A new class of polynomially convex sets

F. Forstnerič and E. L. Stout

A compact subset $E \subset \mathbb{C}^N$ is said to be polynomially convex if for every point $z \in \mathbb{C}^N \setminus E$, there is a holomorphic polynomial $P$ with $P(z) = 1$ and $\|P\|_E = \sup \{|P(w)| : w \in E\} < 1$. In general it is difficult to determine whether a set is polynomially convex, but questions of polynomial convexity come up repeatedly in function-theoretic considerations in $\mathbb{C}^N$. In the present paper we exhibit some new examples of polynomially convex sets in $\mathbb{C}^2$, which are contained in the boundary of the ball $B_2 = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$. This work is closely related to work on removable singularities for $\partial D$.

Given a relatively compact domain $D$ in a Stein manifold $\mathcal{M}$, call the compact subset $E$ of $\partial D$ removable provided $\partial D \setminus E = \Gamma$ is a $\mathcal{C}^1$ submanifold of $\mathcal{M} \setminus E$ such that for every continuous function $f$ on $\Gamma$ that satisfies the tangential Cauchy-Riemann equations in the weak sense, there is $F$ holomorphic on $D$ and continuous on $D \cup \Gamma$ such that $F = f$ on $\Gamma$. Such removable sets have been studied in several recent papers [8, 12—16, 20]. In particular, Jöricke [8] proved that a compact totally real 2-disc of class $\mathcal{C}^2$ in $bB_2$ is removable.

The nexus between the theory of polynomial convexity and that of removable singularities is provided by the result that a compact set $E \subset \partial B_2$ is removable if and only if it is polynomially convex [20, Th. II.10]. An analogous result is valid on strongly pseudoconvex domains: If $D$ is a strongly pseudoconvex domain in a Stein manifold of dimension two, $bD$ smooth, then $E \subset bD$ is removable if and only if it is convex with respect to the algebra $\mathcal{O}(\bar{D})$ of functions holomorphic on $\bar{D}$.

We begin with a general theorem that shows certain sets in strongly pseudoconvex boundaries to be removable and hence to enjoy the convexity property indicated above. It has as an immediate corollary Jöricke's theorem. The proof of Jöricke's theorem thereby obtained is not long, but it does depend in an essential way on some recent work [1] of Bedford and Klingenberg on the hulls of 2-spheres.

* Research supported in part by NSF Grants DMS—8801031 and 8500357.
in Stein manifolds. We also show that certain discs in $b\mathbb{B}_z$ that are not totally real are, nonetheless, polynomially convex, *viz.*, those with at most a finite number of complex tangents each of which is hyperbolic. Our general theorem also yields the polynomial convexity of certain Cantor sets. The paper concludes with a proof that a hyperbolic point in a two-dimensional submanifold of $\mathbb{C}^2$ has a neighborhood basis of polynomially convex sets.

We are indebted to Professor J. Vrabec for help with the topological parts of this paper.

We shall use the notation that if $E$ is a compact subset of the open set $\Omega$, then $\partial(\Omega)$-hull $E$ denotes the set

$$\{z \in \Omega : |f(z)| \leq \sup_{E} |f| \text{ for all } f \in \partial(\Omega)\}.$$

If $E = \partial(\Omega)$-hull $E$, we say that $E$ is $\partial(\Omega)$-convex.

**I. Theorem.** Let $D$ be a relatively compact, strongly pseudoconvex domain with boundary of class $\mathcal{C}^2$ in the two-dimensional Stein manifold $\mathcal{M}$. Let $E \subset bD$ be a compact set with the following properties:

A) There is a Stein neighborhood $\Omega$ of $E$ such that $E$ is $\partial(\Omega)$-convex.

B) If $p \in E$, there exists a neighborhood $U$ of $p$ in $bD$ with $\Omega \supset U$, with $bU \cap E = \emptyset$, and with $U$ homeomorphic to a Euclidean ball in $\mathbb{R}^3$.

Then the set $E$ is removable and so $\partial(\Omega)$-convex.

The condition A implies that the set $E$ is an intersection of Stein domains, so by a result of Lupacciolu [15], if we were in dimension three or above rather than in dimension two, it by itself would suffice to guarantee the removability of the set $E$.

Note also that condition A implies that $E$ is small topologically: Its topological dimension (as distinguished from its metric or Hausdorff dimension) is not more than two: As $E \subset bD$, $\dim E \leq 3$; if $\dim E = 3$, then [7] $E$ contains an open subset of $bD$. Granted that $E$ is $\partial(\Omega)$-convex, this is impossible.

**Proof of the theorem.** We consider a continuous function $f$ on $bD \setminus E$ that satisfies $\bar{\partial}_p f = 0$ in the weak sense; we are to show that there is $F \in \mathcal{O}(D)$ that assumes continuously the boundary values $f$ on $bD \setminus E$.

We show first that $f$ continues holomorphically into $D$ in a neighborhood of each point of the set $E$. Thus, let $p \notin E$, and choose a neighborhood $U_p$ of $p$ in accordance with condition B. As $U_p$ is diffeomorphic to a Euclidean ball in $\mathbb{R}^3$ and as $bU_p$ is disjoint from the set $E$, it follows that there is a smooth two-dimensional sphere $\Sigma_p \subset U_p$ that separates the compact set $bU_p$ from the compact set $E \cap U_p$. (Note: We use here the nontrivial fact that a three-dimensional manifold of class $\mathcal{C}^2$ that is homeomorphic to $\mathbb{R}^3$ is $(\mathcal{C}^2)$-diffeomorphic to $\mathbb{R}^3$. See [18] and the references cited there. The analogous statement for $\mathbb{R}^4$ is known to be false.)
There is a sphere $\Sigma^*$ in $D$ that is of class $C^\infty$ and that is very near $\Sigma^*$, so near that every CR-function $g$ on $bD\setminus E$ extends holomorphically into a neighborhood of $\Sigma^*$. According to Bedford and Klingenberg [1], we can perturb $\Sigma^*$ slightly to obtain a 2-sphere $\Sigma$ that lies very near $\Sigma^*$ and that bounds a Levi-flat 3-ball $\Sigma^+$, which is the envelop of holomorphy of $\Sigma^*$ in the sense that if $h$ is holomorphic on a neighborhood of $\Sigma^*$ (in $M$), then $h$ continues holomorphically into a neighborhood of $\Sigma^+$. If we have made our perturbations sufficiently small, then our function $f$ continues holomorphically into a neighborhood of $\Sigma^+$.

Let $\Omega$ be the domain in $bD$ that is bounded by $\Sigma^*$ and that contains $E \cap U_p$. Let $W_p \subset D$ be a domain obtained as follows: Perturb $\Delta_p \cup \Sigma^*$ into $D$ slightly leaving it fixed near $E$ so as to obtain a smooth manifold with boundary $\bar{\Delta}_p$. If we have made our perturbations of $\Sigma^*$ into $\Delta_p$ sufficiently small, we can obtain $\bar{\Delta}_p$ that is again strongly pseudoconvex, that is so close to $bD$ that each CR-function $h$ on $bD\setminus E$ extends holomorphically into a neighborhood of $\Delta_p$, so that $b\bar{\Delta}_p = \Sigma_p$ and so that $b\bar{\Delta}_p \cap \Sigma^+ = \Sigma_p$. The subdomain $W_p$ of $D$ bounded by $\Delta_p \cup \Sigma^*$ is pseudo-convex. Consequently, there is a strongly pseudoconvex domain $W_p'$ with these properties: 1) $W_p' \subset W_p$, 2) near $E \cap U_p$, $bW_p$ and $bW_p'$ coincide, and 3) outside a neighborhood of $E \cap U_p$, $bW_p'$ is contained in the domain where $f$ is known to be holomorphic.

Since $W_p' \subset \Omega$ and $E$ is $C(\Omega)$-convex, the set $E \cap U_p$ is convex with respect to $C(W_p')$. It follows that $f$ continues holomorphically through $W_p'$. In case $M=C^2$, we may invoke a result of Lupacciolu [14]; in the more general case, we refer to the paper [12] of Laurent-—Thiebaut.

As a consequence of the construction of the neighborhoods $W_p$ and $W_p'$ and the compactness of the set $E$, we see that there are finitely many pairwise disjoint relatively open subsets $V_1, \ldots, V_r$ of $D$ with $E \cap bV_j = \emptyset$ and such that $f$ continues into $V_j \cap D$ as a holomorphic function. There is, in addition, a neighborhood $V$ of $bD\setminus E$ in $\bar{D}\setminus E$ such that the function $f$ continues holomorphically into $V \cap D$. It follows that for some neighborhood $V_0$ of $bD$ in $D$, $f$ continues holomorphically into $V \cap D$. As $D$ is pseudoconvex, $f$ continues through the whole of $D$ — by a version of Hartogs' theorem. (For this, we can cite [4, Th. VII.D.4].)

This completes the proof of the theorem.

Having established the general result I, we turn to some special cases.

**II. Corollary.** Let $D$ be a strongly pseudoconvex domain in a two-dimensional Stein manifold $\mathcal{M}$. Every two-dimensional compact totally real disc $\Delta$ of class $C^2$ in $bD$ is removable and so $\mathcal{O}(\bar{D})$-convex.

In connection with this corollary it is worth noting that not all totally real discs in $C^2$ are polynomially convex. An example was given in Wermer [21]; another example is given in [5]: Define $g(\xi) = (1 - \xi^2)^{\frac{e^{i\xi}}{10}}$. Then $\partial g/\partial \xi$ is zero-free on $C$. 

and \( g(\zeta) = 0 \) if \(|\zeta| = 1\). Thus, the polynomially convex hull of the totally real disc \( E = \{(\zeta, g(\zeta)) : |\zeta| = 1\} \) contains the disc \( \{(\zeta, 0) : |\zeta| = 1\} : E \) is not polynomially convex.

**Proof.** According to [6], there is a neighborhood \( \Omega \) of \( \Delta \) in \( M \) on which there is a nonnegative strongly plurisubharmonic function \( \varphi \) of class \( \mathcal{C}^2 \) with \( \Delta = \{\varphi = 0\} \). If for small \( \varepsilon > 0 \), \( \Omega_\varepsilon = \{z \in \Omega : \varphi(z) < \varepsilon\} \), then for small \( \varepsilon \), the component of \( \Omega_\varepsilon \) that contains \( \Delta \) is a Stein neighborhood of \( \Delta \) with respect to which \( \Delta \) is convex.

It remains only to see that \( \Delta \) has a neighborhood \( U \subset \Omega \cap bD \) that is diffeomorphic to a ball in \( \mathbb{R}^3 \). This is essentially clear. We may suppose given a \( \mathcal{C}^1 \) diffeomorphism \( \psi \) from \( \Delta_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 2\} \) onto a closed submanifold of an open subset of \( bD \) such that \( \psi \) takes the closed disc \( \Delta_\varepsilon \subset \Delta_\varepsilon \) of radius one diffeomorphically onto \( \Delta \). We can then find a \( \mathcal{C}^1 \)-vector field \( \xi \) defined on a neighborhood of \( \Delta \) in \( bD \) that is tangent to \( bD \) and that is transverse to \( \Delta \). If then \( \Psi : \Delta_\varepsilon \times \mathbb{R} \) satisfies \( \Psi(x_1, x_2, 0) = \psi(x_1, x_2) \) and \( \Psi_*(\frac{\partial}{\partial z_3}) = \xi \), then \( \Psi \) takes some neighborhood of \( \Delta_\varepsilon \times \{0\} \subset \Delta_\varepsilon \times \mathbb{R} \) diffeomorphically onto an open neighborhood of \( \Delta \) in \( bD \). As \( \Delta_\varepsilon \times \{0\} \) has arbitrarily small neighborhoods diffeomorphic to a ball, we are done.

If \( \Sigma \) is a two-dimensional submanifold of class \( \mathcal{C}^2 \) in a two-dimensional complex manifold, \( \mathcal{M} \), and if \( \Sigma \) is totally real in a deleted neighborhood of the point \( p_0 \in \Sigma \) but the tangent space \( T_{p_0} \Sigma \) is complex, then it is possible to choose local holomorphic coordinates \((z, w)\) in \( \mathcal{M} \) near \( p_0 \) so that \( p_0 \) is the origin and so that near \( 0, \Sigma \) is given by an equation

\[
w = z\bar{z} + \lambda(z^2 + \bar{z}^2) + o(z^2).
\]

The number \( \lambda \) is uniquely determined, and \( p_0 \) is said to be an **elliptic point** if \( \lambda \in \left[0, \frac{1}{2}\right) \), a **parabolic point** if \( \lambda = \frac{1}{2} \), and a **hyperbolic point** if \( \lambda \in \left(\frac{1}{2}, \infty\right) \). This classification was introduced by Bishop [3].

Near an elliptic point, the local hull of holomorphy of \( \Sigma \) contains a real hypersurface with boundary [11]. We shall see below that near a hyperbolic point, \( \Sigma \) is polynomially convex.

**III. Corollary.** Let \( \Delta \) be a compact two-dimensional disc of class \( \mathcal{C}^2 \) in \( bD \), \( D \) a strongly pseudoconvex domain with \( \mathcal{C}^2 \) boundary in a two-dimensional Stein manifold \( \mathcal{M} \). If \( \Delta \) is totally real except at a finite number of points each of which is a hyperbolic point, then \( \Delta \) is removable and so \( \mathcal{O}(D) \)-convex.

The crux of the corollary, given what has gone before, is to see that \( \Delta \) is the zero locus of a continuous nonnegative plurisubharmonic function.

**IV. Lemma.** With \( \Delta \) as in Corollary III, there is a neighborhood \( \Omega \) of \( \Delta \) in \( \mathcal{M} \) on which is defined a nonnegative plurisubharmonic function \( \varphi \) such that \( \Delta = \{z \in \Omega : \varphi(z) = 0\} \).
Proof. Fix a smooth Hermitian metric on \( M \), and denote by \( \text{dist}_M(p, q) \) the distance from \( p \) to \( q \), \( p, q \in M \), in this metric.

Let \( p_1, \ldots, p_n \) be the points in \( A \) at which the tangent to \( A \) is complex, and let \( A' = A \setminus \{p_1, \ldots, p_n\} \), a totally real closed submanifold with boundary of \( M \setminus \{p_1, \ldots, p_n\} \). The function \( \varphi(z) = \text{dist}_M(z, A) \) is of class \( C^2 \) in a small neighborhood \( \Omega_1 \) of \( A \) in \( M \) and is strongly plurisubharmonic on a smaller open set \( \Omega_2 \subset \Omega_1 \) containing \( A' \). It is not plurisubharmonic near the points \( p_j \), so we will modify it there. The modification will be local near each \( p_j \). Thus, we may fix a \( p_j \) and work in local coordinates in which \( p_j = 0 \).

According to Theorem VI below the set \( A \cap rB_2 \) is polynomially convex for each sufficiently small \( r > 0 \). Hence, there is a smooth plurisubharmonic function \( \varphi: \mathbb{C}^2 \to [0, \infty) \) the zero locus of which is the set \( A \cap rB_2 \). If \( r > 0 \) is sufficiently small, \( A \cap rB_2 \) is totally real except at 0.

Choose numbers \( 0 < r'' < r' < r \), and let \( \tau \) be a smooth function satisfying

(i) \( \text{supp} \tau \subset \subset r'B_2 \), and
(ii) \( \tau < 0 \) near \( A \cap r''B_2 \).

If \( \tau \) is sufficiently small in the \( C^2 \)-norm, then the function \( \varphi_2 = \varphi_1 + \tau \) is still strongly plurisubharmonic in a neighborhood \( \Omega_2 \) of \( A \cap (rB_2 \setminus r'B_2) \). The nonnegative function \( \varphi_3 = \max(\varphi_2, 0) \) is continuous and plurisubharmonic on \( \Omega_2 \) and vanishes on a neighborhood of \( A \cap (r'B_2) \). Thus, we can extend \( \varphi_3 \) as a plurisubharmonic function on a neighborhood \( \Omega_4 \) of \( A \cap rB_2 \) by taking the extension to be zero in a neighborhood of \( A \cap r''B_2 \).

Recall that \( \varphi \) is a nonnegative plurisubharmonic function on \( \mathbb{C}^2 \) vanishing on \( A \cap rB_2 \). The function \( \varphi_4 = \varphi_3 + \varphi \) is a nonnegative continuous plurisubharmonic function on \( \Omega_4 \) that has \( A \cap rB_2 \) as zero locus. Moreover, on \( \Omega_4 \cap (rB_2 \setminus r'B_2) \), \( \varphi_3 \) agrees with \( \varphi_1 \), so on this set, \( \varphi_4 = \varphi_1 + \varphi \). This function is smooth and plurisubharmonic on \( \Omega_4 \cap (rB_2 \setminus r'B_2) \).

Finally, we patch \( \varphi_1 \) and \( \varphi_4 \) to obtain

\[ \varphi = h\varphi_1 + (1-h)\varphi_4 \]

where \( h: \mathbb{C}^2 \to [0, 1] \) is a smooth function that equals 0 on \( r'B_2 \) and equals 1 outside \( rB_2 \). Since both \( \varphi_1 \) and \( \varphi_4 \) are smooth and strongly plurisubharmonic along \( A \cap (rB_2 \setminus r'B_2) \) where the patching occurs, the resulting function \( \varphi \) is also strongly plurisubharmonic there. The function \( \varphi \) is the required modification of \( \varphi_1 \) near \( p_j \).

We repeat this procedure for each \( p_j \) and obtain finally the plurisubharmonic function we seek.

This completes the proof.

There is an extension of the results obtained above for discs to certain more general sets. A subset \( X \) of an \( n \)-dimensional manifold is called cellular if for every
open neighborhood $U$ of $X$ there is an open set $V$ homeomorphic to $\mathbb{R}^n$ with $X \subset V \subset U$. The proof of Theorem I implies that if $D$ is a strongly pseudoconvex domain in a two-dimensional Stein manifold, $bD$ of class $C^2$, then every compact set $X \subset bD$ that is cellular and that is $\emptyset(\Omega)$-convex for some neighborhood $\Omega$ of $X$ is removable.

The final application we make of Theorem I is to certain Cantor sets. By a Cantor set we understand a compact perfect subset of some Euclidean space. Alternatively, a Cantor set is a subset of a Euclidean space that is homeomorphic to the usual Cantor middle-third set constructed in the interval $[0, 1]$. For these matters one can consult the topology text [10].

The following question has not been settled yet: Is every Cantor set in $bB_n$ removable? We shall exhibit below a class of removable Cantor sets.

A Cantor set $E \subset \mathbb{R}^n$ is called tame if there is a homeomorphism of $\mathbb{R}^n$ onto itself that carries $E$ onto the usual middle-third Cantor set contained in a coordinate line of $\mathbb{R}^n$. In [2], Bing gave a condition under which a Cantor set $E$ in $\mathbb{R}^3$ is tame. To state it, we need a definition: Call a set $X \subset \mathbb{R}^3$ 1-LCC (1-locally connected complement) if for each $x \in X$ and for every neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$, $V \subset U$ such that every simple closed curve $\gamma \subset V \setminus X$ is null-homotopic in $U \setminus X$. Bing proves that a 1-LCC Cantor set in $\mathbb{R}^3$ is tame. (Not every Cantor set has this property as is shown by Antoine's necklace [17].) Alternatively, it suffices for each point $x \in X$ to have a neighborhood $V$ that is homeomorphic to $\mathbb{R}^3$ and that has the property that $bV \cap X = \emptyset$.

V. Corollary. Let $D$ be a strongly pseudoconvex domain in a two-dimensional Stein manifold $\mathcal{M}$, $bD$ of class $C^2$. Suppose $E \subset bD$ is a Cantor set such that

A) There is a Stein neighborhood $\Omega$ of $E$ in $\mathcal{M}$ such that $E$ is $\emptyset(\Omega)$-convex.

B) If $p \in E$, there is a neighborhood $V_p$ of $p$ in $bD$ that is homeomorphic to $\mathbb{R}^3$ and that contains a compact neighborhood $E_p$ of $p$ in $E$ that is a tamely embedded Cantor set in $\mathbb{R}^3 = V_p$.

Then $E$ is removable.

Proof. Let $p_0 \in E$. The hypothesis B) yields a homeomorphism $\Phi : V_{p_0} \to \mathbb{R}^3$ that carries $E_{p_0}$ onto the standard Cantor middle-third set $K$ contained in the $x_1$-axis of $\mathbb{R}^3$. Let $p_0$ correspond to $x_0 \in K$ under $\Phi$. There are open Euclidean balls $B$ in $\mathbb{R}^3$ that contain $x_0$ and that satisfy $bB \cap \Phi(E \cap V_{p_0}) = \emptyset$. It follows that in $\Phi^{-1}(B)$ there are 2-spheres of class $C^2$ disjoint from $E$ that bound 3-balls containing $p_0$.

We turn finally to the proof of the result we have used above to the effect that hyperbolic points have polynomially convex neighborhoods.

VI. Theorem. Let $\Sigma$ be the surface in $\mathbb{C}^2$ given by the equation

$$z_2 = z_1 \bar{z}_1 + \gamma(z_1^2 + |z_1|^2) + F(z_1)$$
A new class of polynomially convex sets

with $\gamma > \frac{1}{2}$ and where $F$ is of class $C^2$ and satisfies $F(z_1) = o(z_1^2)$, $z_1 \to 0$. If $r > 0$ is sufficiently small then $\Sigma \cap r \overline{B}_2$ is polynomially convex and satisfies $P(\Sigma \cap r \overline{B}_2) = \emptyset(\Sigma \cap r \overline{B}_2)$.

Here for a compact set $X$ in $\mathbb{C}^2$, $\mathscr{C}(X)$ denotes the space of continuous $\mathbb{C}$-valued functions on $X$, and $P(X)$ is the subspace consisting of those functions that can be approximated uniformly by polynomials.

We are indebted to Sidney Webster for drawing this problem to our attention.

To prove the result, we introduce the surface $\Sigma$ with equation

$$z_2 = z_1 \bar{z}_1 + \gamma (z_1^2 + \bar{z}_1^2)$$

and the totally real two-dimensional planes

$$V_1 = \{(\zeta, \bar{\zeta}) \in \mathbb{C}^2 : \zeta \in \mathbb{C}\}$$

and

$$V_2 = \left\{ \left(\zeta, -\frac{1}{\gamma} \bar{\zeta} - \bar{\zeta} \right) : \zeta \in \mathbb{C}\right\}.$$

In addition, we define a map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\Phi(z_1, z_2) = (z_1, z_1 z_2 + \gamma (z_1^2 + \bar{z}_2^2)).$$

As Dan Burns pointed out to us in a slightly different context, $\Phi$ is a proper map from $\mathbb{C}^2$ onto itself of multiplicity two, which carries $V_1$ and $V_2$ injectively onto $\Sigma$.

Proof. We propose to find surfaces $S_1$ and $S_2$ that osculate $V_1$ and $V_2$, respectively, at the origin, and that satisfy

$$\Phi^{-1}(\Sigma) = S_1 \cup S_2.$$

To this end, it is convenient to introduce $\varphi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\varphi(z_1, z_2) = z_1 \bar{z}_1 + \gamma (z_1^2 + \bar{z}_2^2) + F(z_1) - z_2$$

so that $\varphi = 0$ defines $\Sigma$.

We construct $S_1$ as follows. The surface $S_1$ is to be of the form

$$S_1 = \{(\zeta, \bar{\zeta} + f(\zeta)) : \zeta \in \mathbb{C}, \ z$ small$\}$$

with $f(\zeta) = o(\zeta)$, $\zeta \to 0$. The condition that $\Phi(S_1) \subset \Sigma$ is that $\varphi(\Phi(\zeta, \bar{\zeta} + f(\zeta))) = 0$, which, when written out explicitly, is the quadratic in $f$

$$\gamma f(\zeta)^2 + (2\gamma \frac{\bar{\zeta}}{\zeta} + \zeta) f(\zeta) - F(\zeta) = 0.$$

Solve this for $f$ using the quadratic formula to find

$$f = \frac{1}{2\gamma} \left\{- (\zeta + 2\gamma \bar{\zeta}) \pm \sqrt{(\zeta + 2\gamma \bar{\zeta})^2 + 4\gamma F}\right\}. (1)$$
We want \( f(0) = o(\zeta), \zeta \to 0 \), so we choose the plus sign in (1). With this choice of sign, \( df(0) = 0 \), and \( f \) is of class \( C^1 \) everywhere except possibly at zero. We show that \( f \) is of class \( C^1 \) at 0 as follows. Write

\[
\sqrt{(\zeta + 2\gamma \xi)^2 + 4\gamma F(\zeta)} = (\zeta + 2\gamma \xi) + \hat{f}(\xi).
\]

The function \( \hat{f} \) satisfies the equation

\[
\hat{f}^2 + 2(\zeta + 2\gamma \xi) \hat{f} = 4\gamma F.
\]

Differentiate this with respect to \( \zeta \) to get

(2)

\[
(\zeta + 2\gamma \xi + \hat{f}) \hat{f}_\xi = 2\gamma F_\xi - \hat{f}.
\]

We have \( F(\xi) = o(\xi^2) \) and \( \hat{f}(\xi) = o(\xi), \xi \to 0 \). Consequently, by (2) we see that \( \hat{f}_\xi(\xi) = o(1), \xi \to 0 \). Similarly, \( f_\xi(\xi) = o(1), \xi \to 0 \). It follows that \( f \) is of class \( C^1 \), as we wished to see.

Thus, with \( f \) determined in this way, the surface \( S_1 \) is of class \( C^1 \), and near the origin, it is a small perturbation of the totally real plane \( V_1 \). As such, it is totally real. We know that totally real surfaces are locally polynomially convex and that we have, locally at least, polynomial approximation on such surfaces. Thus, for small \( \delta > 0 \), if

\[
S_1(\delta) = \{(\zeta, \xi + f(\xi)) : |\xi| < \delta\}
\]

with \( f \) given by (1), then \( S_1(\delta) \) is polynomially convex and satisfies \( P(S_1(\delta)) = \Phi(S_1(\delta)) \).

We now make a similar analysis for the surface \( S_2 \). It is to be of the form

\[
S_2 = \left\{ \left(\zeta, \frac{1}{\gamma} \xi - \zeta + g(\xi) \right) : \xi \in \mathbb{C} \right\}
\]

with \( g(\xi) = o(\xi), \xi \to 0 \). The condition that \( \Phi(S_2) \subseteq \Sigma \) is found to be expressed by the equation

\[
\gamma g^2(\xi) - (2\gamma \xi + \xi) g(\xi) - F(\xi) = 0.
\]

The quadratic formula yields

\[
g = \frac{1}{2\gamma} \left\{ (2\gamma \xi + \xi) \pm \sqrt{(2\gamma \xi + \xi)^2 + 4\gamma F} \right\}.
\]

We take the minus sign to obtain \( g(\xi) = o(\xi), \xi \to 0 \). The function \( g \) is continuous, is of class \( C^1 \) except possibly at the origin and satisfies \( dg(0) = 0 \). We claim that \( g \) is of class \( C^1 \) at 0 as well. Write

\[
\sqrt{(2\gamma \xi + \xi)^2 + 4\gamma F} = (2\gamma \xi + \xi) + \bar{g}.
\]
Then the function \( \tilde{g} \) satisfies the quadratic equation
\[
\tilde{g}^2 + 2(2\gamma^2 + \zeta)\tilde{g} = 4\gamma F.
\]

We differentiate this with respect to \( \zeta \) and with respect to \( \zeta \) and find that \( \tilde{g}_\zeta(\zeta) \) and \( \tilde{g}_\zeta^2(\zeta) \) are both \( o(1) \), \( \zeta \to 0 \), whence \( \tilde{g} \) is of class \( \mathcal{C}^1 \) at 0. Thus, \( g \) is of class \( \mathcal{C}^1 \) as we wished.

If for small \( \delta > 0 \),
\[
S_\delta(\delta) = \{ \xi - \frac{1}{\gamma} \zeta - \xi + g(\xi) : |\zeta| \leq \delta \},
\]
then \( S_\delta(\delta) \) is polynomially convex and satisfies \( P(S_\delta(\delta)) = \mathcal{C}(S_\delta(\delta)) \).

Notice that if \( \delta \) is small, then \( S_\delta(\delta) \cap S_\delta(\delta) = \{0\} \) as follows from the assumption that \( \gamma > 1/2 \).

Now fix \( \epsilon > 0 \), define \( \psi : \mathbb{C}^2 \to \mathbb{C} \) by
\[
\psi(z_1, z_2) = \frac{1}{4} (z_1^2 - z_2^2) + \epsilon z_1 z_2,
\]
and consider the sets \( \psi(S_\delta(\delta)) \) and \( \psi(S_\delta(\delta)) \). For small \( \zeta \), we have, taking \( \zeta = \xi + i\eta \),
\[
\psi(\zeta, \xi + f(\zeta)) = i\xi\eta + \epsilon(\xi^2 + \eta^2) + o(\xi^2),
\]
for \( f(\zeta) = o(\zeta), \zeta \to 0 \). The equation (3) implies that for small \( \delta \), \( \psi(S_\delta(\delta)) \) is contained in the cone
\[
\{ u + iv : |v| \leq Cu \}
\]
in the \( u + iv = \psi(z_1, z_2) \)-plane for some \( C > 0 \). We also have
\[
\psi \left( \zeta, -\frac{1}{\gamma} \xi - \xi + g(\zeta) \right) = \frac{1}{4} \left[ \xi^2 - \left( \frac{1}{\gamma} \xi + \xi \right)^2 + \epsilon \zeta \left( -\frac{1}{\gamma} \xi - \xi + g(\xi) \right) \right].
\]
As \( g \) is \( o(\zeta), \zeta \to 0 \), we can write this as
\[
\psi \left( \zeta, -\frac{1}{\gamma} \xi - \xi + g(\zeta) \right) = \frac{1}{4} \left[ \xi^2 - \left( \frac{1}{\gamma} \xi + \xi \right)^2 \right] - \epsilon \zeta \left( \frac{1}{\gamma} \xi + \xi \right) + o(\zeta^2)
\]
\[
= \frac{1}{4} \left[ \left( 1 - \frac{1}{\gamma^2} \right) \xi^2 - \left( \frac{2}{\gamma} - \frac{1}{\gamma^2} \right) \eta^2 + \frac{2i}{\gamma^2} \xi \eta \right] - \epsilon \zeta \left( \frac{1}{\gamma} \xi + \xi \right) + o(\zeta^2).
\]
The coefficient of \( \xi^2 \) is plainly negative, and so is the coefficient of \( \eta^2 \), because \( \gamma > 1/2 \). Thus, provided \( \epsilon \) and \( \delta \) are sufficiently small, we have that \( \text{Re} \psi (\zeta - \frac{1}{\gamma} \xi - \xi + g(\xi)) \leq 0 \)
and that
\[
\left| \text{Im} \psi \left( \zeta - \frac{1}{\gamma} \xi - \xi + g(\xi) \right) \right| \leq -c \text{Re} \psi \left( \zeta - \frac{1}{\gamma} \xi - \xi + g(\xi) \right).
\]
That is, the set \( \psi(S_2(\delta)) \) is contained in a cone with vertex the origin and lying otherwise in the open left half of the \((u+iv)\)-plane.

Notice that \( \psi^{-1}(0) \cap S_1(\delta) \cap S_2(\delta) \) is the polynomially convex set consisting of the origin in \( \mathbb{C}^2 \).

Thus, a theorem of Kallin [9; 19, Lemma 29.21] implies that, provided \( \delta \) is small, the union \( S_1(\delta) \cup S_2(\delta) \) is polynomially convex.

Moreover, as \( P(S_1(\delta)) = \mathcal{C}(S_1(\delta)) \) and \( P(S_2(\delta)) = \mathcal{C}(S_2(\delta)) \), again provided \( \delta \) is small, it follows that

\[
P(S_1(\delta) \cup S_2(\delta)) = \mathcal{C}(S_1(\delta) \cup S_2(\delta)).
\]

It not, there is a nonzero measure \( \mu \) on \( S_1(\delta) \cup S_2(\delta) \) such that \( \mu(P) = 0 \) for every holomorphic polynomial \( P \). Since \( \psi \) takes \( S_1(\delta) \) and \( S_2(\delta) \) into disjoint sectors, Mergelyan's theorem provides a sequence \( \{P_j\}_{j=1,2,...} \) of polynomials such that \( |P_j \circ \psi| \equiv 1 \) on \( S_1(\delta) \cup S_2(\delta) \) and \( \{P_j \circ \psi\}_{j=1,2,...} \) converges pointwise to one on \( S_1(\delta) \) and to zero on \( S_2(\delta) \setminus \{0\} \). If \( P \) is any polynomial, we have

\[
0 = \int (P_j \circ \psi) P d\mu
\]

for all \( j \). By the dominated convergence theorem we may conclude that \( \mu_1 \), the restriction of \( \mu \) to \( S_1(\delta) \), satisfies \( \mu_1(P) = 0 \) for all polynomials \( P \). As \( P(S_1(\delta)) = \mathcal{C}(S_1(\delta)) \) it follows that \( \mu_1 \) is the zero measure. In the same way, \( \mu_2 \), the restriction of \( \mu \) to \( S_2(\delta) \), is the zero measure, and thus \( \mu = 0 \).

The equality (4) follows. It implies that every compact subset of \( S_1(\delta) \cup S_2(\delta) \) is polynomially convex.

If now \( E \subset \Sigma \) is a compact neighborhood in \( \Sigma \) of the origin chosen so small that \( \Phi^{-1}(E) \subset S_1(\delta) \cup S_2(\delta) \), the set \( \Phi^{-1}(E) \) is polynomially convex. But as \( \Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) is a proper holomorphic mapping, the polynomial convexity of \( \Phi^{-1}(E) \) implies that of \( E \). Let \( z_0 \in \mathbb{C}^2 \setminus E \). As \( \Phi^{-1}(E) \) is polynomially convex and disjoint from the set \( \Phi^{-1}(z_0) = E_0 \), which consists of one point or of two points since \( \Phi \) is a two-sheeted branched covering, the set \( E_0 \cup \Phi^{-1}(E) \) is polynomially convex, so there is a polynomial \( P \) on \( \mathbb{C}^2 \) with \( P = 1 \) on \( E_0 \) and \( |P| < 1/2 \) on \( \Phi^{-1}(E) \). The theory of analytic covers [4] yields a polynomial equation

\[
P^2 + (p \circ \Phi) P + q \circ \Phi = 0
\]

for some choice of entire functions, actually polynomials, \( p \) and \( q \) on \( \mathbb{C}^2 \). We have then that \( q \) is holomorphic on \( \mathbb{C}^2 \), \( q(z_0) = 1 \) and \( |q| < 1/4 \) on \( E_0 \) and \( E \). Thus, \( E \) is polynomially convex.

Finally, we must show that \( P(E) = \mathcal{C}(E) \). Granted that \( E \) is polynomially convex, this follows from a result of Wermer's [21]. As Wermer's proof is not simple — of course it covers situations much more general than ours, we offer
A new class of polynomially convex sets

the following short proof. Consider \( h \in C(E) \). The function \( h \circ \Phi \) is in \( C(\Phi^{-1}(E)) \), so there is a sequence \( \{P_j\}_{j=1}^{\infty} \) of polynomials on \( \mathbb{C}^2 \) that converges uniformly on \( \Phi^{-1}(E) \) to \( h \circ \Phi \), since \( C(S_1(\delta) \cup S_2(\delta)) = P(S_1(\delta) \cup S_2(\delta)) \) and \( \Phi^{-1}(E) \subset S_1(\delta) \cup S_2(\delta) \). The function \( Q_j(z) = \frac{1}{2} (P_j(z') + P_j(z'')) \) for all \( z \in C^2 \) such that \( \Phi^{-1}(z) \) consists of exactly two points \( z' \) and \( z'' \) is holomorphic on \( C^2 \) — again we invoke the theory of analytic covers. The sequence \( \{Q_j\}_{j=1}^{\infty} \) converges uniformly on \( E \) to \( h \).

This completes the proof.

References

2. Bing, R. H., Tame Cantor sets in \( E^3 \), Pacific J. Math. 11 (1961), 435--446.
16. Lupaccioli, G. and Stout, E. L., Removable singularities for \( \overline{\partial} \), to appear in the proceedings of the Mittag-Leffler special year in several complex variables.


Received February 6, 1990

F. Forstnerič
E. L. Stout
Department of Mathematics, GN 50
University of Washington
Seattle, Washington 98195
USA