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A TOTALLY REAL THREE SPHERE IN $C^3$
BOUNDING A FAMILY OF ANALYTIC DISKS

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ABSTRACT. We construct a smoothly embedded totally real three-sphere $S$ in $C^3$ and a one-parameter family of analytic disks in $C^3$ that have boundaries in $S$.

1. INTRODUCTION

Denote by $D$ the open unit disk $\{z \in C: |z| < 1\}$ in $C$, by $\overline{D}$ the closed unit disk and by $bD$ its boundary $\{z \in C: |z| = 1\}$. Let $A(D)$ be the algebra of all continuous functions on $\overline{D}$ that are holomorphic on $D$. An analytic disk with boundary in a subset $M \subset C^n$ is a map $f = (f_1, \ldots, f_n): \overline{D} \rightarrow C^n$, $f_j \in A(D)$ ($j = 1, \ldots, n$), such that $f(bD)$ is contained in $M$.

Recall that a real submanifold $M$ of $C^n$ of class $C^1$ is called totally real if for each $x \in M$ the tangent space $T_xM$ of $M$ at $x$ contains no nontrivial complex subspace, i.e., $T_xM \cap iT_xM = \{0\}$. In this note we shall construct an embedded totally real three-sphere $S$ in $C^3$ which bounds a one-parameter family of analytic disks. More precisely, we prove

Theorem 1. There is a smooth totally real submanifold $S$ of $C^3$ diffeomorphic to $\{x \in R^4: |x| = 1\}$ and an embedding $F: \overline{D} \times [-1, 1] \rightarrow C^3$ such that for each $t \in [-1, 1]$ the map $z \rightarrow F(z, t)$, $z \in \overline{D}$, is an analytic disk with boundary in $S$.

The first explicit totally real embedding of the real three-sphere into $C^3$ was given by Ahern and Rudin [2]. The existence of such embeddings also follows from a theorem of Gromov [7, 8, p. 193, 5, Theorem 1.4, 6]. However, it seems difficult to find analytic disks with boundaries in a given submanifold; we do not know whether there are any such disks in the example of Ahern and Rudin [2].

We believe that our example is of interest for the following reason. If $M \subset C^n$ is a smoothly embedded compact lagrange submanifold of $C^n$, i.e., the
pullback of the 2-form $\omega = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ to $M$ vanishes, then for every nonconstant analytic disk $f$ with boundary in $M$ the curve $f: bD \to M$ represents a nontrivial class in the homology group $H_1 M$. To see this assume on the contrary that this path bounds a 2-cycle $\sigma$ in $M$. By a theorem of Čirka [4, p. 293] $f$ is smooth on $\overline{D}$, and the Stokes’s theorem applied to the one-form $\alpha = \sum_{j=1}^n z_j d\bar{z}_j$ yields

$$0 = \int_{\partial} \omega = \int_{\partial} d\alpha = \int_{f(bD)} \alpha = \int_{bD} f^* \alpha = \int_{D} f^* \omega.$$ 

However, since $f^* \omega = |f'|^2 dz \wedge d\bar{z}$ on $D$, the last integral is nonzero, a contradiction. This argument was communicated to me by L. Lempert who raised the question whether the same is true if $M$ is only totally real. (Recall that every lagrange submanifold $M \subset \mathbb{C}^n$ is also totally real.) Our theorem shows that this is not the case: the three-sphere $S$ is simply connected, so $H_1 S = 0$, and yet it may bound analytic disks in $\mathbb{C}^3$.

It is not known whether the three-sphere $S$ admits a lagrange embedding into $\mathbb{C}^3$. In fact it was conjectured that no compact simply connected $n$-dimensional manifold $M$ admits a lagrange embedding into $\mathbb{C}^n$. As for the immersions, every totally real immersion of a compact $n$-manifold $M$ into $\mathbb{C}^n$ is regularly homotopic through totally real immersions to a lagrange immersion of $M$ into $\mathbb{C}^n$ [10, 7, 8, p. 61].

Denote by $\check{M}$ the polynomially convex hull of a set $M \subset \mathbb{C}^n$. If $M$ is a compact embedded totally real submanifold of $\mathbb{C}^n$ of real dimension $n$, it is known [1] that $\check{M}$ has topological dimension at least $n+1$. It would be of interest to know whether every such $M$ bounds analytic disks or analytic varieties in $\mathbb{C}^n$. If so, is $\check{M}\setminus M$ the union of closed analytic subvarieties of $\mathbb{C}^n\setminus M$? Every such subvariety with no zero-dimensional components is contained in the hull of $M$ according to the maximum principle. Note that the general technique of constructing analytic disks due to Bishop [3] and Hill and Taiani [9] does not apply in the totally real case that we are dealing with. Important results in this direction were obtained by Gromov [11].

We shall prove Theorem 1 in §2. In the construction of $S$ we will need an extension theorem for functions that do not annihilate a zero-free complex vector field at any point of an $n$-dimensional cube (Theorem 3). This result of independent interest can be proved by different methods; in §3 we shall prove it using techniques of Gromov. (See [7], §2.4 in [8] and also the exposition in [5].)

I wish to thank L. Lempert for proposing this problem. Thanks also go to S. Webster and E. L. Stout.
2. Proof of Theorem 1

We begin with

Lemma 2. There exists a real-analytic function \( g : \mathbb{C} \to \mathbb{C} \) such that the submanifold \( N = \{(z, g(z)) : z \in \mathbb{C}\} \) of \( \mathbb{C}^2 \) is totally real and \( g(z) = 0 \) for each \( z \in bD \).

Proof. The manifold \( N \) is totally real when the derivative \( \partial g/\partial \bar{z} \) has no zeroes on \( \mathbb{C} \). Instead of simply giving the formula (2.1) for \( g \) we will show how to find such a function.

We set \( g(z) = (z\bar{z} - 1)h(z) \) in order to have \( g(z) = 0 \) when \( |z| = 1 \). Then

\[
\frac{\partial g}{\partial \bar{z}}(z) = zh(z) + (z\bar{z} - 1)\frac{\partial h}{\partial ar{z}}(z).
\]

When \( |z| = 1 \), \( \frac{\partial g}{\partial \bar{z}}(z) = zh(z) \). Since \( \frac{\partial g}{\partial \bar{z}} \) is zero free on \( \mathbb{C} \), its winding number on the circle \( bD = \{ |z| = 1 \} \) equals zero, so the winding number of \( h \) on \( bD \) is \(-1\). To achieve this we set \( h(z) = \bar{z}e^{k(z)} \). Then

\[
\frac{\partial g}{\partial \bar{z}}(z) = e^{k(z)}((2z\bar{z} - 1) + (z\bar{z} - 1)\bar{z} \frac{\partial k}{\partial \bar{z}}(z)).
\]

If we choose \( k(z) = iz\bar{z} \), \( \frac{\partial k}{\partial \bar{z}}(z) = iz \), and set \( t = z\bar{z} \), the expression in the parentheses equals \( (2t - 1) + i(t - 1)t \) which does not vanish for any real \( t \). Thus the function

\[
g(z) = (z\bar{z} - 1)\bar{z}e^{iz\bar{z}}
\]
satisfies Lemma 2. This concludes the proof.

Choose any smooth function \( h : \mathbb{R} \to [0, \infty) \) which equals 0 on \((\infty, 2] \) and is strictly convex on \((2, \infty) \). The real hypersurface \( \Gamma \subset \mathbb{C}^2 \) defined by

\[
\Gamma = \{(z, w) \in \mathbb{C}^2 : r(z, w) = h(z\bar{z} + u^2) + (v - 1)^2 = 1 \}
\]

(here \( w = u + iv \)) is smooth, diffeomorphic to the real three-sphere, and it contains the set \( D \times [-1, 1] \).

Let \( g \) be as in Lemma 2. Define \( f(z, w) = g(z) + w \) on \( z \in \overline{D}, w \in [-1, 1] \) and extend \( f \) smoothly to \( \mathbb{C}^2 \). We will show that the extension of \( f \) to \( \Gamma \) can be chosen in such a way that the real submanifold

\[
S = \{(z, w, f(z, w)) \in \mathbb{C}^3 : (z, w) \in \Gamma \}
\]
of \( \mathbb{C}^3 \) satisfies the conclusion of Theorem 1.

Clearly \( S \) is a smoothly embedded three-sphere which bounds the one parameter family of analytic disks \( F : \overline{D} \times [-1, 1] \to \mathbb{C}^3 \), \( F(z, t) = (z, t, t) \) since \( g \) vanishes on \( |z| = 1 \).

It remains to show that \( S \) is totally real for an appropriate choice of \( f \). Let \( L \) be the tangential \( \overline{\partial} \) operator on \( \Gamma \). Then \( S \) is totally real if and only if \( Lf \) has no zeros on \( \Gamma \). We have

\[
Lf(z, w) = \frac{\partial r}{\partial \bar{z}} \frac{\partial f}{\partial \bar{w}} - \frac{\partial r}{\partial w} \frac{\partial f}{\partial \bar{z}} = h'(z\bar{z} + u^2)z\frac{\partial f}{\partial \bar{w}} - (h'(z\bar{z} + u^2)u + i(v - 1))\frac{\partial f}{\partial \bar{z}}.
\]
On the set \((z, w) \in \overline{D} \times [-1, 1]\) we have \(h'(z\bar{z} + u^2) = 0\), so \(Lf = i\partial g/\partial \bar{z}\) which is nonvanishing. Thus the part of \(S\) lying over \(D \times [-1, 1]\) is totally real. There is an open subset \(U\) of \(\Gamma\) containing \(\overline{D} \times [-1, 1]\) such that \(Lf\) has no zeroes on \(\overline{U}\) and \(\Gamma \setminus U\) is diffeomorphic to a closed three dimensional cube \(I^3 \subset \mathbb{R}^3\). According to Theorem 3 in §3 there is an extension of \(f\) from \(\overline{U}\) to \(\Gamma\) such that \(Lf\) has no zeroes on \(\Gamma\). For such \(f\) the manifold \(S\) given by \((2.3)\) is totally real and Theorem 1 is proved.

3. Functions not annihilating a complex vector field

We denote by \(I^n\) the closed \(n\)-dimensional cube \([0, 1]^n\) in \(\mathbb{R}^n\). Let \(x = (x_1, \ldots, x_n)\) be real coordinates on \(\mathbb{R}^n\). A complex vector field on \(I^n\) with continuous coefficients is an expression \(L = \sum_{j=1}^{n} a_j(x) \partial/\partial x_j\), where \(a_j\) are continuous complex functions on \(I^n\). If \(f\) is a complex \(C^1\) function on \(I^n\), then \(Lf(x) = \sum_{j=1}^{n} a_j(x) \partial f/\partial x_j(x)\). The vector field \(L\) is zero-free on \(I^n\) if for each \(x \in I^n\) at least one number \(a_j(x)\) is nonzero.

**Theorem 3.** Let \(L\) be a zero-free complex vector field with continuous coefficients on \(I^n\) \((n \neq 2)\). For each \(C^1\) function \(f_0\) on \(I^n\) such that \(Lf_0\) is zero-free on the boundary of \(I^n\) there is a \(C^1\) function \(f\) on \(I^n\) which coincides with \(f_0\) near the boundary of \(I^n\) such that \(Lf\) is zero-free on all of \(I^n\).

**Remark.** Theorem 3 is false for \(n = 2\) as the following example shows. Take \(L = \partial/\partial \bar{z} = (\partial/\partial x + i \partial/\partial y)/2\) and \(f(z) = z\bar{z} = x^2 + y^2\). Then \(Lf(z) = z\) is nonvanishing on the boundary of \([-1, 1]^2\), but it can not be extended to a nonvanishing function on \([-1, 1]^2\) since it has positive winding number.

**Proof.** This result follows from Gromov’s Lemma 3.1.3 in [7]. See also §2.4 in [8]. Let \(L = L_1 + iL_2\), where \(L_1 = \sum_{j=1}^{n} a_j(x) \partial/\partial x_j\) and \(L_2 = \sum_{j=1}^{n} b_j(x) \partial/\partial x_j\) are real-valued vector fields on \(I^n\). If \(f = u + iv\), then \(Lf = (L_1u - L_2v) + i(L_1v + L_2u)\). We associate to a function \(f = u + iv\) the section \(x \rightarrow (x; u(x), v(x))\) \((x \in I^n\) of the product bundle \(\pi: X = I^n \times \mathbb{R}^2 \rightarrow I^n\). Let \(X^1\) be the manifold of one-jets of sections of the bundle \(X \rightarrow I^n\). \(X^1\) is isomorphic to the product \(X \times \mathbb{R}^{2n}\); the point \((\alpha, \beta) \in \mathbb{R}^{2n}\) corresponding to a section \(x \rightarrow (x; u(x), v(x))\) of \(X\) is determined by \(\alpha_j = \partial u/\partial x_j\) and \(\beta_j = \partial v/\partial x_j\) \((1 \leq j \leq n)\).

Let \(\Omega \subset X^1\) be the set of all points \((x; q; \alpha, \beta)\) in \(X^1\) \((x \in I^n, q \in \mathbb{R}^2, \alpha, \beta \in \mathbb{R}^n)\) for which at least one of the real numbers

\[
\sum_{i=1}^{n} a_i(x)\alpha_i - b_i(x)\beta_i,
\]

\[
\sum_{i=1}^{n} b_i(x)\alpha_i + a_i(x)\beta_i
\]

is nonzero. In Gromov’s terminology the set \(\Omega\) is an open differential relation of order one on the bundle \(\pi: X \rightarrow I^n\).
Lemma 4. The relation \( \Omega \) defined above is ample in the coordinate directions \( x_1, \ldots, x_n \) on \( I^n \).

Note. For the definition of ampleness see [7, p. 331] or §2 in [5] or [8, p. 180].

Proof. By symmetry it suffices to prove ampleness in the coordinate direction \( x_1 \). Fix a point \( x^0 \in I^n, \ q^0 \in \mathbb{R}^2, \ \alpha' = (\alpha_2, \ldots, \alpha_n), \ \beta' = (\beta_2, \ldots, \beta_n) \) and consider the set

\[
\Omega' = \{(\alpha_1, \beta_1) \in \mathbb{R}^2 : (x_1^0; q_1^0; \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots) \in \Omega\}.
\]

To prove that \( \Omega \) is ample in the direction \( x_1 \) we must show that either \( \Omega' \) is empty or else the convex hull of each of its connected components in \( \mathbb{R}^2 \) equals all of \( \mathbb{R}^2 \).

If \( a_1(x^0) = b_1(x^0) = 0 \), then (3.1) shows that \( \Omega' \) is either empty or \( \mathbb{R}^2 \). If on the other hand at least one of the numbers \( a_1(x^0), b_1(x^0) \) is nonzero, then the system of linear equations

\[
\begin{align*}
    a_1(x^0)\alpha_1 - b_1(x^0)\beta_1 &= c, \\
    b_1(x^0)\alpha_1 + a_1(x^0)\beta_1 &= d
\end{align*}
\]

has determinant \( a_1(x^0)^2 + b_1(x^0)^2 > 0 \) whence it has precisely one solution for each \( (c, d) \in \mathbb{R}^2 \). In this case \( \Omega' \) is the complement of a point in \( \mathbb{R}^2 \). This proves that \( \Omega \) is ample.

Lemma 5. If \( n \neq 2 \) and if \( \alpha_j, \beta_j \) are continuous real-valued functions on the boundary of \( I^n \) such that the expression

\[
F(x) = \sum_{j=1}^{n} (a_j(x)\alpha_j(x) - b_j(x)\beta_j(x)) + i \cdot \sum_{j=1}^{n} (b_j(x)\alpha_j(x) + a_j(x)\beta_j(x))
\]

is zero-free on the boundary of \( I^n \), then there exist continuous extensions of \( \alpha_j, \beta_j \) (\( 1 \leq j \leq n \)) to \( I^n \) such that \( F \) is zero-free on \( I^n \).

Proof. We claim that for \( n \neq 2 \) the map \( F : bI^n \to \mathbb{C}\{0\} \) can be extended to a map \( F : I^n \to \mathbb{C}\{0\} \). If \( n = 1 \) this holds because \( \mathbb{C}\{0\} \) is path-connected. If \( n > 1 \), the obstruction to extending \( F \) is an element of the group \( \pi_{n-1}(\mathbb{C}\{0\}) = \pi_{n-1}(S^1) \) which is trivial when \( n - 1 \geq 2 \). We fix such an extension of \( F \) to \( I^n \).

We subdivide the cube \( I^n \) into smaller closed cubes \( I_1, \ldots, I_r \) with faces parallel to the coordinate axes such that two distinct cubes have at most a face in common and for each \( I_k \) there is an index \( j_k \) for which

\[
a_{j_k}(x)^2 + b_{j_k}(x)^2 > 0, \quad x \in I_k.
\]

We now perform stepwise extension of the functions \( \alpha_j, \beta_j \) to the cubes \( I_k \). On \( I_k \) we extend \( \alpha_j, \beta_j, \ j \neq j_k \) arbitrarily, without changing their values on those faces of \( I_k \) where they have been defined in previous steps. Because
of (3.3) the values of $\alpha_{jk}$ and $\beta_{jk}$ on $I_k$ are now uniquely determined by (3.2). In a finite number of steps we find the desired extensions and Lemma 5 is proved.

We can now conclude the proof of Theorem 3. Let $f_0 = u_0 + iv_0$ be as in the statement of the theorem. Set $\alpha_j = \frac{\partial u_0(x)}{\partial x_j}$ and $\beta_j = \frac{\partial v_0(x)}{\partial x_j}$ for $x \in bI^n$ and $1 \leq j \leq n$. We extend the functions $\alpha_j$, $\beta_j$ ($1 \leq j \leq n$) to $I^n$ using Lemma 5. (Note that $F(x) = Lf_0(x) \neq 0$ for $x \in bI^n$.) The map $\varphi : I^n \to X^1$, $\varphi(x) = (x; f_0(x); \alpha(x), \beta(x))$ is a section of the relation $\Omega$ over $I^n$ which coincides with the one-jet $j^1f_0$ of the section $x \to (x; f_0(x))$ on the boundary $bI^n$. Since $\Omega$ is ample by Lemma 4, Gromov’s Lemma 3.1.3 in [7] implies that there is a $C^1$ function $f : I^n \to C$ whose one-jet $j^1f$ is a section of $\Omega$ over $I^n$ and $j^1f = j^1f_0$ on $bI^n$. This means precisely that $Lf(x)$ is nonvanishing on $I^n$ and $f = f_0$ on $bI^n$. Theorem 3 is proved.

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