Runge tubes

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Runge cylinders in $\mathbb{C}^2$

It was an open question for a long time whether it is possible to embed $\mathbb{C}^* \times \mathbb{C}$ as a **Runge domain** $\Omega \subset \mathbb{C}^2$, i.e., such that holomorphic polynomials are dense in $\mathcal{O}(\Omega)$. (Here, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

Such hypothetical domains have been called **Runge cylinders** in $\mathbb{C}^2$.

The question arose in connection with the classification of Fatou components for Hénon maps by **E. Bedford and J. Smillie (1991)**.

This problem has recently been solved in the affirmative:

Theorem (F. Bracci, J. Raissy, and B. Stensønes, 2017)

For every $n \geq 2$ there exists a (non-polynomial) holomorphic automorphism of $\mathbb{C}^n$ with a parabolic fixed point at 0 whose basin of attraction is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$.

Note that the basin is always a Runge domain.

Existence and plenitude of Runge tubes

In this joint work with **Erlend Fornæss Wold (University of Oslo)** we give a simple proof of the following related result.

Theorem (1; https://arxiv.org/abs/1801.07645)

Let X and Y be Stein manifolds with $\dim X < \dim Y$, and assume that Y has the density property (in particular, we may take $Y = \mathbb{C}^n$).

Suppose that $\theta : X \hookrightarrow Y$ is a holomorphic embedding with $\mathcal{O}(Y)$-convex image (this holds in particular if θ is proper), and let $E \to X$ denote the normal bundle associated to θ.

Then, θ is approximable uniformly on compacts in X by holomorphic embeddings $\tilde{\theta}: E \hookrightarrow Y$ whose images are Runge domains in Y.

Recall that a locally closed subset Z of a complex manifold Y is said to be $\mathcal{O}(Y)$-*convex* if for every compact set $K \subset Z$, its $\mathcal{O}(Y)$-convex hull

$$\widehat{K}_{\mathcal{O}(Y)} = \{y \in Y : |f(y)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(Y)\}$$

is compact and contained in Z.

Runge tubes over open Riemann surfaces

To get a Runge embedding $\mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2$ from Theorem 1, we embed $X = \mathbb{C}^*$ onto the algebraic curve $A = \{zw = 1\} \subset \mathbb{C}^2$ and note that any vector bundle over $\mathbb{C}^*$ (and in fact over any open Riemann surface) is trivial by **Oka's theorem (1939)**.

It is known that every open Riemann surface, X, embeds properly holomorphically into $\mathbb{C}^3$, and a plenitude of them embed into $\mathbb{C}^2$. By Oka's theorem, the normal bundle of any such embedding is trivial.

Corollary (Runge tubes over open Riemann surfaces)

If X is an open Riemann surface which admits a proper holomorphic embedding into $\mathbb{C}^2$, then $X \times \mathbb{C}$ admits a Runge embedding into $\mathbb{C}^2$.

For every open Riemann surface X and every $k \geq 2$, $X \times \mathbb{C}^k$ admits a Runge embedding into $\mathbb{C}^{k+1}$, and more generally into any Stein manifold Y^{k+1} with the density property.

Manifolds with density property

Varolin 2000 A complex manifold Y enjoys the **density property (DP)** if every holomorphic vector field on Y can be approximated by Lie combinations of $\mathbb{C}$-complete holomorphic vector fields.

Similarly, a Lie algebra $\mathfrak{g}$ of holomorphic vector fields on Y enjoys DP if it is densely generated by the complete vector fields that it contains. If Y carries a holomorphic volume form ω, then the density property for the Lie algebra $\mathfrak{g}(\omega)$ of all holomorphic vector fields with vanishing ω-divergence is called the **volume density property (VDP)** of (Y, ω).

Andersén 1990; Andersén & Lempert 1992 $\mathbb{C}^n$ enjoys DP for $n > 1$, and VDP for the volume form $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ for $n \geq 1$.

In fact, **every polynomial holomorphic vector field on $\mathbb{C}^n$ is a finite sum of shear vector fields** given in suitable coordinates $z = (z', z_n)$ by

$$V(z) = f(z') \frac{\partial}{\partial z_n}, \qquad W(z) = f(z') z_n \frac{\partial}{\partial z_n},$$

where $f \in \mathbb{C}[z_1, \ldots, z_{n-1}]$.

Approximating biholomorphisms by automorphisms

Theorem (Andersén-Lempert, Forstnerič-Rosay, Varolin)

Let Y be a Stein manifold with DP. Assume that

$$F_t \colon \Omega_0 \xrightarrow{\cong} \Omega_t \subset Y, \qquad t \in [0,1],$$

is an isotopy of biholomorphic maps between Stein Runge domains in Y, with $F_0 = \mathrm{Id}|_{\Omega_0}$. Then, $F_1 \colon \Omega_0 \to \Omega_1$ is a limit of holomorphic automorphisms of Y, uniformly on compacts in Ω_0.

The analogous result holds for isotopies of biholomorphic maps preserving a holomorphic volume form on a Stein manifold with VDP.

This also applies to isotopies of holomorphic maps $F_t \colon \Omega_0 \to \Omega_t$, defined in a neighborhood of a compact set $K_0 \subset Y$, provided that

$$\text{the set } K_t := F_t(K_0) \text{ is } \mathcal{O}(Y)\text{-convex for every } t \in [0,1].$$

Then, F_1 can be approximated uniformly on K_0 by automorphisms of Y.

Examples of Stein manifolds with DP

- $\mathbb{C}^n$ for $n \geq 1$ satisfies VDP for $dz_1 \wedge \cdots \wedge dz_n$ (**Andersén**).
- $\mathbb{C}^n$ for any $n > 1$ satisfies DP (**Andersén and Lempert**).
- $(\mathbb{C}^*)^n$ with the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ satisfies VDP (**Varolin**). It is not known whether DP holds when $n > 1$.
- If G is a linear algebraic group and $H \subset G$ is a closed proper reductive subgroup, then $Y = G/H$ is a Stein manifold with DP, except when $Y = \mathbb{C}, (\mathbb{C}^*)^n$, or a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$ (**Kaliman, Donzelli & Dvorsky**).
- In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has DP (**Kaliman & Kutzschebauch**).
- If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fibre, then $Y = \{xy = p(z)\}$ has DP (**K&K**). The same is true if the source $\mathbb{C}^n$ of p is an arbitrary Stein manifold with DP.
- A Cartesian product $Y_1 \times Y_2$ of two Stein manifolds Y_1, Y_2 with DP also has DP. The analogous result holds for VDP (**K&K**).

Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2018** A smooth affine algebraic surface Y is a **Gizatullin surface** if $\mathrm{Aut}_{\mathrm{alg}}(Y)$ acts transitively on Y up to finitely many points. Every such surface admits a fibration $\pi\colon Y \to \mathbb{C}$ whose generic fiber equals $\mathbb{C}$ and there is only one exceptional fiber. **If this exceptional fiber is reduced, then Y has DP.**
- **Leuenberger 2016** DP holds for a family of hypersurfaces

$$Y = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2 y = a(z) + x b(z)\},$$

where $x, y \in \mathbb{C}$ and $a, b \in \mathbb{C}[z]$ are polynomials in $z \in \mathbb{C}^{n+1}$. This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to $\mathbb{R}^6$, but is not algebraically isomorphic to $\mathbb{C}^3$ (**Makar-Limanov, Dubouloz**).

It remains an open question whether C is biholomorphic to $\mathbb{C}^3$.

Embeddings into Stein manifolds with DP or VDP

Stein manifolds with the (volume) density property are universal models for proper embeddings of all Stein manifolds.

Theorem (Andrist, F., Ritter, Wold 2016; F. 2017)

Assume that X is a Stein manifold, and Y is a Stein manifold with the density property or the volume density property.

(a) *If $\dim Y > 2\dim X$, then any continuous map $X \to Y$ is homotopic to a proper holomorphic embedding $X \hookrightarrow Y$.*

(b) *If $\dim Y = 2\dim X$, then any continuous map $X \to Y$ is homotopic to a proper holomorphic immersion with simple double points.*

Corollary

Every Stein manifold Y with DP contains a Runge tube $E \hookrightarrow Y$ whose base is an arbitrary Stein manifold X with $2\dim X < \dim Y$.

Proof of Theorem 1: preliminaries

A **domain** D in a complex manifold Y is said to be **Runge** in Y if $\{f|_D : f \in \mathcal{O}(Y)\}$ is a dense subset of $\mathcal{O}(D)$. If both D and Y are Stein, this holds if and only if for every compact subset $K \subset D$ we have that $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{O}(Y)}$. In particular, a domain in a Stein manifold Y which is exhausted by compact $\mathcal{O}(Y)$-convex sets is Runge in Y.

A **holomorphic embedding** $\theta \colon X \hookrightarrow Y$ is said to be **Runge** if the image $Z = \theta(X) \subset Y$ is exhausted by compact $\mathcal{O}(Y)$-convex subsets. If X and Y are Stein, then every proper holomorphic embedding $X \hookrightarrow Y$ is Runge.

Assume that $\pi \colon E \to X$ is a **holomorphic vector bundle** over a Stein manifold X. The total space E is then also a Stein manifold. We write elements of E as $e = (x, v)$, identifying X with the zero section $\{(x, 0) : x \in X\}$ of E. For any $t \in \mathbb{C}^*$ consider a holomorphic automorphism $\psi_t \in \mathrm{Aut}(E)$, with $\psi_t|_X = \mathrm{Id}_X$, given by

$$\psi_t \colon E \to E, \qquad \psi_t(x, v) = (x, tv).$$

A subset $Z \subset E$ is called **radial** if $\psi_t(Z) \subset Z$ holds for every $t \in [0, 1]$.

Proof of Theorem 1: The main lemma

Lemma

Assume that:

- X *is a Stein manifold,*
- $\pi\colon E \to X$ *is a holomorphic vector bundle,*
- $K \subset L$ *are compact radial* $\mathcal{O}(E)$*-convex subsets of* E*,*
- $\Omega \subset E$ *is an open set containing* $X \cup K$*,*
- Y *is a Stein manifold with DP such that* $\dim Y = \dim E$*, and*
- $\theta : \Omega \hookrightarrow Y$ *is a holomorphic embedding such that* $\theta|_X : X \hookrightarrow Y$ *is a Runge embedding and* $\theta(K)$ *is* $\mathcal{O}(Y)$*-convex.*

Then there is a domain $\widetilde{\Omega} \subset E$*, with* $X \cup L \subset \widetilde{\Omega}$*, such that* θ *can be approximated uniformly on* K *by holomorphic embeddings*

$$\tilde{\theta} : \widetilde{\Omega} \hookrightarrow Y$$

such that $\tilde{\theta}|_X : X \hookrightarrow Y$ *is a Runge embedding and* $\tilde{\theta}(L)$ *is* $\mathcal{O}(Y)$*-convex.*

Proof of the lemma

- Recall that $\pi \colon E \to X$. Choose a compact $\mathcal{O}(X)$-convex subset $X_0 \subset X$ such that $\pi(L) \subset X_0$. Since the embedding $\theta|_X \colon X \hookrightarrow Y$ is Runge, the image $Y_0 = \theta(X_0) \subset \theta(X)$ is $\mathcal{O}(Y)$-convex.

- Pick a compact $\mathcal{O}(Y)$-convex neighborhood $N \subset \theta(\Omega)$ of Y_0. Thus, $N = \theta(N_0)$ for a compact set $N_0 \subset \Omega$ with $X_0 \subset \mathring{N}_0$.

- Since $\pi(L) \subset X_0$ and N_0 is a neighborhood of X_0 in E, we can choose $\epsilon > 0$ small enough such that

$$\psi_\epsilon(L) \subset N_0 \subset \Omega.$$

- Since L is $\mathcal{O}(E)$-convex and $\psi_\epsilon \in \mathrm{Aut}(E)$, the set $\psi_\epsilon(L)$ is $\mathcal{O}(E)$-convex, and hence $\mathcal{O}(N_0)$-convex.

- Since $\theta \colon \Omega \to \theta(\Omega)$ is a biholomorphism, it follows that the set $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(N)$-convex, and hence also $\mathcal{O}(Y)$-convex.

Proof of the lemma, 2

Consider the isotopy of injective holomorphic maps σ_t for $t \in [\epsilon, 1]$, defined on an open neighborhood of $\theta(K)$ in Y by the condition

$$\theta \circ \psi_t = \sigma_t \circ \theta, \qquad t \in [\epsilon, 1].$$

Note that the following hold:

(a) $\sigma_1 = \mathrm{Id}$ (since $\psi_1 = \mathrm{Id}$), and

(b) for every $t \in [\epsilon, 1]$ the compact set $\sigma_t(\theta(K)) \subset Y$ is $\mathcal{O}(Y)$-convex.

Condition (b) holds because $\psi_t(K) \subset K$ is clearly $\mathcal{O}(E)$-convex, so

$$\sigma_t(\theta(K)) = \theta(\psi_t(K)) \text{ is } \mathcal{O}(\theta(K))\text{-convex.}$$

Since $\theta(K)$ is $\mathcal{O}(Y)$-convex, it follows that

the set $\sigma_t(\theta(K))$ is $\mathcal{O}(Y)$-convex for every $t \in [\epsilon, 1]$.

Proof of the lemma, 3

Since $\sigma_t(\theta(K))$ is $\mathcal{O}(Y)$-convex for every $t \in [\epsilon, 1]$ and Y has DP,

$$\sigma_\epsilon \text{ can be approximated uniformly on } \theta(K) \text{ by } \phi \in \mathrm{Aut}(Y).$$

Since $\psi_\epsilon(L \cup X) = \psi_\epsilon(L) \cup X \subset \Omega$, there is an open set $\widetilde{\Omega} \subset E$ with

$$L \cup X \subset \widetilde{\Omega}, \qquad \psi_\epsilon(\widetilde{\Omega}) \subset \Omega.$$

We claim that the holomorphic embedding

$$\tilde{\theta} := \phi^{-1} \circ \theta \circ \psi_\epsilon : \widetilde{\Omega} \hookrightarrow Y$$

satisfies the conclusion of the lemma. Indeed:

- $\tilde{\theta}|_X = \phi^{-1} \circ \theta|_X : X \hookrightarrow Y$ is a Runge embedding since $\theta|_X$ is.
- Since $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(Y)$-convex and $\phi \in \mathrm{Aut}(Y)$, the set $\tilde{\theta}(L)$ is also $\mathcal{O}(Y)$-convex.
- On the set $K \subset E$ we have that

  $$\tilde{\theta} = \phi^{-1} \circ \theta \circ \psi_\epsilon = \phi^{-1} \circ \sigma_\epsilon \circ \theta.$$

  Since $\phi^{-1} \circ \sigma_\epsilon$ is close to the identity on $\theta(K)$ by the choice of ϕ, it follows that $\tilde{\theta}$ is close to θ on K.

Proof of Theorem 1

Pick an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = E$ by compact radial $\mathcal{O}(E)$-convex sets.

Let $\theta \colon X \hookrightarrow Y$ be a holomorphic Runge embedding. By a theorem of Docquier and Grauert (1960) there is a neighbourhood $\Omega_0 \subset E$ of the zero section $X \subset E$ such that θ extends to a holomorphic embedding

$$\theta_0 : \Omega_0 \hookrightarrow Y.$$

Set $K_0 = \varnothing$. Applying the main lemma inductively, we find

open neighbourhoods $\Omega_j \subset E$ of $K_j \cup X$, and

holomorphic embeddings $\theta_j \colon \Omega_j \hookrightarrow Y$

satisfying the following conditions for every $j \in \mathbb{N}$:

(a) the compact set $\theta_j(K_j)$ is $\mathcal{O}(Y)$-convex,

(b) the embedding $\theta_j|_X \colon X \hookrightarrow Y$ is Runge, and

(c) θ_j approximates θ_{j-1} as closely as desired on K_{j-1}.

Proof of Theorem 1

If the approximations are close enough, the sequence θ_j converges uniformly on compacts in E to a holomorphic embedding $\tilde{\theta}\colon E \hookrightarrow Y$.

Since $\mathcal{O}(Y)$-convexity of a compact set in a Stein manifold Y is a stable property for compact strongly pseudoconvex domains and every compact $\mathcal{O}(Y)$-convex set can be approximated from the outside by such domains, it follows that the image of each K_j remains $\mathcal{O}(Y)$-convex in the limit provided that all approximations were close enough.

Hence, $\tilde{\theta}(E)$ is a Runge domain in Y. This proves the theorem.

Runge tubes around algebraic submanifolds of $\mathbb{C}^n$

The Runge embeddings $E \hookrightarrow Y$ of the normal bundle in Theorem 1 need not agree with the embedding $\theta : X \hookrightarrow Y$ on the zero section X of E. However, we can ensure this additional condition for algebraic embeddings of codimension at least 2 into $\mathbb{C}^n$.

Theorem (2)

Let X be a Stein manifold and $\theta\colon X \hookrightarrow \mathbb{C}^n$ be proper holomorphic embedding onto an algebraic submanifold $A = \theta(X) \subset \mathbb{C}^n$.
If $n \geq \dim A + 2$, then θ extends to a holomorphic Runge embedding $\tilde{\theta}\colon E \hookrightarrow Y$ of the normal bundle E of θ.

Since every vector bundle over an open Riemann surface is trivial, we get

Corollary

Let X be an affine algebraic curve. Every proper algebraic embedding $\theta\colon X \hookrightarrow \mathbb{C}^{n+1}$ for $n \geq 2$ extends to a holomorphic embedding $\tilde{\theta} : X \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ onto a Runge domain in $\mathbb{C}^{n+1}$.

The proof of Theorem 2 requires the following

Addendum to the main lemma:
If $Y = \mathbb{C}^n$ with $n \geq \dim X + 2$, $\theta \colon \Omega \hookrightarrow Y$ is a holomorphic embedding (where $\Omega \subset E$ is an open neighborhood of $K \cup X$), and $A = \theta(X) \subset \mathbb{C}^n$ is a closed algebraic submanifold of $\mathbb{C}^n$, then the approximating holomorphic embedding $\tilde{\theta} : \widetilde{\Omega} \hookrightarrow \mathbb{C}^n$ can be chosen to agree with θ on X.

The proof of this addendum uses the following result.

Theorem (Kaliman and Kutzschebauch, 2008)

If $A \subset \mathbb{C}^n$ is an algebraic submanifold with $n \geq \dim A + 2$, then every polynomial vector field on $\mathbb{C}^n$ that vanishes on A is a Lie combination of complete polynomial vector fields vanishing on A.

By using this result and Serre's Theorem A and B, we can approximate the biholomorphism σ_ϵ (in the proof of Theorem 1) by an automorphism $\phi \in \mathrm{Aut}(Y)$ such that $\phi(z) = z$ for all $z \in A$.

There are no algebraic Runge tubes

Note that holomorphic Runge embeddings of the normal bundle, furnished by the previous Theorem and Corollary, can never be algebraic.

Indeed, if $E \to X$ is the algebraic normal bundle of an algebraic submanifold $X \subset \mathbb{C}^n$ and $F\colon E \hookrightarrow \mathbb{C}^n$ is an algebraic embedding, then $\Omega = F(E) \subset \mathbb{C}^n$ is a Zariski open set in $\mathbb{C}^n$ and its complement $A = \mathbb{C}^n \setminus \Omega$ is a Zariski closed set, i.e., an algebraic subvariety of $\mathbb{C}^n$ (**Chevalley 1958**).

Since Ω is a Stein domain, $A = \mathbb{C}^n \setminus \Omega$ is of pure codimension one, and hence $A = \{f = 0\}$ for some entire function $f \in \mathcal{O}(\mathbb{C}^n)$.

Clearly, the function $1/f \in \mathcal{O}(\Omega)$ cannot be approximated uniformly on compacts in Ω by entire functions, and hence the domain $\Omega = F(E)$ is not Runge in $\mathbb{C}^n$.

THANK YOU

FOR YOUR ATTENTION