^+(\phi) = 0$.*

Proof. (a) By $\mathfrak{I}$-invariance of ϕ we have

$$\int_{\mathfrak{I}_*\delta_j} \phi = \int_{\delta_j} \mathfrak{I}^*\phi = \int_{\delta_j} \overline{\phi}, \qquad j = 1, \ldots, \ell.$$

Therefore, $\mathcal{P}^+(\phi) = 0$ implies that ϕ has vanishing periods over all curves in $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ and hence is exact. The converse is obvious.

(b) Likewise, $\mathcal{P}^+(\Re\phi) = \Re\mathcal{P}^+(\phi) = 0$ implies that $\Re\phi$ is exact. The imaginary periods (the flux) of ϕ may be arbitrary, subject to the conditions

$$\int_{\mathfrak{I}_*\delta_j} \Im\phi = -\int_{\delta_j} \Im\phi, \qquad j = 1, \ldots, \ell.$$

In particular, we have $\int_{\delta_1} \Im\phi = 0$ since $\mathfrak{I}_*\delta_1 = \delta_1$.

$\mathfrak{I}$-invariant period dominating sprays

Lemma (2)

Let $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ be a basis of $H_1(M;\mathbb{Z})$ furnished by Lemma 1, and let $\mathcal{P}^+ \colon \mathscr{A}(M,\mathbb{C}^n) \to (\mathbb{C}^n)^\ell$ denote the associated period map:

$$\mathcal{P}^+(f) = \left(\int_{\gamma_i} f\theta\right)_{i=1,\dots,\ell} \in (\mathbb{C}^n)^\ell.$$

For every nonflat, $\mathfrak{I}$-invariant map $f\colon M \to \mathcal{A}_*$ of class $\mathscr{A}(M) = \mathscr{C}(M) \cap \mathscr{O}(\mathring{M})$ there exists a dominating $\mathfrak{I}$-invariant spray

$$F\colon M \times r\mathbb{B}^N \to \mathcal{A}_*$$

of class $\mathscr{A}(M)$ which is **period dominating***, in the sense that the differential*

$$\frac{\partial}{\partial\zeta}\bigg|_{\zeta=0} \mathcal{P}^+(F(\cdot\,,\zeta))\colon \mathbb{C}^N \to (\mathbb{C}^n)^\ell$$

maps $\mathbb{R}^N$ (the real part of $\mathbb{C}^N$) surjectively onto $\mathbb{R}^n \times (\mathbb{C}^n)^{\ell-1}$.

(Very) special Cartan pairs

Definition

Let M be an open Riemann surface with a fixed-point-free antiholomorphic involution $\mathfrak{I} \colon M \to M$.

A pair (A, B) of compact sets in M is a **$\mathfrak{I}$-invariant Cartan pair** if

(a) the sets $A, B, A \cap B$ and $A \cup B$ are $\mathfrak{I}$-invariant with $\mathscr{C}^1$ boundaries;

(b) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ (the separation property).

A $\mathfrak{I}$-invariant Cartan pair (A, B) is **special** if $B = B' \cup \mathfrak{I}(B')$, where B' is a compact set with $\mathscr{C}^1$ boundary in M and $B' \cap \mathfrak{I}(B') = \emptyset$.

A special Cartan pair (A, B) is **very special** if the sets B' and $A \cap B'$ are disks (hence, $\mathfrak{I}(B')$ and $A \cap \mathfrak{I}(B')$ are also disks).

Gluing pairs of $\mathfrak{I}$-invariant sprays

Lemma (3)

Let $(M, \mathfrak{I})$ be an open Riemann surface with a fixed-point-free antiholomorphic involution. Assume that

- *(A, B) is a special $\mathfrak{I}$-invariant Cartan pair in M,*
- *$\epsilon > 0$ and $r > 0$ are real number, and*
- *$F\colon A \times r\mathbb{B}^N \to \mathcal{A}_*$ is a $\mathfrak{I}$-invariant spray of class $\mathscr{A}(A)$ which is dominating over the set $C = A \cap B$.*

Then, there exist numbers $\delta > 0$ and $r' \in (0, r)$ such that for every $\mathfrak{I}$-invariant spray $G\colon B \times r\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(B)$ satisfying

$$||F - G||_{0, C \times r\mathbb{B}^N} < \delta$$

there is a $\mathfrak{I}$-invariant spray $H\colon (A \cup B) \times r'\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(A \cup B)$ satisfying

$$||H - F||_{0, A \times r'\mathbb{B}^N} < \epsilon.$$

Main approximation lemma for $\mathfrak{I}$-invariant maps

Lemma (4)

Let $(M, \mathfrak{I})$ be as above, and let (A, B) be a **very special $\mathfrak{I}$-invariant Cartan pair** *in M. Let $\mathcal{P}^+$ denote the period map on A (cf. Lemma 2).*

Then, every $\mathfrak{I}$-invariant map $f\colon A \to \mathcal{A}_*$ of class $\mathscr{A}(A)$ can be approximated, uniformly on A, by $\mathfrak{I}$-invariant holomorphic maps $\tilde{f}\colon A \cup B \to \mathcal{A}_*$ satisfying $\mathcal{P}^+(\tilde{f}) = \mathcal{P}^+(f)$.

Remark: Lemma 4 gives the corresponding approximation theorem for $\mathfrak{I}$-invariant conformal minimal immersions. Indeed, if $\mathcal{P}^+(f) = 0$ then

$$\widetilde{\mathbf{X}} = 2\int \Re(\tilde{f}\theta)\colon A \cup B \to \mathbb{R}^n$$

is an $\mathfrak{I}$-invariant conformal minimal immersion approximating the immersion $\mathbf{X} = 2\int \Re(f\theta)$ on A.

Proof. By Lemma 2, there exists a $\mathfrak{I}$-invariant dominating and period dominating spray $F\colon A \times r\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(A)$ with $F(\cdot\,, 0) = f$.

Proof of Lemma 4

By the definition of a very special Cartan pair, $B = B' \cup \mathfrak{I}(B')$ is the union of two disjoint disks, and $C' = A \cap B' \subset B'$ is a disk.

Pick a number $r' \in (0, r)$. Since $\mathcal{A}_*$ is complex homogeneous, and hence an Oka manifold, it is possible to approximate F, uniformly on $C' \times r'\mathbb{B}^N$, by a holomorphic spray $G\colon B' \times r'\mathbb{B}^N \to \mathcal{A}_*$.

We extend the spray G to $\mathfrak{I}(B') \times r'\mathbb{B}^N$ by symmetrization:

$$G(p, \zeta) = G(\mathfrak{I}(p), \bar{\zeta}) \quad \text{for } p \in \mathfrak{I}(B') \text{ and } \zeta \in r'\mathbb{B}^N.$$

It follows that G is an $\mathfrak{I}$-invariant spray on $B \times r'\mathbb{B}^N$ which approximates F on $(A \cap B) \times r'\mathbb{B}^N$. Lemma 3 furnishes an $\mathfrak{I}$-invariant spray

$$H\colon (A \cup B) \times r''\mathbb{B}^N \to \mathcal{A}_*$$

for some $r'' \in (0, r')$ which approximates F on $A \times r''\mathbb{B}^N$. By the period domination of F, there exists $\zeta_0 \in r''\mathbb{B}^N \cap \mathbb{R}^N$ such that the $\mathfrak{I}$-invariant map $\tilde{f} = H(\cdot\,, \zeta_0)\colon A \cup B \to \mathcal{A}_*$ satisfies $\mathcal{P}^+(f) = \mathcal{P}^+(\tilde{f})$. $\square$

Change of topology of the domain

The following procedure is employed to handle the change of topology. Let $A \subset M$ be an $\mathfrak{I}$-invariant domain and $\mathbf{X}\colon A \to \mathbb{R}^n$ be a $\mathfrak{I}$-invariant conformal minimal immersion. Attach to A a couple of arcs $E = E_1 \cup E_2$, with $\mathfrak{I}(E_1) = E_2$ and $E_1 \cap E_2 = \varnothing$, and proceed as follows.

1. Extend the derivative $2\partial\mathbf{X}/\theta\colon A \to \mathcal{A}_*$ to a map $f\colon A \cup E \to \mathcal{A}_*$ satisfying $f \circ \mathfrak{I} = \overline{f}$ and $\int_{E_1} \Re(f\theta) = \mathbf{X}(q) - \mathbf{X}(p)$, where $\partial E_1 = \{p, q\}$. By Lemma 2, there is a period-dominating $\mathfrak{I}$-invariant spray $F\colon (A \cup E) \times r\mathbb{B}^N \to \mathcal{A}_*$ with $F(\cdot\,, 0) = f$.

2. Choose a small tubular neighborhood V_1 of the arc E_1. Approximate F over $(A \cup E) \cap V_1$ by a spray G defined over V_1. Extend G to $V_2 := \mathfrak{I}(V_1) \supset E_2$ by setting $G(p, \zeta) = G(\mathfrak{I}(p), \bar{\zeta})$. By Lemma 3 we can glue F and G into an $\mathfrak{I}$-invariant spray $\widetilde{F}\colon D \times r'\mathbb{B}^N \to \mathcal{A}_*$ over an $\mathfrak{I}$-invariant domain $D \supset A \cup E$ for some $r' \in (0, r)$.

3. The period domination property of F furnishes a parameter value $\zeta_0 \in r'\mathbb{B}^N \cap \mathbb{R}^N$ such that the map $\widetilde{F}(\cdot\,, \zeta_0)\colon D \to \mathcal{A}_*$ integrates to an $\mathfrak{I}$-invariant conformal minimal immersion $\widetilde{\mathbf{X}}\colon D \to \mathbb{R}^n$.

Conclusion

The **Runge-Mergelyan approximation theorem** (Theorem 1) is proved recursively, using an $\mathfrak{I}$-invariant strongly subharmonic exhaustion function $\rho\colon M \to \mathbb{R}$. The noncritical case is handled by Lemma 4; this amounts to attaching bumps. The critical points of ρ (where the topology of the sublevel set $\{\rho \leq c\}$ changes) are handled as explained above.

The **general position theorem** (also included in Theorem 1) is obtained by combining these methods with transversality arguments.

To obtain **complete conformal minimal surfaces with Jordan boundaries** in $\mathbb{R}^n$ (cf. Theorem 2) and **proper conformal minimal surfaces** in (minimally) convex domains (cf. Theorem 3), we also use approximate solutions to the **Riemann-Hilbert boundary value problem** for conformal minimal surfaces and holomorphic null curves.