The length of a set in the sphere whose polynomial hull contains the origin*

by Franc Forstnerič

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Communicated by Prof. J. Korevaar at the meeting of February 24, 1992

ABSTRACT

Let $X$ be a compact subset of the unit sphere in the complex Euclidean space $\mathbb{C}^n$ such that the origin $0 \in \mathbb{C}^n$ belongs to the polynomial hull of $X$. Assuming that $X$ is rectifiable in the Hausdorff $(\mathbb{R}^1,1)$-sense, it is shown that the length of $X$ is at least $2n$.

INTRODUCTION

It is well known that every pure one-dimensional complex variety $A \subset \mathbb{C}^n$ with reasonably "nice" boundary $bA$ satisfies the isoperimetric inequality

$$(\text{Length}(bA))^2 \leq 4\pi \text{Area}(A).$$

Moreover, if $A$ contains the origin $0 \in \mathbb{C}^n$ while $bA$ lies outside the ball $B(r) = \{z \in \mathbb{C}^n : |z| < r\}$ of radius $r$, then $\text{Area}(A) \geq \pi r^2$ and therefore

$$\text{Length}(bA) \geq 2\pi r.$$ 

(See Chirka [6], p. 180 and p. 195, and Bishop [4].) The same inequalities hold if $A$ is an immersed minimal surface with connected boundary; see Section 7.3 in [5]. The constants in these inequalities are the best possible.

H. Alexander [3] extended the isoperimetric inequality to closed Jordan curves $X \subset \mathbb{C}^n$: If $X$ is not polynomially convex, then the set $A = \bar{X} \setminus X$ is an irreducible one-dimensional subvariety of $\mathbb{C}^n \setminus X$, and the isoperimetric ine-

*Supported in part by the Research Council of the Republic of Slovenia.
quality holds when the length and the area are computed using the Hausdorff measures $\mathcal{H}^1$ resp. $\mathcal{H}^2$ on $\mathbb{C}^n$. (For Hausdorff measures see Federer [7, p. 171].) In particular, if $X$ is a closed Jordan curve in the unit sphere $S = \{ z \in \mathbb{C}^n : |z| = 1 \}$ whose polynomial hull $\tilde{X}$ contains the origin $0 \in \mathbb{C}^n$, then we have

$$\mathcal{H}^1(X) \geq 2\sqrt{\pi}(\mathcal{H}^2(\tilde{X}))^{1/2} \geq 2\pi.$$ 

Recall that the polynomial hull of $X$ is the set

$$\tilde{X} = \{ z \in \mathbb{C}^n : |f(z)| \leq \sup_{X} |f| \text{ for all } f \in \mathcal{O}(\mathbb{C}^n) \}.$$ 

In this article we shall consider the following question that was raised by Stout [8, problem 4.2.2]:

*If $X \subset S$ is a compact set in the unit sphere whose polynomially convex hull $\tilde{X}$ contains the origin $0 \in \mathbb{C}^n$, must the length of $X$ be at least $2\pi$?*

In [11] Stout proved a weaker inequality $\mathcal{H}^1(X) \geq \sqrt{2}\pi$ that improved the previously known result $\mathcal{H}^1(X) \geq 2$ by Sibony [9]. (The result of Sibony applies also to sets that are not contained in a sphere.)

Here we give a very simple proof of the inequality

$$(*) \quad \mathcal{H}^1(X) \geq 2\pi \text{ when } X \subset S \text{ and } 0 \in \tilde{X}$$

for $(\mathcal{H}^1,1)$-rectifiable compact sets $X \subset S$. Together with a new result of Mark Lawrence this settles the general case as well.

**THE RESULT**

**DEFINITION.** (Federer [7, p.251].)

(a) A set $X \subset \mathbb{R}^k$ is 1-rectifiable if it is the image of a bounded subset $U \subset \mathbb{R}$ under a Lipschitz continuous mapping $f: U \rightarrow \mathbb{R}^k$.

(b) $X$ is $(\mathcal{H}^1,1)$-rectifiable if $\mathcal{H}^1(X) < \infty$ and almost all of $X$ (with respect to the length $\mathcal{H}^1$) can be covered by a countable union of 1-rectifiable sets.

Our main result is

**THEOREM.** If $X$ is a compact $(\mathcal{H}^1,1)$-rectifiable subset of the unit sphere $S \subset \mathbb{C}^n$ such that the origin $0 \in \mathbb{C}^n$ belongs to the polynomial hull $\tilde{X}$, then $\mathcal{H}^1(X) \geq 2\pi$.

Clearly the result can be stated for the ball $r \mathbb{B}$ of radius $r$: If $X \subset rS$ is compact and $(\mathcal{H}^1,1)$-rectifiable, and if $0 \in \tilde{X}$, then $\mathcal{H}^1(X) \geq 2\pi r$.

The isoperimetric inequality does not hold in the context of our Theorem. Namely, Alexander constructed in [2] a compact disconnected set $X \subset \mathbb{C}^2$ of finite length whose polynomial hull $\tilde{X}$ has infinite area. His set $X$ is not highly pathological, it consists of a countable disjoint union of real-analytic simple closed curves, and $A = \tilde{X} \setminus X$ is countable union of analytic subsets of $\mathbb{C}^2 \setminus X$. Obviously $X$ is $(\mathcal{H}^1,1)$-rectifiable.
It is unknown whether the isoperimetric inequality holds for compact connected sets \( X \subset \mathbb{C}^n \) of finite length. Recall that \( A = \hat{X} \setminus X \) is then a pure one-dimensional analytic subvariety of \( \mathbb{C}^n \setminus X \) (if not empty) according to Alexander [1]. For smooth curves \( X \) this had been proved by Stolzenberg [10].

**Remark added to the proof.** It suffices to prove the inequality (*) for sets \( X \subset S \) of finite length \( (\mathcal{H}^1(X) < \infty) \) that are minimal, in the sense that no proper compact subset of \( X \) contains 0 in its polynomial hull. Recently Mark Lawrence (private communication) informed me of his new result that such a set is necessarily \( (\mathcal{H}^1, 1) \)-rectifiable. Together with our theorem this implies

**Corollary.** If \( X \) is a compact subset of \( S \) and \( 0 \in \hat{X} \) then \( \mathcal{H}^1(X) \geq 2\pi \).

Also, after the completion of the first version of this article, H. Alexander [12] and N. Poletski (private communication) informed me that they had independently proved the estimate (*) by different methods.

**Proof of the Theorem.**

The result will follow from the following Lemma and an integral geometric formula (Crofton formula) from Federer [7, p. 284].

**Lemma.** If \( X \) is a compact subset of \( \mathbb{C}^n \setminus \{0\} \) of finite length such that \( 0 \in \hat{X} \), then almost every real hyperplane \( \Sigma \subset \mathbb{C}^n \) passing through the origin intersects \( X \) at least at two points.

Here, "almost every" is meant with respect to the volume measure on the Grassman manifold of real hypersurfaces \( 0 \in \Sigma \subset \mathbb{C}^n \).

**Proof.** Since \( X \) has finite length and \( 0 \notin X \), Fubini's theorem implies that almost every complex hyperplane \( L \subset \mathbb{C}^n \) passing through 0 misses \( X \). Here, "almost every" refers to the volume measure on the Grassman manifold of complex \((n-1)\)-dimensional subspaces of \( \mathbb{C}^n \).

Fix such a hyperplane \( L \), and let \( \Sigma \) be any real hyperplane in \( \mathbb{C}^n \) containing \( L \). Then \( L \) splits \( \Sigma \) in two open half-planes \( \Sigma_+ \) and \( \Sigma_- \).

We claim that both \( \Sigma_+ \) and \( \Sigma_- \) intersect \( X \). To see this, let \( \pi : \mathbb{C}^n \to L^\perp \) be the orthogonal projection onto the complex line \( L^\perp \) orthogonal to \( L \). If \( X \) is disjoint from \( \Sigma_+ \) then \( \pi(X) \subset L^\perp \) is disjoint from the real half-line \( \pi(\Sigma_+) \), hence \( 0 \in L^\perp \) lies in the unbounded component of \( L^\perp \setminus \pi(X) \). Thus \( 0 \) is not in the polynomial hull of \( \pi(X) \) in \( L^\perp \) and hence \( 0 \) is not in the hull of \( X \) in \( \mathbb{C}^n \), a contradiction. (This also follows from Oka's criterion for polynomial convexity [10, p. 263]: since \( 0 \) belongs to \( \hat{X} \), we can not move \( L \) continuously to infinity without hitting \( X \).)

This shows that every such real hyperplane \( \Sigma \) intersects \( X \) at least two points. Since the remaining set of real hyperplanes through the origin in \( \mathbb{C}^n \) has measure zero, the lemma is proved.
We now apply Theorem 3.2.48 in Federer [7, p. 284] as follows. Let \( B \) be the intersection of a real hyperplane through the origin in \( \mathbb{C}^n \) with the sphere \( S \). By our hypothesis \( X \) is \( (\mathcal{H}^1, 1) \) rectifiable, it is \( \mathcal{H}^1 \) measurable since Hausdorff measures are Borel regular, and \( \mathcal{H}^1(X) < \infty \). Clearly \( B \) is \( m=(2n-2) \)-rectifiable and \( \mathcal{H}^m \)-measurable since it is an \( m \)-manifold. Applying Theorem 3.2.48 in [7] to the constant functions \( \alpha = 1 \) on \( A = X \) and \( \beta = 1 \) on \( B \) we get

\[
\int \int \mathcal{H}^0 d\theta_{2n}(g) = C \cdot \mathcal{H}^1(X) \cdot \mathcal{H}^m(B)
\]

for some constant \( C \) depending only on \( m \) and \( n \).

Recall that \( \mathcal{H}^0 \) is just the counting measure. The lemma implies \( \mathcal{H}^0(X \cap g(B)) \geq 2 \) for almost all \( g \in \Omega(2n) \) with respect to the volume measure \( \theta_{2n} \). Hence we get

\[
\mathcal{H}^1(X) \geq 2\theta_{2n}(\Omega(2n))/C\mathcal{H}^m(B).
\]

To calculate the constant on the right hand side we choose \( X \) to be the intersection of \( S \) with a complex line through the origin, hence \( \mathcal{H}^1(X) = 2\pi \). In this case \( X \cap g(B) \) contains exactly two points for most \( g \in \Omega(2n) \), hence the inequality above is actually an equality. Thus the value of the right hand side equals \( \mathcal{H}^1(X) = 2\pi \).

This completes the proof of the Theorem.

REFERENCES