# Runge tubes

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# Runge cylinders in $\mathbb{C}^2$

It was an open question for a long time whether it is possible to embed $\mathbb{C}^* \times \mathbb{C}$ as a **Runge domain** $\Omega \subset \mathbb{C}^2$, i.e., such that holomorphic polynomials are dense in $\mathcal{O}(\Omega)$. (Here, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

Such hypothetical domains have been called **Runge cylinders** in $\mathbb{C}^2$.

The question arose in connection with the classification of Fatou components for Hénon maps by **E. Bedford and J. Smillie (1991)**.

This problem has recently been solved in the affirmative:

**Theorem (F. Bracci, J. Raissy, and B. Stensønes, 2017)**

*For every $n \geq 2$ there exists a (non-polynomial) holomorphic automorphism of $\mathbb{C}^n$ with a parabolic fixed point at 0 whose basin of attraction is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$.*

Note that the basin is always a Runge domain. The basin of an *attracting* fixed point is biholomorphic to $\mathbb{C}^n$.

# Existence and plenitude of Runge tubes

In this joint work with **Erlend Fornæss Wold (University of Oslo)** we give a simple proof of the following related result.

**Theorem (1; Proc. Amer. Math. Soc. 2018)**

*Let $X$ and $Y$ be Stein manifolds with $\dim X < \dim Y$, and assume that $Y$ has the density property (in particular, we may take $Y = \mathbb{C}^n$, $n > 1$).*

*Suppose that $\theta : X \hookrightarrow Y$ is a holomorphic embedding with $\mathcal{O}(Y)$-convex image (this holds in particular if $\theta$ is a proper holomorphic embedding), and let $E \to X$ denote the normal bundle associated to $\theta$.*

*Then, $\theta$ is approximable uniformly on compacts in $X$ by holomorphic embeddings $\tilde{\theta} : E \hookrightarrow Y$ whose images are Runge domains in $Y$.*

Recall that a locally closed subset $Z$ of a complex manifold $Y$ is said to be *$\mathcal{O}(Y)$-convex* if for every compact set $K \subset Z$ its $\mathcal{O}(Y)$-convex hull

$$\widehat{K}_{\mathcal{O}(Y)} = \{y \in Y : |f(y)| \le \sup_K |f| \quad \forall f \in \mathcal{O}(Y)\}$$

is compact and contained in $Z$.

# Runge tubes over open Riemann surfaces

It is known that every open Riemann surface, $X$, embeds properly holomorphically into $\mathbb{C}^3$, and some (all?) embed into $\mathbb{C}^2$.

Since every holomorphic vector bundle over an open Riemann surface is trivial by **Oka's theorem (1939)**, we get the following corollary.

**Corollary (Runge tubes over open Riemann surfaces)**

*If $X$ is an open Riemann surface which admits a proper holomorphic embedding into $\mathbb{C}^2$, then $X \times \mathbb{C}$ admits a Runge embedding into $\mathbb{C}^2$.*

*For every open Riemann surface $X$ and integer $k \geq 2$, $X \times \mathbb{C}^k$ admits a Runge embedding into $\mathbb{C}^{k+1}$, and more generally into any Stein manifold $Y^{k+1}$ with the density property.*

In particular, to get a Runge embedding $\mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2$, we embed $X = \mathbb{C}^*$ onto the algebraic curve $\{zw = 1\} \subset \mathbb{C}^2$.

# An example

The following example shows that it is in general impossible to **extend** a proper holomorphic embedding $\theta : X \hookrightarrow Y$ to a holomorphic embedding $E \hookrightarrow Y$ of the normal bundle $E$ of $\theta$, even a non-Runge one.

## Example

For every pair of integers $1 \leq k < n$ there exists a proper holomorphic embedding $\theta : X = \mathbb{C}^k \hookrightarrow Y = \mathbb{C}^n$ such that every holomorphic map $\mathbb{C}^{n-k} \to \mathbb{C}^n \setminus \theta(\mathbb{C}^k)$ is degenerate (of rank $< n - k$ at each point).

**Buzzard and Fornæss 1996 for $k = 1, n = 2$; Forstnerič 1999; Borell and Kutzschebauch 2006.**

Since the normal bundle of $\theta$ is the trivial bundle $E = \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^k$ which admits nondegenerate holomorphic maps

$$\mathbb{C}^{n-k} \to \mathbb{C}^n \setminus (\mathbb{C}^k \times \{0\}^{n-k}) = \mathbb{C}^k \times (\mathbb{C}^{n-k} \setminus \{0\}),$$

$\theta$ does not extend to a holomorphic embedding $E = \mathbb{C}^n \hookrightarrow \mathbb{C}^n$.

# Runge tubes around algebraic submanifolds of $\mathbb{C}^n$

In spite of the above example, we have the following **extendibility result for affine algebraic submanifolds** of codimension $\geq 2$ in $\mathbb{C}^n$.

**Theorem (2)**

*Let $\theta\colon X \hookrightarrow \mathbb{C}^n$ be an affine algebraic submanifold.*
*If $n \geq \dim X + 2$, then $\theta$ extends to a holomorphic Runge embedding $\tilde\theta\colon E \hookrightarrow Y$ of the normal bundle $E$ of $\theta$.*

*In particular, if $A \subset \mathbb{C}^{n+1}$ $(n \geq 2)$ is a smooth affine algebraic curve then there is a holomorphic Runge embedding $A \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ extending the inclusion map $A = A \times \{0\}^n \hookrightarrow \mathbb{C}^{n+1}$.*

It is easily seen that the embedding $\tilde\theta\colon E \hookrightarrow Y$ cannot be algebraic.

# The density property

**Varolin 2000** A complex manifold $Y$ enjoys the **density property (DP)** if every holomorphic vector field on $Y$ can be approximated by Lie combinations of $\mathbb{C}$-complete holomorphic vector fields.

Similarly, a Lie algebra $\mathfrak{g}$ of holomorphic vector fields on $Y$ enjoys DP if it is densely generated by the complete vector fields that it contains. If $Y$ carries a holomorphic volume form $\omega$, then the density property for the Lie algebra of all holomorphic vector fields with vanishing $\omega$-divergence is called the **volume density property (VDP)** of $(Y, \omega)$.

**Andersén 1990; Andersén & Lempert 1992** $\mathbb{C}^n$ enjoys DP for $n > 1$, and VDP for the volume form $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ for $n \geq 1$.

In fact, **every polynomial holomorphic vector field on $\mathbb{C}^n$ is a finite sum of shear vector fields** given in suitable coordinates $z = (z', z_n)$ by

$$V(z) = f(z') \frac{\partial}{\partial z_n}, \qquad W(z) = f(z') z_n \frac{\partial}{\partial z_n},$$

where $f \in \mathbb{C}[z_1, \ldots, z_{n-1}]$.

# Approximating biholomorphisms by automorphisms

**Theorem (Andersén-Lempert, Forstnerič-Rosay, Varolin)**

*Let $Y$ be a Stein manifold with DP. Assume that*

$$F_t \colon \Omega_0 \xrightarrow{\cong} \Omega_t \subset Y, \qquad t \in [0,1],$$

*is an isotopy of biholomorphic maps between Stein Runge domains in $Y$, with $F_0 = \mathrm{Id}|_{\Omega_0}$. Then, $F_1 \colon \Omega_0 \to \Omega_1$ is a limit of holomorphic automorphisms of $Y$, uniformly on compacts in $\Omega_0$.*

*The analogous result holds for isotopies of biholomorphic maps preserving a holomorphic volume form on a Stein manifold with VDP.*

This also applies to isotopies of holomorphic maps $F_t \colon \Omega_0 \to \Omega_t$, defined in a neighborhood of a compact set $K_0 \subset Y$, provided that

$$\text{the set } K_t := F_t(K_0) \text{ is } \mathcal{O}(Y)\text{-convex for every } t \in [0,1].$$

Then, $F_1$ can be approximated uniformly on $K_0$ by automorphisms of $Y$.

# Examples of Stein manifolds with DP

- $\mathbb{C}^n$ for $n \geq 1$ satisfies VDP for $dz_1 \wedge \cdots \wedge dz_n$ (**Andersén**).
- $\mathbb{C}^n$ for any $n > 1$ satisfies DP (**Andersén and Lempert**).
- $(\mathbb{C}^*)^n$ with the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ satisfies VDP (**Varolin**). It is not known whether DP holds when $n > 1$.
- If $G$ is a linear algebraic group and $H \subset G$ is a closed proper reductive subgroup, then $Y = G/H$ is a Stein manifold with DP, except when $Y = \mathbb{C}, (\mathbb{C}^*)^n$, or a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$ (**Kaliman, Donzelli & Dvorsky**).
- In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has DP (**Kaliman & Kutzschebauch**).
- If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fibre, then $Y = \{xy = p(z)\}$ has DP (**K&K**). The same is true if the source $\mathbb{C}^n$ of $p$ is an arbitrary Stein manifold with DP.
- A Cartesian product $Y_1 \times Y_2$ of two Stein manifolds $Y_1, Y_2$ with DP also has DP. The analogous result holds for VDP (**K&K**).

# Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2018** A smooth affine algebraic surface $Y$ is a **Gizatullin surface** if $\mathrm{Aut}_{\mathrm{alg}}(Y)$ acts transitively on $Y$ up to finitely many points. Every such surface admits a fibration $\pi\colon Y \to \mathbb{C}$ whose generic fiber equals $\mathbb{C}$ and there is only one exceptional fiber. **If this exceptional fiber is reduced, then $Y$ has DP.**
- **Leuenberger 2016** DP holds for a family of hypersurfaces

$$Y = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2 y = a(z) + xb(z)\},$$

where $x, y \in \mathbb{C}$ and $a, b \in \mathbb{C}[z]$ are polynomials in $z \in \mathbb{C}^{n+1}$. This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to $\mathbb{R}^6$, but is not algebraically isomorphic to $\mathbb{C}^3$ (**Makar-Limanov, Dubouloz**).

**It remains an open question whether $C$ is biholomorphic to $\mathbb{C}^3$.**

# Embeddings into Stein manifolds with DP or VDP

Stein manifolds with the (volume) density property are universal embedding spaces for all Stein manifolds.

**Theorem (Andrist, F., Ritter, Wold 2016; F. 2017)**

*Assume that $X$ is a Stein manifold, and $Y$ is a Stein manifold with the density property or the volume density property.*

(a) *If* $\dim Y > 2 \dim X$, *then any continuous map* $X \to Y$ *is homotopic to a proper holomorphic embedding* $X \hookrightarrow Y$.

(b) *If* $\dim Y = 2 \dim X$, *then any continuous map* $X \to Y$ *is homotopic to a proper holomorphic immersion with simple double points.*

**Corollary**

*A Stein manifold $Y$ with DP contains a Runge tube $E \hookrightarrow Y$ whose base is an arbitrary Stein manifold $X$ with* $2 \dim X < \dim Y$.

# Proof of Theorem 1: preliminaries

A **domain** $\Omega$ in a complex manifold $Y$ is said to be **Runge** in $Y$ if $\{f|_\Omega : f \in \mathcal{O}(Y)\}$ is a dense subset of $\mathcal{O}(\Omega)$. If both $\Omega$ and $Y$ are Stein, this holds if and only if for every compact subset $K \subset \Omega$ we have that $\widehat{K}_{\mathcal{O}(\Omega)} = \widehat{K}_{\mathcal{O}(Y)}$. In particular, a domain in a Stein manifold $Y$ which is exhausted by compact $\mathcal{O}(Y)$-convex sets is Runge in $Y$.

A **holomorphic embedding** $\theta\colon X \hookrightarrow Y$ is said to be **Runge** if the image $Z = \theta(X) \subset Y$ is exhausted by compact $\mathcal{O}(Y)$-convex subsets. If $X$ and $Y$ are Stein, then every proper holomorphic embedding $X \hookrightarrow Y$ is Runge.

Assume that $\pi\colon E \to X$ is a **holomorphic vector bundle** over a Stein manifold $X$. The total space $E$ is then also a Stein manifold. We write elements of $E$ as $e = (x, v)$, identifying $X$ with the zero section $\{(x, 0) : x \in X\}$ of $E$. For any $t \in \mathbb{C}^*$ consider a holomorphic automorphism $\psi_t \in \mathrm{Aut}(E)$, with $\psi_t|_X = \mathrm{Id}_X$, given by

$$\psi_t\colon E \to E, \qquad \psi_t(x, v) = (x, tv).$$

A subset $Z \subset E$ is called **radial** if $\psi_t(Z) \subset Z$ holds for every $t \in [0, 1]$.

# Proof of Theorem 1: The main lemma

## Lemma

*Assume that:*

- $X$ *is a Stein manifold,*
- $\pi\colon E \to X$ *is a holomorphic vector bundle,*
- $K \subset L$ *are compact radial* $\mathcal{O}(E)$*-convex subsets of* $E$*,*
- $\Omega \subset E$ *is an open set containing* $X \cup K$*,*
- $Y$ *is a Stein manifold with DP such that* $\dim Y = \dim E$*, and*
- $\theta : \Omega \hookrightarrow Y$ *is a holomorphic embedding such that* $\theta|_X : X \hookrightarrow Y$ *is a Runge embedding and* $\theta(K)$ *is* $\mathcal{O}(Y)$*-convex.*

*Then there is a domain* $\widetilde{\Omega} \subset E$ *containing* $X \cup L$ *such that* $\theta$ *can be approximated uniformly on* $K$ *by holomorphic embeddings*

$$\tilde{\theta} : \widetilde{\Omega} \longhookrightarrow Y$$

*such that* $\tilde{\theta}|_X : X \hookrightarrow Y$ *is a Runge embedding and* $\tilde{\theta}(L)$ *is* $\mathcal{O}(Y)$*-convex.*

# Proof of the lemma

- Recall that $\pi\colon E \to X$. Choose a compact $\mathcal{O}(X)$-convex subset $X_0 \subset X$ such that $\pi(L) \subset X_0$. Since the embedding $\theta|_X\colon X \hookrightarrow Y$ is Runge, the image $Y_0 = \theta(X_0) \subset \theta(X)$ is $\mathcal{O}(Y)$-convex.
- Pick a compact $\mathcal{O}(Y)$-convex neighborhood $N \subset \theta(\Omega)$ of $Y_0$. Thus, $N = \theta(N_0)$ for a compact set $N_0 \subset \Omega$ with $X_0 \subset \mathring{N}_0$.
- Recall that $\psi_t\colon E \to E$, $\psi_t(x, v) = (x, tv)$. Since $\pi(L) \subset X_0 \subset \mathring{N}_0$, we can choose $\epsilon > 0$ small enough such that

$$\psi_\epsilon(L) \subset N_0 \subset \Omega.$$

- Since $L$ is $\mathcal{O}(E)$-convex and $\psi_\epsilon \in \mathrm{Aut}(E)$, the set $\psi_\epsilon(L)$ is $\mathcal{O}(E)$-convex, and hence $\mathcal{O}(N_0)$-convex.
- Since $\theta\colon \Omega \to \theta(\Omega)$ is a biholomorphism, it follows that the set $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(N)$-convex, and hence also $\mathcal{O}(Y)$-convex.

# Proof of the lemma, 2

Consider the isotopy of injective holomorphic maps $\sigma_t$ for $t \in [\epsilon, 1]$, defined on an open neighborhood of $\theta(K)$ in $Y$ by the condition

$$\theta \circ \psi_t = \sigma_t \circ \theta, \qquad t \in [\epsilon, 1].$$

Note that the following hold:

(a) $\sigma_1 = \mathrm{Id}$ (since $\psi_1 = \mathrm{Id}$), and

(b) for every $t \in [\epsilon, 1]$ the compact set $\sigma_t(\theta(K)) \subset Y$ is $\mathcal{O}(Y)$-convex.

Condition (b) holds because $\psi_t(K) \subset K$ is clearly $\mathcal{O}(E)$-convex, so

$$\sigma_t(\theta(K)) = \theta(\psi_t(K)) \text{ is } \mathcal{O}(\theta(K))\text{-convex.}$$

Since $\theta(K)$ is $\mathcal{O}(Y)$-convex, it follows that

the set $\sigma_t(\theta(K))$ is $\mathcal{O}(Y)$-convex for every $t \in [\epsilon, 1]$.

# Proof of the lemma, 3

The AL-theorem applied to the isotopy $\sigma_t$ $(t \in [\epsilon, 1])$ shows that

$$\sigma_\epsilon \text{ can be approximated uniformly on } \theta(K) \text{ by } \phi \in \mathrm{Aut}(Y).$$

Since $\psi_\epsilon(L \cup X) = \psi_\epsilon(L) \cup X \subset \Omega$, there is an open set $\widetilde{\Omega} \subset E$ with

$$L \cup X \subset \widetilde{\Omega}, \qquad \psi_\epsilon(\widetilde{\Omega}) \subset \Omega.$$

We claim that the holomorphic embedding

$$\tilde{\theta} := \phi^{-1} \circ \theta \circ \psi_\epsilon : \widetilde{\Omega} \hookrightarrow Y$$

satisfies the conclusion of the lemma. Indeed:

- $\tilde{\theta}|_X = \phi^{-1} \circ \theta|_X : X \hookrightarrow Y$ is a Runge embedding since $\theta|_X$ is.
- Since $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(Y)$-convex and $\phi \in \mathrm{Aut}(Y)$, the set $\tilde{\theta}(L)$ is also $\mathcal{O}(Y)$-convex.
- On the set $K \subset E$ we have that

  $$\tilde{\theta} = \phi^{-1} \circ \theta \circ \psi_\epsilon = \phi^{-1} \circ \sigma_\epsilon \circ \theta.$$

  Since $\phi^{-1} \circ \sigma_\epsilon$ is close to the identity on $\theta(K)$, $\tilde{\theta}$ is close to $\theta$ on $K$.

# Proof of Theorem 1

Pick an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = E$ by compact radial $\mathcal{O}(E)$-convex sets.

Let $\theta\colon X \hookrightarrow Y$ be a holomorphic Runge embedding. By **Docquier and Grauert (1960)** there is a neighbourhood $\Omega_0 \subset E$ of the zero section $X \subset E$ such that $\theta$ extends to a holomorphic Runge embedding

$$\theta_0 : \Omega_0 \hookrightarrow Y.$$

Set $K_0 = \varnothing$. Applying the main lemma inductively, we find

open neighbourhoods $\Omega_j \subset E$ of $K_j \cup X$, and

holomorphic embeddings $\theta_j\colon \Omega_j \hookrightarrow Y$,

satisfying the following conditions for every $j \in \mathbb{N}$:

(a) the compact set $\theta_j(K_j)$ is $\mathcal{O}(Y)$-convex,
(b) the embedding $\theta_j|_X\colon X \hookrightarrow Y$ is Runge, and
(c) $\theta_j$ approximates $\theta_{j-1}$ as closely as desired on $K_{j-1}$.

# Proof of Theorem 1

If the approximations are close enough, the sequence $\theta_j$ converges uniformly on compacts in $E$ to a holomorphic embedding $\tilde{\theta}\colon E \hookrightarrow Y$.

Since $\mathcal{O}(Y)$-convexity of a compact set in a Stein manifold $Y$ is a stable property for compact strongly pseudoconvex domains and every compact $\mathcal{O}(Y)$-convex set can be approximated from the outside by such domains, it follows that the image of each $K_j$ remains $\mathcal{O}(Y)$-convex in the limit provided that all approximations were close enough.

Hence, $\tilde{\theta}(E)$ is a Runge domain in $Y$. This proves the theorem.

# Recall: Theorem 2

**Theorem (2)**

*Let* $\theta\colon X \hookrightarrow \mathbb{C}^n$ *be an affine algebraic submanifold.*

*If* $n \geq \dim X + 2$*, then* $\theta$ **extends** *to a holomorphic Runge embedding* $\tilde{\theta}\colon E \hookrightarrow Y$ *of the normal bundle* $E$ *of* $\theta$*.*

# Runge tubes around algebraic submanifolds of $\mathbb{C}^n$

The proof of Theorem 2 requires the following

**Addendum to the main lemma:**
If $Y = \mathbb{C}^n$ with $n \geq \dim X + 2$, $\theta\colon \Omega \hookrightarrow Y$ is a holomorphic embedding (where $\Omega \subset E$ is an open neighborhood of $K \cup X$), and $A = \theta(X) \subset \mathbb{C}^n$ is a closed algebraic submanifold of $\mathbb{C}^n$, then the holomorphic embedding $\tilde{\theta} : \widetilde{\Omega} \hookrightarrow \mathbb{C}^n$ can be chosen to agree with $\theta$ on $X$.

The proof of this addendum uses the following result.

**Theorem (Kaliman and Kutzschebauch, 2008)**

*If $A \subset \mathbb{C}^n$ is an algebraic submanifold with $n \geq \dim A + 2$, then every polynomial vector field on $\mathbb{C}^n$ that vanishes on $A$ is a Lie combination of complete polynomial vector fields vanishing on $A$.*

By using this result and Serre's Theorem A and B, we can approximate the biholomorphism $\sigma_\epsilon$ (in the proof of Theorem 1) by an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ such that $\phi(z) = z$ for all $z \in A$. This ensures that the every embedding obtained in the inductive process agrees with $\theta$ on $X$.

# There are no algebraic Runge tubes

**Holomorphic Runge embeddings in Theorem 2 cannot be algebraic.**

Indeed, if $E \to X$ is the algebraic normal bundle of an algebraic submanifold $X \subset \mathbb{C}^n$ and $F\colon E \hookrightarrow \mathbb{C}^n$ is an algebraic embedding, then $\Omega = F(E) \subset \mathbb{C}^n$ is a Zariski open set in $\mathbb{C}^n$ and its complement $\Sigma = \mathbb{C}^n \setminus \Omega$ is a Zariski closed set, i.e., an algebraic subvariety of $\mathbb{C}^n$ (**Chevalley 1958**).

Since $\Omega$ is a Stein domain, $\Sigma$ must be of pure codimension one, so $\Sigma = \{f = 0\}$ for some entire function $f \in \mathcal{O}(\mathbb{C}^n)$. Clearly, the function $1/f \in \mathcal{O}(\Omega)$ cannot be approximated uniformly on compacts in $\Omega$ by entire functions, and hence the domain $\Omega = F(E)$ is not Runge in $\mathbb{C}^n$.

# A problem

**Problem**

*Is there a Runge embedding of the (trivial) normal bundle* $E = H \times \mathbb{C} \cong \mathbb{C}^* \times \mathbb{C}$ *of the hyperbola* $H = \{(z, w) \in \mathbb{C}^2 : zw = 1\}$ **extending** *the inclusion map* $H \hookrightarrow \mathbb{C}^2$*?*

The method of proof of Theorem 2 breaks down at the point where one would need to know that the Lie algebra of holomorphic vector fields vanishing on $H$ has the density property.

To decide about this is a notoriously hard problem well known and open since decades, as is the problem about the density property of $(\mathbb{C}^*)^n$ for $n > 1$.

THANK YOU

FOR YOUR ATTENTION