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Intersections of Analytic and Smooth Discs

FRANC FORSTNERIČ

Dedicated to Walter Rudin.

ABSTRACT. If A is an embedded analytic disc in $\mathbf{C}^2$ and M is an embedded smooth disc in $\mathbf{C}^2$ with isolated complex tangents that has the same boundary as A and is totally real near the boundary, then the intersection number $A\cdot M$ is related to the number $I_+(M)$ of positively oriented complex tangents (counted with algebraic multiplicities) by the formula $A\cdot M = I_+(M) - 1$. In particular, if M is totally real or if it only has hyperbolic complex tangents, then the discs A and M must also intersect at an interior point.

Introduction.

Several years ago Vitushkin [14] raised the following question:

Does there exist a bounded strongly pseudoconvex domain $D \subset \mathbf{C}^2$ homeomorphic to the ball and an embedded analytic disc $A \subset \mathbf{C}^2\backslash\overline{D}$, with the boundary of A contained in the boundary of D ?

An embedded analytic disc in $\mathbf{C}^2$ is the image of a smooth embedding $F\colon\overline{\Delta} = \{z \in \mathbf{C}\colon |z| \le 1\} \to \mathbf{C}^2$ that is holomorphic in the open disc Δ.

It is essential to require that D be a topological cell (or at least to be simply connected), for one could otherwise take D to be a suitable tubular neighborhood of any smooth curve in $\mathbf{C}^2$ that bounds an analytic disc.

While it is easy to find such pairs (D, A) in $\mathbf{C}^3$, Vitushkin conjectured that there are no such configurations in $\mathbf{C}^2$. He believed that knowing this may perhaps be useful in solving the well-known Jacobian conjecture.

Later M. Gromov claimed that such a configuration existed. This was communicated to me by N. Sibony and Kruzhilin in May 1991. According to Gromov one should simply find an embedded analytic disc $A \subset \mathbf{C}^2$ and an embedded

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totally real disc $M \subset \mathbf{C}^2$ such that $\overline{A} \cap \overline{M} = bA = bM$, i.e., the two discs have common boundary but they do not intersect at any interior point. Then, by taking D to be a suitable strongly pseudoconvex tubular neighborhood of M and by replacing A by a slightly smaller disc, one would obtain the required configuration.

Recall that a smooth embedded real surface $M \subset \mathbf{C}^2$ (with or without boundary) is said to be *totally real* at a point $p \in M$ if the tangent space T_pM spans $\mathrm{T}_p\mathbf{C}^2$ over the field $\mathbf{C}$. If this fails than T_pM is a complex line in $\mathrm{T}_p\mathbf{C}^2$, and p is said to be a *complex tangent* of M. Recall also [9] that a totally real submanifold $M \subset \mathbf{C}^n$ is the zero set of a strongly plurisubharmonic function $\rho \geq 0$ defined in a neighborhood U of M, with $d\rho \neq 0$ on $U \backslash M$, hence the sublevel sets $\{\rho < \epsilon\}$ for sufficiently small $\epsilon > 0$ are strongly pseudoconvex domains homotopic to M.

It seems that the first example of an embedded analytic disc $A \subset \mathbf{C}^2$ and an embedded totally real disc $M \subset \mathbf{C}^2$ with common boundaries was found by Wermer [17]. It suffices to take $A = \{(z,0): |z| \leq 1\}$ and $M = \{(z, f(z)): |z| \leq 1\}$, where f is a smooth function that vanishes on the circle $|z| = 1$ and such that $\partial f/\partial\overline{z}$ is zero-free; for instance, one may take

$$f(z) = (z\overline{z} - 1)\overline{z}\exp(iz\overline{z}).$$

The problem with this particular example is that the two discs A and M also intersect at the interior point (0,0), so when we fatten M to obtain a strongly pseudoconvex domain D, the complement $D \backslash A$ is an annulus with boundary in D rather than a disc.

At first this just seemed an unfortunate choice of the example. In [5] we showed that this is not so by proving

THEOREM. *An embedded analytic disc A and an embedded totally real disc M in $\mathbf{C}^2$ with common boundary $bA = bM$ must intersect at an interior point.*

We explicitly constructed a deformation of the analytic disc A into a totally real disc $\tilde{A}$, without introducing new intersections with M, such that the two discs glue smoothly along bM into an immersed totally real sphere $S = \tilde{A} \cup M$ in $\mathbf{C}^2$. As it is well known that every totally real embedded n-manifold in $\mathbf{C}^n$ must have Euler number zero [16] while $\chi(S^2) = 2$, we conclude that our sphere must have self-intersections and therefore A and M intersect at an interior point.

Totally real discs are not the only ones that have a Stein neighborhood basis: according to [8] the same is true for discs with isolated *hyperbolic* complex tangents (in the sense of Bishop [3]). Call such discs hyperbolic. It is well known that *elliptic* complex tangents are not allowed since near such points the surface has a nontrivial local envelope of holomorphy, see [3] and [12].

Thus one might hope that the construction suggested by Gromov is possible using a pair of an embedded analytic disc and an embedded hyperbolic disc that intersect transversely along their boundaries but have disjoint interiors.

In this article we show that this is not the case, by proving a formula (1) that relates the intersection number $A \cdot M$ and the number $I_+(M)$ of positive complex tangents of M, counted with appropriate algebraic multiplicities. If M is totally real, then $A \cdot M = -1$ (with an appropriate choice of orientations). If M has m isolated positive hyperbolic complex tangents and no elliptic complex tangents, then $A \cdot M = -m - 1 \leq -1$, hence the two discs intersect at an interior point as well.

Our proof here is substantially simpler than the one in [5].

While we are not able to settle Vitushkin's question at this moment, we show that the existence of such a configuration (with just slightly stronger hypotheses) would have the following interesting consequence (Theorem 2): One would be able to find an embedded analytic disc $A \subset \mathbf{C}^2$ and an embedded disc $M \subset \mathbf{C}^2$ such that

(i) $\overline{M} \cap \overline{A} = bA = bM$,

(ii) M is totally real except at an elliptic complex tangent $p \in M$, and

(iii) the envelope of holomorphy of $\overline{M}$ does not contain A.

Whether or not this is possible in $\mathbf{C}^2$ seems to be another interesting question.

Results.

Let M be an embedded real surface in $\mathbf{C}^2$ and let $p \in M$ be an isolated complex tangent. We recall the definition of *index* of p in M. (See [15] and [7] for details.) Locally near p we orient M coherently with the standard orientation of T_pM as a complex line. Choose an oriented neighborhood $U \subset M$ of p, U homeomorphic to the two-disc, and let $\tau: U \to \mathcal{G}$ be the Gauss map $\tau(q) = T_qM$ into the Grassman manifold of oriented real 2-planes in $\mathbf{C}^2 = \mathbf{R}^4$. It is well-known (see Chern and Spanier [4]) that $\mathcal{G}$ is the product of two spheres $\mathcal{G} = S_1 \times S_2$ such that the set of complex lines in $\mathbf{C}^2$ equals $H_+ \cup H_-$, where $H_+ = \{\pi_1\} \times S_2$ is the set of positively oriented lines and $H_- = \{\pi_2\} \times S_2$ is the set of negatively oriented complex lines in $\mathbf{C}^2$.

DEFINITION. *(Notation as above.) The index $I(p)$ of an isolated complex tangent $p \in M$ is defined to be the local intersection number of $\tau: U \to \mathcal{G}$ with H_+ at the point $\tau(p) \in H_+$.*

There are several equivalent definitions of the index, see [7]. The simplest way to compute $I(p)$ is to write M in suitable local holomorphic coordinates (z, w) near p as a graph $w = f(z)$, with p corresponding to $(0,0)$. The condition that p is an isolated complex tangent of M is equivalent to $z = 0$ being an isolated zero of the function $\overline{\partial} f = \partial f / \partial \overline{z}$; the index $I(p)$ then equals the winding numbers of $\overline{\partial} f$ around the origin. See [15] or [7] for the details. Recall that every elliptic complex tangent $p \in M$ (in the sense of Bishop [3]) has index $+1$, and every hyperbolic complex tangent has index -1. A totally real point has index zero.

Suppose now that the surface M is orientable and choose an orientation on M. Denote by $\tau: M \to \mathcal{G}$ the induced Gauss map. If M has boundary bM, we shall assume that M is totally real along bM. Then we have well-defined global intersection numbers $I_+(M)$ resp. $I_-(M)$ of the map τ with submanifolds H_+

resp. H_- of $\mathcal{G}$. Clearly $I_+(M)$ is just the sum of the indices of all positively oriented complex tangents $p \in M$, and similarly $I_-(M)$ is the sum of indices of negatively oriented complex tangents. Their sum

$$I(M) = I_+(M) + I_-(M)$$

is called the *index* of M, and their difference

$$I(bM; M) = I_+(M) - I_-(M)$$

is the *Maslov index* of the boundary bM in M, see [7] and the definition below. For a closed surface $M \subset \mathbf{C}^2$ we have $I_+(M) = I_-(M) = I(M)/2$ [6], [7].

It is obvious how to extend these definitions to immersions of a given surface M into $\mathbf{C}^2$. If $F_t: M \to \mathbf{C}^2$, $0 \le t \le 1$, is a regular homotopy of immersions (or embeddings) such that each F_t is totally real near the boundary of M (this condition is vacuous if $bM = \emptyset$), then the numbers I_+, I_-, I are the same for each immersion in the family.

Suppose now that A and M are embedded closed discs in $\mathbf{C}^2$ of class $\mathcal{C}^1$ with the common boundary $bA = bM$, and such that the interior of A is complex-analytic while M is totally real near bM. (This implies that A and M intersect transversely along their joint boundary.) We choose on A the canonical orientation induced by its complex structure; this uniquely determines an orientation on M such that the pairs (A, bA) and $(M, bM) = (M, bA)$ are oriented manifolds with boundary. Denote by $A \cdot M$ the oriented intersection number of A and M; this number is invariant under smooth deformations of A and M for which the two discs intersect transversely along $bA = bM$ at each step of the deformation, but the interior of each disc does not cross the boundary.

THEOREM 1. *With the notation as above we have*

$$A \cdot M = I_+(M) - 1. \tag{1}$$

COROLLARY. *If the disc M is hyperbolic in the sense that it only has isolated complex tangents p of index $I(p) \le 0$, then $A \cdot M \le -1$, hence A and M intersect at an interior point. In particular, we have $A \cdot M = -1$ whenever the disc M is totally real.*

REMARK. There is an apparent lack of symmetry in the formula (1) above since $I_-(M)$ does not appear. We shall see in the proof of Theorem 1 that

$$A \cdot M = I_-(M) + I_+(\tilde{A}) - 1,$$

where $\tilde{A}$ is a disc obtained by gluing A smoothly with a thin collar of M near bA.

The Corollary implies that one can not construct a pair (D, A) of Vitushkin's type using Gromov's suggestion. Although we do not know at present what is the answer to Vitushkin's question, the following result shows that it is rather unlikely for a configuration (D, A) of the required type to exist in $\mathbf{C}^2$.

THEOREM 2. *Let $D \subset \mathbf{C}^2$ be a domain whose closure $\overline{D}$ is diffeomorphic to the closed four-ball, and let $A \subset \mathbf{C}^2 \backslash D$ be an embedded analytic disc of class C^k up to the boundary ($k \geq 1$) intersecting $\overline{D}$ transversely along the curve $bA \subset bD$.*

Then there exists an embedded C^∞ disc $M \subset D$ of class C^k up to the boundary, with $bM = bA$, such that M is totally real except at one elliptic complex tangent $p \in M$. Moreover, we can choose M such that locally near p it is given in suitable affine complex coordinates by $w = z\overline{z}$.

QUESTION. *If A and M are as in the conclusion of Theorem 2, does the envelope of holomorphy of $\overline{M}$ necessarily contain A ?*

If the answer is positive, then the domain D in Theorem 2 cannot be strongly pseudoconvex for the obvious reason that the envelope of holomorphy of $\overline{M}$ would then be contained in $\overline{D}$. Thus a positive answer to this question proves Vitushkin's conjecture that there are no such configurations in $\mathbf{C}^2$.

Recall that near the elliptic complex tangent $p \in M$ the local envelope of holomorphy consists of a one-parameter family of analytic discs with boundaries in M [12], [3]. One would expect that this family of discs continues all the way to the disc A. Results of this type have been proved by Bedford and Gaveau [1], Bedford and Klingenberg [2], and Gromov [11] in the case when M is a part of a 'generic' smooth 2-sphere contained in the boundary of a strongly pseudoconvex domain.

Our last result is about the existence of totally real discs in $\mathbf{C}^2$ with prescribed one-jet at the boundary. Given a smooth oriented simple closed curve $\gamma \subset \mathbf{C}^2$ and a pair of smooth vector fields X, Y along γ such that $X(p)$ and $Y(p)$ are $\mathbf{C}$-linearly independent at each $p \in \gamma$, we define

DEFINITION. *The Maslov index $I(X,Y)$ equals the winding number of the function $p \rightarrow \det(X(p), Y(p))$ as p traces γ once in the positive direction.*

Although the sign of $I(X,Y)$ depends on the orientation of γ, the condition $I(X,Y) = 0$ is independent of the orientation. Notice that the two-by-two determinant above is nonzero at p precisely when the two vectors are complex-linearly independent at p.

If $M \subset \mathbf{C}^2$ is an embedded disc with boundary $bM = \gamma$, we let X be the unit tangent to γ and let Y be the unit inward radial tangent vector field to M along γ. If M is a totally real disc, a simple argument with winding numbers shows that $I(X,Y) = 0$. This also follows from the formula $I(X,Y) = I_+(M) - I_-(M)$ proved in [7]. The converse of this is also true, and it may be useful in constructions of totally real discs:

PROPOSITION 3. *If $\gamma \subset \mathbf{C}^2$ is a smooth simple closed curve with tangent vector field X, and if Y is a smooth vector field along γ such that $I(X,Y) = 0$ (this requires that X and Y are $\mathbf{C}$-independent), then there exists a smooth embedded totally real disc $M \subset \mathbf{C}^2$ such that $bM = \gamma$ and Y is the inner radial tangent vector field to M along γ.*

Proof of Theorem 1.

Step 1: Reduction to the case when A is analytic past bA.

Let X be a vector field along bA that is tangent to A and points towards the interior of A. We extend X to a vector field on $\mathbf{C}^2$ that is smooth on $\mathbf{C}^2\backslash bA$ and is supported in a small neighborhood U of bA. We now flow the disc M for a short time $t > 0$ in the direction of X to obtain a new disc $\tilde{M}$ with boundary $b\tilde{M}$ contained in the interior of A. If $t > 0$ is sufficiently small, we do not introduce any new intersections of A and M and we do not affect the indices $I_{\pm}(M)$. Thus we may replace M by $\tilde{M}$ and assume that A is contained in a larger analytic disc A_0. After an additional small perturbation of M we may also assume that $bM = bA$ is a smooth real-analytic curve.

Step 2: Reduction to the case when M is a graph.

Denote by $\Delta(r) = \{z \in \mathbf{C}: |z| < r\}$ the open unit disc of radius r. There is a biholomorphic mapping $\Phi: U_\epsilon \subset \mathbf{C}^2 \to V \subset \mathbf{C}^2$ from a polydisc $U_\epsilon = \Delta(1+\epsilon)\times\Delta(1)$ onto an open neighborhood V of A such that $\Phi(\Delta(1)\times\{0\}) = A$ and $\Phi(T\times\{0\}) = bA = bM$, where $T = b\Delta(1)$ is the unit circle.

Let $M_0 = \Phi^{-1}(M\cap V)$. If the neighborhood V is chosen sufficiently small, then M_0 is a totally real annular region in U_ϵ, with one of its boundary components equal to $T\times\{0\}$. Moreover, its intersection with every real 3-plane $\Pi_\theta = \mathbf{R}e^{i\theta}\times\mathbf{C}$ is a disjoint union of two arcs that project one-to-one onto the plane $z = 0$.

The tangent space to M_0 at $(e^{i\theta}, 0)$ is spanned by vectors $X(\theta) = (ie^{i\theta}, 0)$ and $Y(\theta) = (a(\theta)e^{i\theta}, b(\theta))$, where $a(\theta)$ is real-valued and $b(\theta) \neq 0$. We can assume that $|b(\theta)| = 1$ for all θ.

Choose a smooth function $k \geq 0$ on $[0,\infty) \subset \mathbf{R}$ such that $k'(0) + a(\theta) > 0$ for all θ, $\operatorname{supp} k \subset [0, 1/2]$, and $\max k < \epsilon$. Denote by $\Psi_t: \mathbf{C}^2 \to \mathbf{C}^2$ the transformation

$$\Psi_t(z,w) = \big(z(1+tk(|w|)), w\big), \qquad 0 \leq t \leq 1.$$

Then $M_t = \Psi_t(M_0)$ is a smooth deformation of M_0 within the polydisc U_ϵ such that the tangent space to the surface M_1 at $(e^{i\theta}, 0) \in bM_1$ is spanned by the vectors $X(\theta)$ and $Y_1(\theta) = \big(a_1(\theta)e^{i\theta}, b_1(\theta)\big)$, where $a_1(\theta) > 0$ for all θ. Moreover, M_t is totally real near the circle $T\times\{0\} \subset bM_t$ for each t and it coincides with M_0 near the boundary of U_ϵ.

The condition $a_1(\theta) > 0$ implies that near $T\times\{0\}$, M_1 is a totally real graph over the annulus $A_\delta = \Delta(1+\delta)\backslash\Delta(1)$ for a suitable $\delta > 0$. Denote this part of M_1 by N. Let $\gamma \subset M_1$ be the curve that projects onto the circle $|z| = 1+\delta$, so $bN = \gamma\cup(T\times\{0\})$.

Step 3: Gluing M with A.

Now we choose a smooth function $g: \overline{\Delta}(1+\delta) \to \mathbf{C}$ such that its graph

$$K = \{(z, g(z)): |z| \leq 1+\delta\}$$

glues smoothly with M_1 along the curve γ, i.e., the set $M' = (M_1\backslash N)\cup K$ is a smooth disc in U_ϵ. We leave out the obvious details of the construction of g.

Now we go back to the neighborhood V of A. Set

$$\tilde{A} = \Phi(K), \qquad \tilde{M} = (M \backslash V) \cup \Phi(M_1 \backslash N), \qquad \lambda = \Phi(\gamma).$$

By construction, $\tilde{M}$ and $\tilde{A}$ are smoothly embedded discs in $\mathbf{C}^2$ that are smoothly glued along their joint boundary λ into an immersed 2-sphere $S = \tilde{M} \cup \tilde{A}$ in $\mathbf{C}^2$. By construction we have $A \cdot M = \tilde{A} \cdot \tilde{M}$ and $I_\pm(\tilde{M}) = I_\pm(M)$ since the modification of M into $\tilde{M}$ was totally real near the boundary. After a generic small perturbation we may assume that S only has isolated complex tangents.

Step 4: An application of the index formula.

If S is a smoothly immersed closed oriented real 2-surface in $\mathbf{C}^2$ with isolated complex tangents and with transverse self-intersections, then we have the *index formula*

$$I(S) = \chi(S) - 2d(S),$$

where $I(S) = I_+(S) + I_-(S)$ is the total index of S (the sum of indices of all complex tangents), $\chi(S)$ is the Euler number, and $d(S)$ is the (Whitney's) oriented self-intersection number of S. (See [15], [7], and [6].) If S is non-orientable, the formula above holds modulo 2.

We apply this to our immersed sphere $S = \tilde{M} \cup \tilde{A}$. First we recall that

$$I_+(\tilde{M}) - I_-(\tilde{M}) = I_+(\tilde{A}) - I_-(\tilde{A})$$

since both numbers equal the Maslov index of the curve λ in S. Now comes the important point: Since K is a graph over the z-axis, all complex tangents of K and therefore of $\tilde{A} = \Phi(K)$ are positively oriented, so $I_-(\tilde{A}) = 0$. Using the last formula we thus get

$$I(S) = I_+(\tilde{M}) + I_-(\tilde{M}) + I_+(\tilde{A}) = 2I_+(\tilde{M}) = 2I_+(M).$$

Inserting this into the index formula and noting that $\chi(S) = 2$ we conclude

$$I_+(M) = 1 - d(S) = 1 + \tilde{A} \cdot \tilde{M} = 1 + A \cdot M.$$

The change of sign is due to the fact that the chosen orientations on $\tilde{A}$ and $\tilde{M}$ do not add up to an orientation on S.

This completes the proof of Theorem 1.

PROOF OF THEOREM 2.

First we prove that there exist a smooth embedded disc $M_0 \subset D \cup bA$ intersecting bD transversely along the curve γ.

Let S^4 be the standard one point compactification of $\mathbf{C}^2 = \mathbf{R}^4$ into a four-sphere. We represent S^4 as the union $B_+ \cup B_-$ of two closed four-balls that intersect in a 3-sphere (the equator). Since $\overline{D} \subset S^4$ is diffeomorphic to $B_+ \subset S^4$, Theorem 3.1 in [10, p.185] implies that there is a diffeomorphism $\Psi: S^4 \to S^4$ carying $\overline{D}$ onto B_+, hence Ψ carries the complement $S^4 \backslash D$ onto B_-.

Let $\tau: S^4 \to S^4$ be the reflection about the equator S^3 that interchanges B_+ with B_-. Then $M_0 = \Psi^{-1} \circ \tau \circ \Psi(A) \subset \overline{D}$ is the embedded disc in $\overline{D}$ with the required properties. This justifies our claim.

Recall that the disc A is the image of a $\mathcal{C}^k$ embedding $F: \overline{\Delta} \to A \subset \mathbf{C}^2$ that is holomorphic on the open disc Δ. We can extend F to a diffeomorphic mapping $\Phi: U_\epsilon \to V$ of class $\mathcal{C}^k$ from a polydisc $U_\epsilon = \Delta(1+\epsilon) \times \Delta(1) \subset \mathbf{C}^2$ onto a neighborhood V of A such that Φ is biholomorphic in a neighborhood of the open disc $\Delta(1) \times \{0\}$.

The map $\gamma(\theta) = F(e^{i\theta})$ parametrizes the boundary curve $bA = bM_0$. Let $X(\theta)$ be the unit vector at $\gamma(\theta)$ that is tangent to M_0 and real orthogonal to $\gamma'(\theta)$; among the two possible choices we take the *inner* radial vector to M_0 pointing into D. Also let

$$Y(\theta) = \mathrm{D}\Phi(e^{i\theta}, 0) \cdot (e^{i\theta}, \delta) \in \mathrm{T}_{\gamma(\theta)}\mathbf{C}^2.$$

Choose $\delta > 0$ sufficiently small that $Y(\theta)$ points into D for each θ.

We now deform the vector field X continuously into Y by setting $X_t(\theta) = (1-t)X(\theta) + tY(\theta)$ for $0 \le t \le 1$. By construction $X_t(\theta)$ points into D for each $t \in [0,1]$. We can follow the deformation $\{X_t : t \in [0,1]\}$ by an isotopy $\{M_t : t \in [0,1]\}$ consisting of embedded $\mathcal{C}^k$ discs in $D \cup bM_0$ such that M_t is tangent to X_t along the boundary $bM_t = bM_0$. We leave out the simple details.

Consider now the disc M_1. Its tangent bundle along the boundary is spanned by the vector fields Y and γ'. We claim that the Maslov index $I(bM_1, M_1)$ of the boundary curve bM_1 in M_1 equals one. Recall from [7] that the Maslov index is just the winding number m of the function $\theta \to \det(Y(\theta), \gamma'(\theta))$, that is, $I(bM_1, M_1) = I(Y, \gamma')$. Since these vectors are the images of the vectors $\mu(\theta) = (e^{i\theta}, \delta)$ resp. $\nu(\theta) = (ie^{i\theta}, 0)$ by the derivative $\mathrm{D}\Phi(e^{i\theta}, 0)$, and since the derivative $\mathrm{D}\Phi$ is $\mathbf{C}$-linear and nonsingular at every point of the disc $\overline{\Delta} \times \{0\} \subset \mathbf{C}^2$, it follows that m equals the winding number of the function

$$\theta \to \det(\mu(\theta), \nu(\theta)) = \det \begin{pmatrix} e^{i\theta} & ie^{i\theta} \\ \delta & 0 \end{pmatrix} = -i\delta e^{i\theta}$$

which equals one. This verifies the claim.

The calculation above also shows that the disc M_1 is totally real near the boundary $bM_1 = bA$. After a small generic perturbation of M_1 we can assume that M_1 only has isolated complex tangents.

We choose orientations on A and M_1 as in Theorem 1. Since $A \cdot M_1 = 0$ by construction, Theorem 1 implies $I_+(M_1) = 1$. Also, $I_+(M_1) - I_-(M_1) = I(bM_1, M_1) = 1$, hence $I_-(M_1) = 0$.

The main result of [7] implies that we can find a $\mathcal{C}^0$-small perturbation M of M_1 in D which coincides with M_1 near the boundary of M_1 and such that M has precisely one positive complex tangent of index one (elliptic complex tangent) and no negative complex tangents. This is achieved by cancelling complex tangents of the same sign in pairs as explained in [7]. Moreover, we can specify the local form of M near the elliptic complex tangent.

This completes the proof of Theorem 2.

The proof shows that for every integer m we can find an embedded disc $M \subset D$ as above with one positive elliptic complex tangent and with one negative complex tangent of index m. Alternatively, M can have exactly m negative complex tangents that are elliptic if $m > 0$ and hyperbolic if $m < 0$.

PROOF OF PROPOSITION 3.

Choose an embedded totally real disc $M_0 \subset \mathbf{C}^2$. Let $\gamma_0 = bM_0$ and let Y_0 be the vector field tangent to M_0 along γ_0 and inner radial to M_0.

There is a smooth isotopy $\{\gamma_t : 0 \leq t \leq 1\}$ of $\gamma_0 = bM_0$ into $\gamma_1 = \gamma$ in $\mathbf{C}^2$. Let X_t be the tangent vector field to γ_t, depending smoothly on t. We can also find a vector field Y_t along γ_t, depending smoothly on $t \in [0,1]$, such that $Y_1 = Y$ and $I(X_t, Y_t) = 0$ for all t. (This requires in particular that X_t and Y_t are $\mathbf{C}$-independent for each t.)

The isotopy (γ_t, Y_t) can be extended to an isotopy A_t of a small annular collar $A_0 \subset M_0$ of γ_0 in M_0 into a similar annular collar $A = A_1$ along γ.

According to the isotopy extension theorem [10, p.180], the isotopy $t \to A_t \subset \mathbf{C}^2$ can be realized by a smooth diffeotopy $\Phi_t : \mathbf{C}^2 \to \mathbf{C}^2$, with Φ_0 the identity. Let $M_t = \Phi_t(M_0)$. By construction every M_t is totally real near the boundary $bM_t = \gamma_t$, hence the indices $I_{\pm}(M_t)$ are independent of t. Since M_0 is totally real, we conclude that $I_{\pm}(M_1) = 0$.

Using the results of [7] we can find a C^0-small smooth deformation M of M_1 that agrees with M_1 near γ such that M is totally real. Clearly M is the required disc.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706