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REGULARITY OF VARIETIES IN STRICTLY PSEUDOCONVEX DOMAINS

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Abstract

We prove a theorem on the boundary regularity of a purely p-dimensional complex subvariety of a relatively compact, strictly pseudoconvex domain in a Stein manifold. Some applications describing the structure of the polynomial hull of closed curves in $\mathbb{C}^n$ are also given.

Introduction

Let X be a complex manifold, $M \subset X$ a connected $(2p-1)$-dimensional submanifold of X of class C^k $(k \geq 1, p \geq 1)$, and A a closed complex subvariety of $X \backslash M$ of pure dimension p such that $\overline{A} \subset A \cup M$. Then either $\overline{A}$ is a complex subvariety of X or else there exists a closed subset $E \subset A$ of $(2p-1)$-dimensional Hausdorff measure $\mathcal{H}_{2p-1}(E) = 0$ such that the pair $(A \backslash E, M \backslash E)$ is a C^k submanifold with boundary [**2**, p.190]. In the second case A has locally finite $2p$ dimensional volume in X, and M can be oriented such that the pair (A, M) satisfies the theorem of Stokes [**2**, p.192], [**6**], [**8**]. Consequently M is a maximally complex submanifold of X, i.e., the maximal complex subspace $\mathrm{T}_z^{\mathbb{C}} M$ of the real tangent space $\mathrm{T}_z M$ to M at z has real codimension one in $\mathrm{T}_z M$.

There is a converse of this due to Harvey and Lawson [**6**]: If X is a Stein manifold and M is a closed, compact, maximally complex submanifold of X of dimension $2p-1$ $(p \geq 2)$, then M bounds (in the sense of currents) a purely p-dimensional complex subvariety $A \subset X \backslash M$, with boundary regularity as above.

We are interested in the boundary regularity of a purely p-dimensional complex subvariety of a relatively compact, strictly pseudoconvex domain $\Omega \subset X$ with C^2 boundary. We shall give a simple proof of the following

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Theorem 1. *Assume that*

(1) X is a Stein manifold;

(2) Ω is a relatively compact, strictly pseudoconvex domain with C^2 boundary in X;

(3) M is a closed $(2p-1)$-dimensional submanifold of X of class C^k $(p \geq 1, k \geq 2)$ contained in the boundary $b\Omega$ of Ω;

(4) A is a purely p-dimensional complex subvariety of Ω such that $\overline{A} \subset \subset A \cup M$, and $\overline{A}$ intersects every connected component of M.

Then there exists an open neighborhood U of M such that the pair $(A \cap U, M)$ is a C^k manifold with boundary, and $\overline{A}$ intersects $b\Omega$ transversely in the set M.

Consequently A has at most finitely many singularities in Ω. The manifold M is maximally complex, and its tangent space $T_z M$ is not contained in the maximal complex tangent space $T_z^{\mathbb{C}} b\Omega$ to the boundary of Ω for any $z \in M$.

We obtain an interesting consequence concerning holomorphic convexity of closed curves. We shall state the result only for $X = \mathbb{C}^n$. Recall that the *polynomially convex hull* of a compact set $K \subset \mathbb{C}^n$ is

$$\hat{K} = \{z \in \mathbb{C}^n : |f(z)| \leq \sup_K |f| \text{ for all holomorphic polynomials } f\}.$$

If M is a rectifiable closed Jordan curve in $\mathbb{C}^n$, then either M is polynomially convex, $M = \hat{M}$, or else $A = \hat{M} \backslash M$ is a purely one-dimensional analytic variety according to Wermer [**10**], [**11**, p.71], Stolzenberg [**9**], and Alexander [**1**].

Corollary 2. *Let Ω be a bounded C^2 strictly pseudoconvex domain in $\mathbb{C}^n$ with polynomially convex closure, and let M be a simple closed curve of class C^k, $k \geq 2$, contained in the boundary of Ω. If M is not polynomially convex, then the one-dimensional complex variety $A = \hat{M} \backslash M$ has at most finitely many singularities.*

Proof: Since $\overline{\Omega}$ is polynomially convex, A is contained in $\overline{\Omega}$. Every point $p \in b\Omega$ is a peak point for Ω, so the maximum principle implies that A is contained in Ω. Therefore the corollary follows from Theorem 1. ∎

We shall say that a submanifold $M \subset b\Omega$ of class C^1 is *complex tangential* at the point $z \in M$ if

$$T_z M \text{ is contained in } T_z^{\mathbb{C}} b\Omega. \tag{1}$$

Here, $T_z^{\mathbb{C}} b\Omega = T_z b\Omega \cap \sqrt{-1}\, T_z b\Omega$. We shall say that M is *complex transverse* at z if it is not complex tangential.

Corollary 3. *Let $\Omega \subset \mathbb{C}^n$ be as in Corollary 2. If $M \subset b\Omega$ is a simple closed curve of class C^2 that is complex tangential at least at one point, then M is polynomially convex.*

Proof: If M is not polynomially convex, Theorem 1 implies that the polynomial hull $\hat{M} = A \cup M \subset \overline{\Omega}$ is a complex variety with smooth boundary near every point $z \in M$, and $\hat{M}$ intersects $b\Omega$ transversely in M. This implies that M is complex transverse in $b\Omega$ and the corollary follows. ■

Example. If M is a simple closed C^2 curve in the sphere $\{z \in \mathbb{C}^n : |z| = 1\}$ parametrized by the map $r(t) = (r_1(t), \dots, r_n(t))$ with nonvanishing derivative, and if

$$\sum_{j=1}^{n} r_j'(t)\overline{r_j(t)} = 0$$

for some value of the parameter t, then M is polynomially convex

It seems rather surprising that a condition at one point of the curve guaranties its polynomial convexity, as long as the curve stays inside the given strictly pseudoconvex boundary.

Remarks.

1. Theorem 1 is stated in [**2**, p.203], but the proof given there does not appear to be complete.

2. If one knows already that M is the boundary of $A = \hat{M}\backslash M$ in the sense of currents and if $p \geq 2$, then Theorem 1 is a special case of Theorem 10.3 in [**6**, p.275].

3. In the case when $p = 1$ and the variety A is a proper holomorphic image of the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, Theorem 1 follows from the more general results of Čirka [**3**] concerning the regularity of one-dimensional complex varieties in the complement of a totally real submanifold of the ambient space.

4. In the case $p \geq 2$, Theorem 1 was proved by the author in [**4**]. Our new proof is simpler and includes the case $p = 1$ when M is a curve. We first show that the pair (A, M) is a manifold with boundary in a neighborhood of each point $z \in M$ at which M is complex transversal, i.e., the condition (1) fails. The proof in this case is the same as in [**4**]. The main difficulty in [**4**] was to show that M can not be complex tangential at any point if it bounds a p-dimensional variety. In this paper we prove this by a very simple perturbation argument.

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Proof of Theorem 1

By the embedding theorem of Fornæss and Khenkin [**7**, p.112] we may assume that $X = \mathbb{C}^n$ and Ω is a strictly convex domain in $\mathbb{C}^n$.

It suffices to prove that each point $z^0 \in \overline{A} \cap M$ has an open neighborhood U such that the pair $(A \cap U, M \cap U)$ is a smooth manifold with boundary. We first prove this in the case when M is complex transverse at z^0, i.e., condition (1) fails. This part of the argument is the same as in [**4**]. We include it for the convenience of the reader.

By an affine change of coordinates in $\mathbb{C}^n$ we may assume that

(i) $z^0 = 0$,

(ii) $T_0 b\Omega = \{\Re z_1 = 0\}$ and $T_0^{\mathbb{C}} b\Omega = \{z_1 = 0\}$, and

(iii) the domain Ω is contained in $\{\Re z_1 > 0\}$.

Recall that $T_0 M$ is a real $(2p-1)$-dimensional subspace of $\{\Re z_1 = 0\}$ that is not contained in $\{z_1 = 0\}$. Thus the orthogonal projection of $T_0 M$ onto the z_1 axis is a real line, and the intersection $W = T_0 M \cap \{z_1 = 0\}$ has real dimension $2p - 2$.

We can choose a complex $(p-1)$-dimensional subspace L contained in $\{z_1 = 0\}$ such that the orthogonal projection $\mathbb{C}^n \to L$ maps W surjectively onto L. After a unitary change of coordinates $z_2, \ldots, z_n$ we may assume that $L = \{z_1 = z_{p+1} = \ldots = z_n = 0\}$.

Let $\pi: \mathbb{C}^n \to \mathbb{C}^p = \{z_{p+1} = 0, \ldots, z_n = 0\}$ be the orthogonal projection. Since $b\Omega$ is strictly convex, we can find an open polydisc neighborhood $U = U' \times U''$ of 0 in $\mathbb{C}^n$, with $U' \subset \mathbb{C}^p$ and $U'' \subset \mathbb{C}^{n-p}$, such that $\pi: U \cap \overline{\Omega} \to U'$ is a proper mapping. Our choice of L implies that $\pi: T_0 M \to \mathbb{C}^p$ is injective. Shrinking U if necessary it follows that π maps $M \cap U$ diffeomorphically onto a real hypersurface $\Gamma \subset U'$ of class C^k that splits $U' \backslash \Gamma$ in two connected components Γ^+ and Γ^-. Let Γ^+ be the region contained in $\{\Re z_1 > 0\}$. Since $M \cap U$ is contained in the strictly convex boundary $b\Omega \cap U$ and $\mathbb{C}^p \times \{0\}$ contains the normal vector $(1, 0, \ldots, 0)$ to $b\Omega$ at 0, the projection $\pi(M \cap U) = \Gamma$ is hypersurface in $\mathbb{C}^p$ which is strictly convex from the side Γ^+, provided that the neighborhood U is sufficiently small.

Since $\pi: \overline{\Omega} \cap U \to U'$ is proper and the set $(A \cup M) \cap U$ is closed in U, the restriction

$$\pi: (A \cup M) \cap U \to U'$$

is also proper. The convexity of Γ^+ along Γ implies that $\pi(A \cap U)$ is contained in Γ^+ according to the maximum principle. Hence the mapping

$$\pi: A \cap U \to \Gamma^+ \tag{2}$$

is an analytic cover [**5**, p.101].

Denote by s the number of sheets of this analytic cover, i.e., the number of points in the generic fiber. Notice that all sheets converge to the common edge M as we approach Γ. We claim that this implies $s = 1$. We only give a sketch of proof since the details can be found in [**4**].

Let $z = (z', z'')$, where $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$. There is a linear function $w = w(z'')$ that separates points of $\pi^{-1}(z') \cap A \cap U$ for all points $z' \in \Gamma^+$ outside a proper complex subvariety $\sigma \subset \Gamma^+$. For each $z' \in \Gamma^+ \backslash \sigma$ we denote by $w_1(z'), \ldots, w_s(z')$ the values of w at the points of $\pi^{-1}(z') \cap A \cap U$. Let $P(w, z')$ be the polynomial in w defined by

$$P(w, z') = \prod_{j=1}^{s} \big(w - w_j(z')\big) = w^s + a_1(z') w^{s-1} + \cdots + a_s(z'), \quad z' \in \Gamma^+ \backslash \sigma.$$

Its coefficients $a_j(z')$ are bounded holomorphic functions on $\Gamma^+\backslash\sigma$, so they extend to bounded functions on Γ^+. The discriminant $\delta(z')$ of $P(.,z')$ is also a bounded holomorphic function on Γ^+ since it is a polynomial expression in the coefficients a_j of P. Recall that $\delta(z') = 0$ if and only if $P(.,z')$ has multiple roots.

If $s > 1$, the hypothesis $\overline{A} \subset A \cup M$ implies that the nontangential boundary values of δ on Γ equal zero almost everywhere since the different sheets of (2) converge together to M. This implies $\delta \equiv 0$ on Γ^+, a contradiction. Thus $s = 1$ as claimed.

It follows that the projection (2) is a bijection, so $(A \cup M) \cap U$ is a graph of the form

$$(A \cup M) \cap U = \{(z', f(z')) : z' \in \Gamma^+ \cup \Gamma\}.$$

Since A is complex analytic and M is of class C^k, it follows that f is holomorphic on Γ^+ and of class C^k on Γ. Clearly f is also continuous on $\Gamma^+ \cup \Gamma$. The regularity theorem [6, p.249] implies that f is of class C^k on $\Gamma^+ \cup \Gamma$. This proves that $(A \cup M) \cap U$ is a C^k manifold with boundary intersecting $b\Omega$ transversely.

It remains to show that the manifold M is complex transverse at each point $z \in M \cap \overline{A}$ so that the first part of the proof applies. The following argument is considerably simpler than the one in [4], and it also applies in the case $p = 1$.

Assume that the condition (1) is satisfied for some $z = z^0 \in M \cap \overline{A}$. Let $\Lambda \subset T^{\mathbb{C}}_{z^0} b\Omega$ be the smallest complex subspace of $\mathbb{C}^n$ containing $T_{z^0} M$. Since $T_{z^0} M$ is not a complex subspace, there is a vector $b \in \Lambda \backslash T_{z^0} M$. We can choose a function h of class C^2, supported on a neighborhood of z^0 in $\mathbb{C}^n$, such that $h|_M \equiv 0$, but the derivative of h at z^0 in the direction b is nonzero.

Let ρ be a strictly convex defining function of class C^2 for Ω, so $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $d\rho \neq 0$ on $b\Omega$. If $\epsilon > 0$ is sufficiently small, the domain

$$\Omega_\epsilon = \{z \in \mathbb{C}^n : \rho(z) + \epsilon h(z) < 0\}$$

is of class C^2 and strictly convex. Fix such an ϵ. Since h vanishes on M, M is contained in the boundary of Ω_ϵ. Thus we have $A \subset \hat{M} \subset \hat{\Omega}_\epsilon = \overline{\Omega}_\epsilon$, and the maximum principle implies $A \subset \Omega_\epsilon$.

Our choice of h implies that $T_{z^0} b\Omega_\epsilon$ does not contain Λ, so $T^{\mathbb{C}}_{z^0} b\Omega_\epsilon$ does not contain $T_{z^0} M$. This means that M is complex transverse in $b\Omega_\epsilon$ at the point z^0. By the first part of the proof, with Ω replaced by Ω_ϵ, the set $\overline{A}$ is a local C^k manifold with boundary M near z^0.

We have proved that the pair (A, M) is a local manifold with boundary near every point $z \in \overline{A} \cap M$. This implies that $\overline{A} \cap M$ is an open and closed subset of M. Since we assumed that $\overline{A}$ intersects every connected component of M, it follows that $\overline{A} = A \cup M$.

It remains to show that $\overline{A}$ intersects $b\Omega$ transversely. The restriction $\rho' = \rho|_{\overline{A}}$ of the plurisubharmonic defining function ρ of Ω to $\overline{A}$ is a negative subharmonic function of class C^2 on the complex manifold with boundary $\overline{A}$. The Hopf lemma implies

$$\rho(z) \leq -c\,\mathrm{dist}(z, M), \quad z \in A$$

for some $c > 0$. Here, dist denotes the Euclidean distance. Since $-\rho(z)$ is proportional to the distance of z to $b\Omega$, we conclude that $\mathrm{dist}(z, M)$ is proportional to $\mathrm{dist}(z, b\Omega)$ for $z \in A$. Hence $\overline{A}$ intersects $b\Omega$ transversely at each point of M. Thus the condition (1) fails and M is everywhere complex transverse.

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