Math. Z. 213, 49–64 (1993)

Mathematische Zeitschrift

A reflection principle on strongly pseudoconvex domains with generic corners

Franc Forstneric*

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Received 10 June 1992; in final form 16 July 1992

1 Introduction

In this paper we prove a version of the 'reflection principle' for biholomorphic mappings between certain domains in the complex Euclidean space $\mathbf{C}^n$ with generic real-analytic (or smooth) corners, under the condition that the map extends continuously up to the corner manifolds. Our method is similar to the one in our previous article [14]. The new ingredient here is a novel way of applying the Hopf Lemma in order to obtain estimates on the behavior of the mapping near a corner (Proposition 2.1 below). This allows us to assume less on the corner manifolds, provided that we assume somewhat more on the domains away from corners. We also obtain a global result, Theorem 1.3, for mappings between real-analytic, strongly pseudoconvex domains with generic corners.

Let $\mathbf{C}^n$ be the complex Euclidean space of dimension n, with complex coordinates $z=(z_1, z_2, \ldots, z_n)$, $z_j=x_j+iy_j$. Let U be a bounded open set in $\mathbf{C}^n$, and let $\rho_j(z)$, $1 \leqq j \leqq d$, be real valued functions of class $\mathscr{C}^2$ on U, satisfying

$$\partial\rho_1 \wedge \partial\rho_2 \wedge \ldots \wedge \partial\rho_d \neq 0 \quad \text{on } U; \tag{1}$$

that is, the *complex* gradients of the ρ_j should be independent. Here,

$$\partial\rho(z) = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(z)\, dz_j = \sum_{j=1}^{n} \frac{1}{2}\left(\frac{\partial\rho}{\partial x_j} - i\frac{\partial\rho}{\partial y_j}\right) dz_j.$$

Condition (1) implies that the set

$$M = \{z \in U : \rho_1(z) = \rho_2(z) = \ldots = \rho_d(z) = 0\} \tag{2}$$

is a generic Cauchy-Riemann (C-R) submanifold of U, of real dimension $2n-d$ and of C-R dimension $m=n-d$. This means that for each point $p \in M$, the maximal complex tangent space $\mathrm{T}_p^{\mathbf{C}} M = \mathrm{T}_p M \cap i\mathrm{T}_p M$ to M at p has complex

* Supported in part by the Research Council of Republic of Slovenia

dimension $n-d$ and real codimension d in $T_p M$. We denote by $T^C M$ the complex bundle over M of rank m, with fibers $T_p^C M$. The domain

$$D=\{z\in U: \rho_j(z)<0 \quad \text{for } 1\leqq j\leqq d\} \tag{3}$$

contains M in its boundary bD and has a generic corner (edge) at M. If, in addition, the functions $\rho_1, \ldots, \rho_d$ are (strongly) plurisubharmonic, we call D a (strongly) pseudoconvex domain with a generic corner M. Sometimes domains D of this type are called 'wedges', and the manifold M is the 'edge' of the wedge D.

If $k=r+\alpha$, with $r\in \mathbf{Z}_+$ and $0\leqq\alpha<1$, we denote by $\mathscr{C}^k=\mathscr{C}^{r,\alpha}$ the class of all functions that are r times continuously differentiable, and whose derivatives of order r satisfy the Hölder condition of order α. Also, $\mathscr{C}^{k-0}=\mathscr{C}^k$ if k is not an integer, and $\mathscr{C}^{k-0}=\bigcup_{0<\alpha<1}\mathscr{C}^{k-1,\alpha}$ if $k\in\mathbf{Z}_+$.

Our first main result is the following theorem.

Theorem 1.1 *Suppose that D resp. D' are strongly pseudoconvex domains in $\mathbf{C}^n$ of the form* (3), *with generic corners $M\subset bD$ resp. $M'\subset bD'$* (2) *of class $\mathscr{C}^k$, $k>3$. If $F: D\to D'$ is a biholomorphic mapping that extends to a homeomorphism $F: D\cup M\to D'\cup M'$ and maps M to M', then $F: M\to M'$ is a smooth diffeomorphism of class $\mathscr{C}^{k-1-0}$. If the manifolds M and M' are real-analytic, then F extends biholomorphically to a neighborhood of every point $p\in M$.*

Note that a holomorphic function in D that is continuous on $D\cup M$ and smooth on M is also smooth on $D_1\cup M$ for every domain $D_1\subset D$ that has a smaller opening at the corner M [9].

Theorem 1.1 is a special case of the following result. Let

$$\mathscr{L}_p\,\rho(v)=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\,\partial\bar z_k}(p)\,v_j\,\bar v_k$$

be the Levi form of a function ρ at $p\in\mathbf{C}^n$, in direction of the vector $v=(v_1, \ldots, v_n)$.

Theorem 1.2 *Let D and D' be domains in $\mathbf{C}^n$ of the form* (3), *defined by plurisubharmonic functions ρ_j resp. ρ'_j of class $\mathscr{C}^2$, with generic corners M resp. M' of class $\mathscr{C}^k$, $k>3$. Suppose that the Levi form of every ρ_j $(1\leqq j\leqq d)$ is positive definite on the complex tangent bundle $T^C M$, and the analogous condition holds for M'. If $F: D\to D'$ is a biholomorphic map that extends to a homeomorphism $F: D\cup M\to D'\cup M'$ and maps M to M', then $F: M\to M'$ is a smooth diffeomorphism of class $\mathscr{C}^{k-1-0}$. If the manifolds M and M' are real-analytic, then F extends biholomorphically to a neighborhood of every point $p\in M$.*

Remark 1 It is not required that the surfaces $\rho_j=0$ be smooth or real-analytic.

Remark 2 Conditions of Theorem 1.1 and 1.2 imply that the manifolds M and M' are strongly pseudoconvex, in the sense that they are locally contained in a strongly pseudoconvex hypersurface. Several equivalent characterizations of this condition can be found in [14]; see also Sect. 3 below.

The question whether Theorem 1.1 holds if the functions ρ_j, ρ'_j are merely weakly plurisubharmonic, but the manifolds M and M' are strongly pseudoconvex, remains open.

Remark 3 The requirement that D and D' be *finite* intersections of domains with $\mathscr{C}^2$ boundaries is too strong. However, our conditions are convenient since they are simple to state, and the class of such domains arises naturally in many contexts. It will be apparent from the proof that our method applies to biholomorphic mappings between open sets in $\mathbf{C}^n$ that are osculated from outside to second order by domains of this type. Also, our methods apply to more general wedges with generic, strongly pseudoconvex edges. (See Theorem 3.1 in Sect. 3 below).

Theorems 1.1 and 1.2 imply the corresponding global results. As a simple case we consider biholomorphic mappings between bounded, strongly pseudoconvex domains in $\mathbf{C}^n$ with piecewise smooth, real-analytic boundaries, such that all corners in the boundary are generic C-R manifolds. This means that in a small neighborhood U of every boundary point $p \in bD$, $D \cap U$ is a domain of the form (3), where the ρ_j are real-analytic, strongly plurisubharmonic functions satisfying (1).

Range proved [27] that biholomorphic mappings between domains of this type extend Hölder continuously to the boundary. (In fact, he proved this with condition (1) replaced by the weaker condition $d\rho_1 \wedge d\rho_2 \wedge \ldots \wedge d\rho_d \neq 0$. If (1) holds then the proof is almost the same as for smoothly bounded domains, and the mapping is Hölder continous with the exponent 1/2.) Together with Theorem 1.1 this implies the following result.

Theorem 1.3 *Every biholomorphic mapping $F: D \to D'$ of bounded, real-analytic, strongly pseudoconvex domains in $\mathbf{C}^n$ with generic corners extends to a biholomorphic map on a neighborhood of $\bar{D}$.*

In the proof of Theorem 1.3 we will also need the following result that generalizes a theorem of Pinchuk [23].

Proposition 1.4 *Let D and D' be domains in $\mathbf{C}^n$ of the form* (3), *defined by $\mathscr{C}^2$ plurisubharmonic functions ρ_j resp. ρ'_j, with generic corners M resp. M' of class $\mathscr{C}^k$, $k > 3$. Suppose that the Levi form of every function ρ_j is positive definite on the complex tangent bundle $T^{\mathbf{C}}M$, and the analogous condition holds for M'. If $F: D \to D'$ is a biholomorphic map that extends continuously to $D \cup M$ and satisfies $F(M) \subset M'$, then M and M' have the same dimension.*

Theorems 1.1 and 1.2 are new when $1 < d < n$, i.e., when M and M' are generic C-R manifolds of positive C-R dimension and of real codimension at least two. The two extreme case $d = n$ and $d = 1$ are well undestood. We shall now briefly recall the main known results in order to put our new theorems in context.

When $d = n$, the corners M and M' are generic, *totally real* submanifolds of $\mathbf{C}^n$. If such a manifold M is real-analytic, it is locally biholomorphically equivalent to the totally real subspace $\mathbf{R}^n \subset \mathbf{C}^n$. If $F: D \cup M \to \mathbf{C}^n$ is a continuous map that is holomorphic on D and maps M to M', the extendability of F follows by 'reflecting' F across M and M' and applying the classical edge-of-the-wedge theorem. (It suffices to assume that M' is totally real.) If M and M' are merely smooth, one applies the smooth version of the edge-of-the-wedge theorem as in [25] and [14]; the conclusion is that F is smooth on M.

The other extreme case is $d = 1$; M and M' are then hypersurfaces in $\mathbf{C}^n$ that form an open part of the boundary of D resp. D'. If $F: D \to D'$ is a locally biholomorphic (or proper holomorphic) mapping that takes M to M', and if M and M' are pseudoconvex of finite type, then F extends holomorphically to a neighborhood of p in $\mathbf{C}^n$. This result (and its smooth analogue) is due to several authors: Fefferman [13], Bell and Catlin [4, 5], Diederich and Fornæss [10, 11], Baouendi and Rothschild [2, 3], and others. (See the survey [17].) Most proofs of smooth extendability of F to M rely on the Bergman kernel and the $\bar{\partial}$-Neumann methods. Simpler proofs for mappings between strongly pseudoconvex domains were given in [15, 20, 22, 25, 26]; in this case the optimal loss of smoothness is just a bit more than 1/2 according to [19].

The intermediate case $1<d<n$ has not been studied very much up to now. If M' has non-degenerate Levi form at $F(p)$ for some $p \in M$, and if F is continuously differentiable on $D \cup M$ with DF_p non-degenerate on $T_p^{\mathbf{C}} M$, then F extends to a neighborhood of p according to Webster [33, 34]. A related result for analytic sets was obtained in [24].

However, it is well known that for Levi nondegenerate manifolds the most difficult part of the regularity problem is to obtain some small initial amount of regularity of F on M, for instance, to prove that F is $\mathscr{C}^1$ up to M. This point was already apparent in the paper by Nirenberg et al. [22] which offered the first proof of Fefferman's theorem using the reflection principle. For manifolds with degenerate Levi form one needs a higher degree of initial smoothness of the mapping in order to get the reflection principle going; see the papers [1, 2 11].

It seems that the only results in which $1<d<n$ and F is merely *continuous* up to the edge to begin with are due to Tumanov and Khenkin [32] and the author [14]. Both these papers considered the regularity of local homeomorphisms $F: M \to M'$ for certain class of generic C-R manifolds, under the condition that F and its inverse F^{-1} satisfy the tangential Cauchy-Riemann equations on M resp. M' in the weak sense. Such mappings are called C-R homeomorphisms.

If the manifold M is minimal at a point $p \in M$ in the sense of Tumanov [31], then the local envelope of holomorphy of M near p contains a wedge domain with edge M according to [31] (see Sect. 3 below). If M' is also minimal at the point $F(p)$, it follows that F extends to a biholomorphic mapping between domains that can be closely approximated by wedges with edge M resp. M'. This explains the connection with the problem considered here.

In [32] the manifolds M and M' were assumed to be defined by quadratic hermitian polynomials (quadrics). Under suitable assumptions on the Levi form it was proved that every local C-R homeomorphism $F: M \to M'$ extends holomorphically to a neighborhood of M. In the subsequent papers [16] and [30] it was shown that such a mapping extends to a complex rational mapping on $\mathbf{C}^n$. The main idea of the proof in [32] was to exploit a special family of complex balls of dimension at least two, contained in the boundary of the envelope of holomorphy of the quadric M resp. M'.

In [14] we established the regularity of C-R homeomorphisms for strongly pseudoconvex C-R manifolds (2) that are over-extendable. Recall from [14] that M is said to be over-extendable at a point $p \in M$ if every C-R function, defined on a neighborhood of p in M, can be extended in some smaller neighborhood of p to a holomorphic function on a wedge domain $\mathscr{W}(\Gamma, U)$ with edge M (Sect. 3) whose cone Γ is larger than the Levi cone $C_p(M)$ of M at p. (See Sect. 3 for the definition of $C_p(M)$). The main result of [14] was the following.

Theorem. *If M and M' are smooth (resp. real-analytic)* C-R *manifolds* (2) *that are strongly pseudoconvex and over-extendable at points $p \in M$ resp. $q \in M'$, then every local* C-R *homeomorphism $F: M \to M'$, $F(p)=q$, is smooth (resp. real-analytic) in a neighborhood of p in M.*

The condition of over-extendability is too strong since it does not hold for quadrics, and it is difficult to verify. It holds for instance when the third order homogeneous part of the Taylor expansion of the defining function of M at p is sufficiently independent of the second order part (the Levi form); cf. [14, Theorem 2.2].

Our results in this paper do not depend on over-extendability, but we pay the price by assuming that F is a biholomorphic mapping between domains whose boundaries are reasonably nice away from the corners. In Sect. 3 we show that the results in our present paper, as well as the Theorem of [14] stated above, can be deduced from the same technical result (Theorem 3.1) that was proved in [14], although it was not stated there explicitly.

It would be interesting to see whether the regularity results obtained here can also be obtained by the Bergman kernel methods that were very successful in the hypersurface case. In this connection we remark that the sup norm estimates for the $\bar{\partial}$-equation have been established on domains with generic corners by Range and Siu [28]. (See also [21].)

The paper is organized as follows. In Sect. 2 we obtain the relevant information on the behavior of F near corners (Proposition 2.1 and Corollary 2.2), and we prove Proposition 1.4. In Sect. 3 we indicate how the methods of [14] give a proof of Theorems 1.1 and 1.2. Finally we prove Theorem 1.3.

2 Behavior of the mapping near a generic corner

Before stating the results of this section we establish some notation that will be used throughout the paper.

Let $M \subset \mathbf{C}^n$ be a smooth manifold of the form (2), where the functions ρ_j satisfy (1). We shall identify the real tangent space $\mathrm{T}_p\mathbf{C}^n$ with $\mathbf{C}^n$ in the usual way. The almost complex operator J on $\mathrm{T}_p\mathbf{C}^n$ then corresponds to multiplication by $i=\sqrt{-1}$ on $\mathbf{C}^n$. Condition (1) implies that for each $p \in M$, $\mathrm{T}_p M$ is a direct sum $\mathrm{T}_p M = \mathrm{T}_p^{\mathbf{C}} M \oplus L_p$, where $\mathrm{T}_p^{\mathbf{C}} M = \mathrm{T}_p M \cap i\mathrm{T}_p M$ is a complex subspace of dimension $m=n-d$ and L_p is a real subspace of dimension d that is real orthogonal to $\mathrm{T}_p^{\mathbf{C}} M$. We will take $N_p = iL_p$ as the canonically chosen normal space to M at p. Clearly we have a decomposition

$$\mathrm{T}_p\mathbf{C}^n = \mathrm{T}_p M \oplus N_p = \mathrm{T}_p^{\mathbf{C}} M \oplus L_p \oplus N_p. \tag{4}$$

There exists a family of linear transformations $U_p \in \mathbf{GL}(n, \mathbf{C})$, depending continuously (even smoothly) on the point $p \in M$, satisfying

$$U_p(\mathrm{T}_p^{\mathbf{C}} M) = \mathbf{C}^m \times \{0\}^d, \qquad U_p(L_p) = \{0\}^m \times \mathbf{R}^d, \qquad U_p(N_p) = \{0\}^m \times i\mathbf{R}^d. \tag{5}$$

The fibers N_p form the normal bundle $N \mapsto M$ in $\mathbf{C}^n$. By the tubular neighborhood theorem there is a neighborhood V of M in $U \subset \mathbf{C}^n$ that is diffeomorphic to a neighborhood of the zero section in N. We denote by $\pi: V \to M$ the projection onto M with fibers $\pi^{-1}(p)=N_p\cap V$, where we identify N_p is the obvious way with the corresponding affine subspace of $\mathbf{C}^n$ at p.

Correction to [14] In the paper [14] we erroneously claimed that there exists a family of *unitary* transformations U_p $(p\in M)$ satisfying (5). Specifically, such transformations were used in [14, Propositions 3.2, 3.3, 5.3, 6.1, and 6.2]. However, the proofs given there remain valid since they do not depend on this extra condition. We apologize to the readers of [14] for the inaccurate claim.

Let $D\subset U\subset\mathbf{C}^n$ be a domain (3) with corner M. We denote by $d(z,M)$ the distance from a point $z\in U$ to M. Clearly $d(z,M)$ is proportional to the function $\max\{-\rho_j(z)\colon 1\leqq j\leqq d\}$ on D, in the sense that their quotient is bounded from above and bounded away from zero on D.

For $\varepsilon>0$ small we set

$$D_\varepsilon=\{z\in D\colon |\rho_j(z)|>\varepsilon d(z,M),\ 1\leqq j\leqq d\}. \tag{6}$$

D_ε is a domain with corner M that has strictly smaller opening along M than D. As $\varepsilon\to 0$, the domains D_ε increase to D.

The following proposition is of crucial importance.

Proposition 2.1 *Let $D, D'\subset\mathbf{C}^n$ be domains of the form* (3) *with generic corners $M\subset bD$ resp. $M'\subset bD'$ of class $\mathscr{C}^k$, $k>3$. Suppose that D' is defined by plurisubharmonic functions ρ'_j, $1\leqq j\leqq d'$, and the Levi form of every ρ'_j is positive definite on $T^{\mathbf{C}}M'$. If $F\colon D\to D'$ is a holomorphic mapping that extends continuously to $D\cup M$ and satisfies $F(M)\subset M'$, then for each $\varepsilon>0$ sufficiently small and each set $U_0\Subset U$ there exist constants $C>0$, $\delta>0$ and a subset $U'_0\Subset U'$ such that*

(a) *$d(F(z),M')\geqq C d(z,M)$ for $z\in D_\varepsilon\cap U_0$, and*
(b) *$F(D_\varepsilon\cap U_0)\subset D'_\delta\cap U'_0$.*

Remark. We shall see in the proof that the assumption on the Levi form of the functions ρ'_j on $T^{C}M'$ and the continuity of F up to M is required only in the proof of (b). Part (a) holds without these assumptions.

Corollary 2.2 *Let D and D' be domains with generic corners M resp. M' as in Proposition* 2.1. *If $F\colon D\to D'$ is a biholomorphic map that extends continuously to $D\cup M$ and satisfies $F(M)\subset M'$, there exists for each $\varepsilon>0$ and $U_0\Subset U$ a constant $C>0$ such that*

$$(1/C)\,d(F(z),M')\leqq d(z,M)\leqq C d(F(z),M'),\qquad z\in D_\varepsilon\cap U_0. \tag{7}$$

Proof. The right inequality is Proposition 2.1(a). By Proposition 2.1(b), F maps $D_\varepsilon\cap U_0$ into $D'_\delta\cap U'_0$ for some $\delta>0$ and $U'_0\Subset U'$. Applying Proposition 2.1(a) to F^{-1} on $D'_\delta\cap U'_0$ we obtain the left estimate in (7). This proves Corollary 2.2.

In order to prove Proposition 2.1 we need the following Hopf Lemma near corners.

Lemma 2.3 (Hopf Lemma near a generic corner.) *Let $D\subset U$ be domain* (3) *with generic corner $M\subset bD(2)$ of class $\mathscr{C}^k$, $k>3$. Fix $U_0\Subset U$ and $\varepsilon>0$.*

(a) *If ρ is a continuous, negative, plurisubharmonic function on D, there is a constant $C>0$ such that*

$$\rho(z)\leqq -C d(z,M),\qquad z\in D_\varepsilon\cap U_0.$$

(b) *If ρ is a continuous, plurisubharmonic function on D (not necessarily negative!) that extends continuously to $D \cup M$ with $\rho|_M = 0$, there is a constant $C > 0$ such that*

$$\rho(z) \leqq C\, d(z, M), \qquad z \in D_\varepsilon \cap U_0.$$

Remark. In several places in literature, the Hopf Lemma has been used on domains D with little boundary regularity (e.g., piecewise smooth or convex domains) by considering negative plurisubharmonic functions on cones $\Gamma_p \subset D$ with a given vertex $p \in bD$. In this case we have a rate of decay $\rho(z) \leqq -C\, d(z, p)^a$, where the exponent $a \geqq 1$ depends on the opening (angle) of the cone. This, however, only gives fractional estimates in (7) which are not good enough for our purposes.

Proof of Lemma 2.3 The proof relies on the fact that for a domain D with a generic corner $M \subset bD$, every smaller domain $D_\varepsilon \cap U_0 \subset D$ of the form (6) can be exhausted, at least in some neighborhood of M, by analytic discs of uniform size, contained in D, that abut the edge M along an open piece of their boundary (again of uniform size). Actually, for (a) we only need linear discs in D that touch M at a single point; this was already proved in [14, Corollary 3.4]. In the proof of (b) we need discs that have a boundary arc contained in M.

Let $L_p \subset T_p M$ and $N_p = i L_p$ be as in (4). Denote by S_p be the unit sphere of L_p (in the Euclidean norm). The disjoint union of sets S_p for $p \in M$ is a sphere bundle $S \mapsto M$ over M. We shall denote the points of S by (p, v), where $p \in M$ and $v \in S_p$ is a vector tangent to M at p, of unit length, orthogonal to $T_p^{\mathbf{C}} M$. Similarly, points of the normal bundle N will be written in the form (p, itv), where $p \in M$, $v \in S_p \subset L_p$, $t \geqq 0$, and $i = \sqrt{-1}$. Set

$$\Delta_+(r) = \{\zeta = x + iy \in \mathbf{C}\colon\ -r < x < r,\, 0 < y < r\} \subset \mathbf{C}.$$

Lemma 2.4 *Let $S \mapsto M$ be the sphere bundle defined above. For each relatively compact subset $M_0 \Subset M$ there exists a mapping $\Phi\colon S \times \bar{\Delta}_+(r) \to \mathbf{C}^n$ of class $\mathscr{C}^1$ for some $r > 0$, satisfying the following properties for each $(p, v) \in S$:*

(a) The map $\Phi(p, v, \cdot)\colon \bar{\Delta}_+(r) \to \mathbf{C}^n$ *is an embedding that is holomorphic on* $\Delta_+(r)$,

(b) $\Phi(p, v, 0) = p$ *and* $(\partial \Phi / \partial x)(p, v, 0) = v$ $(x = \Re \zeta)$, *and*

(c) $\Phi(p, v, x) \in M$ *for* $-r < x < r$.

Intuitively speaking, the image $A(p, v) \subset \mathbf{C}^n$ of $\Delta_+(r)$ by $\Phi(p, v, \cdot)$ is an embedded analytic disc in $\mathbf{C}^n$, $\mathscr{C}^1$ up to the boundary, with the boundary arc

$$E(p, v) = \{\Phi(p, v, x)\colon\ -r < x < r\} \subset M$$

contained in M. The arc $E(p, v)$ passes through p, and its tangent vector at p is v. In applications, M_0 will be a neighborhood of a chosen point $p_0 \in M$ in M.

We shall omit the construction of Φ because similar constructions have been done elsewhere and are well known by now. One uses the Bishop's method [6], together with the implicit function theorem approach developed by Hill and Taiani [18]. See for instance the papers [7, 8, 12], and [29].

Let us mention that the construction of such analytic discs is entirely elementary when M is real-analytic. It suffices to take a family of smooth real-analytic arcs $E(p, v)\subset M$, depending smoothly on $(p, v)\in S$, such that $p\in E(p, v)$ and the tangent vector to $E(p, v)$ at p equals v. We complexify each of these curves to obtain a family of local complex curves $\Lambda(p, v)$ in $\mathbf{C}^n$, each of them divided in two half-spaces by M. The half-space of $\Lambda(p, v)$ determined by the vector $iv\in N_p$ is the required disc $A(p, v)$.

We continue with the proof of Lemma 2.3. Consider the mapping $\Psi: \Omega\subset N \to \mathbf{C}^n$, defined on a suitable neighborhood Ω of the zero section in the normal bundle N by

$$\Psi(p, iyv)=\Phi(p, v, iy), \qquad p\in M_0, \quad v\in S_p, \quad y\geqq 0.$$

The properties of Φ imply that Ψ is an embedding of class $\mathscr{C}^1$ of Ω onto a neighborhood $\Psi(\Omega)$ of M_0 in $\mathbf{C}^n$. Along the zero section of N, Ψ is tangent to the map $(p, iyv)\in N\to p+iyv\in\mathbf{C}^n$ that is linear on fibers N_p of N.

Inside each space L_p we let Γ_p be the set consisting of all tangent vectors $v\in L_p$ such that the vector $iv\in N_p$ points to the interior of the domain D. To give a precise definition, let $\rho_1, \ldots, \rho_d$ be the defining functions of D (3). Since the form $d\rho_j=\partial\rho_j+\bar\partial\rho_j$ annihilates $T_p M$, the form $i\partial\rho_j$ is real valued on L_p for each $j=1, \ldots, d$, and Γ_p can be defined by

$$\Gamma_p=\left\{v=(v_1, \ldots, v_n)\in L_p: \sum_{k=1}^{n} i\frac{\partial\rho_j}{\partial z_k}(p)\, v_k<0, \quad 1\leqq j\leqq d\right\}. \tag{8}$$

Clearly Γ_p is an open cone in L_p with vertex at the origin. Similarly we can define cones $\Gamma_p(\varepsilon)$ corresponding to domains D_ε (6).

Choose numbers $0<\varepsilon_2<\varepsilon_1<\varepsilon$, and let $D_\varepsilon\subset D_{\varepsilon_1}\subset D_{\varepsilon_2}\subset D$ be the corresponding domains (6) with corner M. Let $\Gamma_p(\varepsilon_1)$ be the cone (8) at $p\in M$, associated to the domain D_{ε_1}. The following properties of Φ and Ψ follow from Lemma 2.4 and the implicit function theorem:

(i) there is a number r_1, $0<r_1<r$, such that $\Phi(p, v, \zeta)\in D_{\varepsilon_2}$ for each $p\in M_0$, $v\in\Gamma_p(\varepsilon_1)\cap S_p$, and $\zeta\in\Delta_+(r_1)$;

(ii) the image of the set

$$\Omega(\varepsilon_1)=\{(p, iyv): p\in M_0, v\in\Gamma_p(\varepsilon_1)\cap S_p,\ 0<y<r_1\}\subset N$$

by the mapping Ψ contains $D_\varepsilon\cap V_0$ for some open set $V_0\subset\mathbf{C}^n$ with $V_0\cap M=M_0$.

Note that $\Omega(\varepsilon_1)$ is a wedge in the bundle N whose edge is the zero section of N. The map Ψ takes $\Omega(\varepsilon_1)$ onto a wedge-like domain that contains $D_\varepsilon\cap V_0$ and is itself contained in D.

We can now complete the proof of Lemma 2.3. According to (ii) every point $z\in D_\varepsilon\cap V_0$ is of the form $z=\Phi(p, v, iy)\in A(p, v)$ for some $p\in M_0$ and $v\in\Gamma_p(\varepsilon_1)\cap S_p$. We apply the one-variable Hopf Lemma to the subharmonic function $\rho\circ\Phi(p, v, \cdot)$ on $\Delta_+(r_1)$. In case (a) when ρ is negative on D we get

$$\rho(\Phi(p, v, iy))\leqq -Cy, \qquad 0<y<r_1,$$

for some $C>0$. Note that the set

$$K=\{\Phi(p, v, x+ir_1): p\in M_0, v\in\Gamma_p(\varepsilon_1)\cap S_p, |x|<r_1\}$$

is relatively compact in D, and hence $\rho \leqq -C' < 0$ on K. This implies that the constant C in the estimate above can be chosen independently of (p, v). By construction of our discs, the distance $d(z, M)$ from $z = \Phi(p, v, iy)$ to the edge M is proportional to y, uniformly with respect to p and v. Thus the estimate (a) follows.

In case (b) we take into account that $\rho = 0$ on M, and consequently ρ is bounded from above near M. Since $\rho(\Phi(p, v, x)) = 0$ for $-r < x < r$, the Hopf Lemma implies

$$\rho(\Phi(p, v, iy)) \leqq Cy, \quad 0 < y < r_1$$

for some $r_1 > 0$, with a constant $C > 0$ independent of $p \in M_0$ and $v \in \Gamma_p(\varepsilon_1) \cap S_p$. This implies the estimate (b). Lemma 2.3 is proved.

Proof of Proposition 2.1 The function $\rho' = \sum_{j=1}^{d} \rho'_j$ is negative plurisubharmonic on D', and $-\rho'(z)$ is proportional to the distance $d(z, M')$ from the edge. Applying Lemma 2.3 to the continuous, negative, plurisubharmonic function $\rho' \circ F$ on D we get $\rho'(F(z)) \leqq -C d(z, M)$ for $z \in D_\varepsilon \cap U_0$, which is equivalent to (a). Similarly we obtain estimates $\rho'_j(F(z)) \leqq -C d(z, M)$ for $1 \leqq j \leqq d$. Notice that we did not use the hypothesis that F is continuous up to M and maps M to M'.

In order to prove (b), we fix a point $p \in M$ and choose local holomorphic coordinates near the point $q = F(p) \in M'$ in which $q = 0$, and $T_0 M' = \mathbf{C}^m \times \mathbf{R}^d$. Write the coordinates in the form $z = (z', z'')$, where $z' \in \mathbf{C}^m$ and $z'' = u'' + iv'' \in \mathbf{C}^d$ $(u'', v'' \in \mathbf{R}^d)$. Locally near 0, M' is of the form $v'' = \phi(z', \bar{z}', u'')$, where ϕ has values in $\mathbf{R}^d$ and it vanishes to second order at 0. Set

$$\tau(z) = |v'' - \phi(z', \bar{z}', u'')|^2 = |v''|^2 + (\text{terms of order} \geqq 3).$$

The Levi form of τ at 0 is strictly positive definite on $\{0\}^m \times \mathbf{C}^d$ and it vanishes on $\mathbf{C}^m \times \{0\}$. Since the Levi form of every ρ'_j is assumed to be strictly positive definite on $T_0^{\mathbf{C}} M' = \mathbf{C}^m \times \{0\}^d$, it follows that the function $\rho'_j + c\tau$ is strongly plurisubharmonic near 0 for a sufficiently large constant $c > 0$. Choose $c > 0$ and a neighborhood V_0 of q in $\mathbf{C}^n$ such that $\rho'_j + c\tau$ is strongly plurisubharmonic on $\bar{V}_0$ for every $1 \leqq j \leqq d$. Let $U_0 \Subset U$ be a neighborhood of p such that $F(U_0 \cap D)$ is relatively compact in V_0.

If $c_1 > 0$ is sufficiently small then the function $-c_1 \rho'_k + \rho'_j + c\tau$ is still strongly plurisubharmonic on V_0, and it equals zero on $M' \cap V_0$. Its composition with F is a plurisubharmonic function on $D \cap U_0$ that equals zero on $M \cap U_0$. Applying Lemma 2.3 (b) on a smaller domain $D_\varepsilon \cap U_1$ $(U_1 \Subset U_0)$ we get

$$-c_1 \rho'_k(F(z)) + \rho'_j(F(z)) + c\tau(F(z)) \leqq c_2 d(z, M), \quad z \in D_\varepsilon \cap U_1.$$

Dividing this inequality by $-\rho'_j(F(z)) > 0$ and deleting the last term on the left hand side (that is positive) we obtain

$$c_1 \rho'_k(F(z))/\rho'_j(F(z)) \leqq 1 + c_2 d(z, M)/(-\rho'_j(F(z)) \leqq c_3, \quad z \in D_\varepsilon \cap U_1.$$

We used the previous estimate $\rho'_j(F(z)) \leqq -C d(z, M)$.

The upshot is that each quotient $(\rho'_k \circ F)/(\rho'_j \circ F)$ is uniformly bounded on $D_\varepsilon \cap U_1$. Clearly this implies that the point $F(z)$ lies in a domain $D'_\delta \subset D'$ for a suitable $\delta>0$. From the construction it follows that the constants involved in these estimates can be chosen uniformly for points p in a relatively compact subset of M, so the part (b) follows. This completes the proof of Proposition 2.1.

Proof of Proposition 1.4 The proof is essentially the one given by Pinchuk [23]. We shall estimate the Bergman kernel function $K_D(z)$ of D, restricted to the diagonal, when z approaches the corner M within a smaller domain $D_\varepsilon \subset D$ of the form (6).

Denote by Δ^k the unit complex polydisc of dimension k. Let M (2) be of real codimension d in $\mathbf{C}^n$. For each $p\in M$ we denote by N_p the normal space to M at p as in (4), considered as an affine subspace of $\mathbf{C}^n$ passing through p. Choose numbers $0<\varepsilon_1<\varepsilon$. Fix a point $p_0\in M$.

As in [23, Lemma 1.4], we can find a neighborhood $\Omega\subset\mathbf{C}^n$ of p_0 and families of one-to-one holomorphic maps $f_p, g_p\colon \Omega\to\mathbf{C}^n$ (local changes of coordinates), depending continuously on $p\in\Omega\cap M$, such that

$$f_p(D_{\varepsilon_1}\cap\Omega\cap N_p)\subset\Delta^{d-1}\times\mathbf{B}^{n-d+1}\subset f_p(D\cap\Omega),$$
$$g_p(D\cap\Omega)\subset\Delta^{d-1}\times\mathbf{B}^{n-d+1},$$

and

$$f_p(p)=g_p(p)=(\overbrace{1,\ldots,1}^{d\,\text{times}},\overbrace{0,\ldots,0}^{n-d\,\text{times}})=e_0.$$

For each point $z\in D\cap\Omega$ close to p_0 we let $p=\pi(z)\in M\cap\Omega$ be its projection onto M. The distance from points $f_p(z)$ and $g_p(z)$ to e_0 is then comparable to $d(z, M)$. Moreover, the left inclusion for f_p implies that for every point $z\in D_\varepsilon\cap\Omega$ close to p_0, the distance from $f_p(z)$ $(p=\pi(z))$ to each face of the domain $\Delta^{d-1}\times\mathbf{B}^{n-d+1}$ is comparable to $d(z, M)$.

By comparing the Bergman kernel function $K_D(z)$ with the Bergman kernel of the domain $\Delta^{d-1}\times\mathbf{B}^{n-d+1}$ we now get the estimates

$$C_1\, d(z, M)^{-(n+d)}\leqq K_D(z)\leqq C_2\, d(z, M)^{-(n+d)}, \qquad z\in D_\varepsilon\cap\Omega,$$

for some constants C_1 and C_2 independent of z. The left estimate follows from the inclusion involving g_p, with $p=\pi(z)$, and the right estimate follows from inclusions involving f_p.

Let $F\colon D\to D'$ be a biholomorphic map as in Proposition 1.4, mapping the corner $M\subset bD$ of codimension d into corner $M'\subset D'$ of codimension d'. The Bergman kernel functions of D and D' are related by

$$K_{D'}(F(z))\,|J(F)(z)|^2=K_D(z),$$

where $J(F)$ is the Jacobian determinant of F. We must prove that $d=d'$. To reach a contradiction suppose that $d\neq d'$.

Consider first the case $d<d'$. Let $z\in D_\varepsilon\cap\Omega$. By Proposition 2.1 and Corollary 2.2, the image $w=F(z)$ lies in D'_δ for some $\delta>0$, and the distances $d(z, M)$ and $d(w, M')$ are comparable. Combining the estimate on $K_D(z)$ from above with the estimate on $K_{D'}(w)$ from below we get

$$|J(F)(z)|^2\leqq C_3\, d(z, M)^{-(n+d)}/d(w, M')^{-(n+d')}\leqq C_4\, d(z, M)^{d'-d}.$$

Since $d' > d$, it follows that $J(F)(z)$ is bounded near M and has limit zero on M. Since $M \subset bD$ is a generating submanifold of $\mathbf{C}^n$, the uniqueness theorem of Sadullaev [29] implies that $J(F) = 0$ on D, a contradiction.

Similarly, if $d > d'$, we combine the estimate on $K_D(z)$ from below with the estimate on $K_{D'}(w)$ from above ($w = F(z) \in D'_\delta$) to get

$$|J(F)(z)|^{-2} \leqq C_5\, d(w, M')^{-(n+d')}/d(z, M)^{-(n+d)} \leqq C_6\, d(z, M)^{d-d'}.$$

It follows that the holomorphic function $1/J(F)(z)$ on D has limit zero on $M \subset bD$. Thus the function $1/J(F)$ is zero on D, a contradiction. Proposition 1.4 is proved.

Remark. Instead of using the Bergman kernel we could also use the osculating maps above and the estimates on the infinitesimal Kobayashi metric (or the Kobayashi volume).

3 A regularity theorem for C-R homeomorphisms

In this section we will explain how Theorems 1.1 and 1.2 can be proved by using the methods developed in [14].

We must recall the terminology of [14]. Let M be a C-R manifold of the form (2). In suitable local holomorphic coordinates (z, w) $(z \in \mathbf{C}^m,\ w \in \mathbf{C}^d,\ m + d = n)$ defined in a neighborhood U of a chosen point $p \in M$ (which we take to be the origin $p = 0$), the manifold M is given by

(9) $$M = \{(z, w) \in U : \Im w_j = Q_j(z, \bar{z}) + R_j(z, \bar{z}, \Re w),\ 1 \leqq j \leqq d\},$$

where $Q = (Q_1, \ldots, Q_d)$ is a hermitian quadratic form on $\mathbf{C}^m$ with values in $\mathbf{R}^d$, called the *Levi form* of M at 0, and $R = (R_1, \ldots, R_d)$ is a smooth function with values in $\mathbf{R}^d$ that vanishes to third order at the origin. We associate to M the *Levi cone* at the origin, defined by

$$C_0(M) = \mathrm{Co}\{\mathbf{Q}(z, \bar{z}) : z \in \mathbf{C}^m \backslash \{0\}\}.$$

(Co denotes the linearly convex hull.)

Definition. (a) A convex cone $\Gamma \subset \mathbf{R}^d$ is *strongly convex* if there is a vector $\sigma \in \mathbf{R}^d$ such that $\sigma \cdot x > 0$ for all $x \in \Gamma$, and if the closure $\bar{\Gamma}$ contains no line through the origin in $\mathbf{R}^d$.
(b) The manifold M (9) is *strongly pseudoconvex at* 0 if the Levi cone $C_0(M)$ is strongly convex.

Notice that a strongly convex cone does not contain the origin. It was shown in [14, Proposition 4.4], that M is strongly pseudoconvex at the origin if and only if M is contained, locally near the origin, in a strongly pseudoconvex hypersurface. Thus the two definitions of strong pseudoconvexity of M are equivalent. It is easily seen that the cone $C_0(M)$ is strongly convex if and only if there is a $\sigma \in \mathbf{R}^d$ satisfying

$$\sum_{j=1}^{d} \sigma_j Q_j(z, \bar{z}) > 0 \qquad \text{for all } z \in \mathbf{C}^m \backslash \{0\}.$$

To every open cone $\Gamma \subset \mathbf{R}^d$ with vertex 0 we associate the *wedge domain*

$$\mathscr{W}(\Gamma, U) = \{(z, w) \in U : \Im w - (Q(z, \bar{z}) + R(z, \bar{z}, \Re w)) \in \Gamma\} \tag{10}$$

with edge M. Let S be the unit sphere of $\mathbf{R}^d$. If $\Gamma_0 \subset \Gamma_1 \subset \mathbf{R}^d$ are cones (always with vertex 0) such that $S \cap \Gamma_0$ is relatively compact in $S \cap \Gamma_1$, we say that the cone Γ_0 is finer than Γ_1 and denote this relation by $\Gamma_0 < \Gamma_1$. The notation $\mathscr{W}(\Gamma_0, U_0) < \mathscr{W}(\Gamma_1, U_1)$ means that $\Gamma_0 < \Gamma_1$ and $U_0 \Subset U_1$.

The following result was proved in [14], although it was not stated explicitly. All sets and mappings should be understood as germs at the origin.

Theorem 3.1 *Let $F: M \to M'$, $F(0) = 0$, be a homeomorphic mapping between smooth manifolds $M, M' \subset \mathbf{C}^n$ of the form* (9). *Assume that there exist wedges $\mathscr{W}_1 < \mathscr{W}_2 < \mathscr{W}$ (resp. $\mathscr{W}_1' < \mathscr{W}_2' < \mathscr{W}'$) of the form* (10), *with edge M (resp. M'), such that the following hold near the origin:*

(a) *The cone Γ of $\mathscr{W}$ (resp. Γ' of $\mathscr{W}'$) is such that* $\mathrm{Co}(\Gamma \cup C_0(M))$ *(resp.* $\mathrm{Co}(\Gamma' \cup C_0(M'))$) *is strongly convex.*
(b) *F extends to a continuous map on $\mathscr{W}_2 \cup M$ that is holomorphic on $\mathscr{W}_2$ and satisfies $F(\mathscr{W}_1) \subset \mathscr{W}_1'$, $F(\mathscr{W}_2) \subset \mathscr{W}'$.*
(c) *F^{-1} extends to a continuous map on $\mathscr{W}_2' \cup M'$ that is holomorphic on $\mathscr{W}_2'$ and satisfies $F^{-1}(\mathscr{W}_2') \subset \mathscr{W}$.*

Then $F: M \to M'$ is a smooth diffeomorphism near the origin. If M and M' are smooth of order $k > 3$, then $F|_M$ is smooth of order $\mathscr{C}^{k-1-0}$ near 0. If M and M' are real-analytic, then F extends to a biholomorphic mapping in a neighborhood of the origin in $\mathbf{C}^n$.

We give a brief outline of the proof of Theorem 3.1. In [14] we assumed that the manifolds M and M' are over-extendable at the origin. Using this assumption we proved in [14, Proposition 5.1], that F and F^{-1} extend to holomorphic mappings on suitable wedges such that the conditions of Theorem 3.1 are satisfied. This was the only place in [14] where the over-extendability was used.

In Sects. 5 and 6 of [14] we only used the conditions given in Theorem 3.1 above. First we proved linear distance estimates comparing $d(z, M)$ with $d(F(z), M')$ for points $z \in \mathscr{W}_1$ [14, Proposition 5.2]. This is replaced by Corollary 2.2 in the present paper. The condition that the cone $\mathrm{Co}(\Gamma \cup C_0(M))$ is strongly convex implies that, in suitable local holomorphic coordinates near the origin, there exist d strongly convex 'barrier functions' $\rho_1^*, \ldots, \rho_d^*$, with linearly independent complex gradients at the origin, satisfying $\rho_j^*|_M = 0$ and $\rho_j^* < 0$ on $\mathscr{W}$ for $1 \leqq j \leqq d$. Similar barriers exist for the wedge $\mathscr{W}'$. This is the content of Proposition 4.3 in [14].

Using the distance estimates for F and the barrier functions as above, we obtained in [14, Proposition 5.3] the relevant estimates on the derivative $DF(z)$ of F at points $z \in \mathscr{W}_1$.

Finally, in Sect. 6 of [14] we used these estimates, together with a 'reflection principle' for generic C-R manifolds that was developed by Webster in [33] and [34], to prove that F is smooth on M. The same method was explained, for the case of hypersurfaces, in our paper [15]. We recall briefly the main idea.

Let $\mathscr{G} = \mathscr{G}(m, n)$ be the Grassman manifold of complex m-dimensional subspaces of $\mathbf{C}^n$, where $m = n - d$ is the C-R dimension of M and M'. We associate to M resp. M' a submanifold $\tilde{M}$ resp. $\tilde{M}'$ in the complex manifold $X=\mathbf{C}^n \times \mathcal{G}(m, n)$ by setting

$$\tilde{M}=\{(z, \mathrm{T}_z^{\mathbf{C}} M): z\in M\},$$

and similarly for $\tilde{M}'$. Notice that $\tilde{M}$ projects one-to-one onto M.

Webster proved in [33] that $\tilde{M}$ is totally real at the point $(p, \mathrm{T}_p^{\mathbf{C}} M)$ over $p\in M$ if and only if the Levi form of M at p is nondegenerate, which is the case here. However, $\tilde{M}$ is not generic in X unless $d=1$, because its dimension is too small.

The map $F: \mathcal{W} \to \mathcal{W}'$ is covered by the holomorphic map $\tilde{F}: \mathcal{W} \times \mathcal{G} \to \mathcal{W}' \times \mathcal{G}$, defined by

$$\tilde{F}(z, \Lambda)=(F(z), \mathrm{D}F(z)(\Lambda)).$$

Here, $\mathrm{D}F(z)(\Lambda)$ is the image of $\Lambda\in\mathcal{G}$ by the derivative of F at z.

Using the estimates on F as explained above, we proved in Sect. 6 of [14] that there exists a wedge domain $\tilde{\mathcal{W}} \subset \mathcal{W}_1 \times \mathcal{G} \subset X$ with edge (corner) $\tilde{M}$ such that $\tilde{F}$ extends continuously to $\tilde{\mathcal{W}} \cup \tilde{M}$ and maps $\tilde{M}$ to $\tilde{M}'$. More precisely, the boundary value of $\tilde{F}$ on $\tilde{M}$ is given by

$$\tilde{F}(p, \mathrm{T}_p^{\mathbf{C}} M)=(z, \mathrm{T}_z^{\mathbf{C}} M'), \quad p\in M, \quad q=F(p)\in M'.$$

This is the content of Proposition 6.2 in [14].

When the manifolds M and M' are real-analytic, we completed the proof by intersecting the wedge $\tilde{\mathcal{W}}$ with the complexification $\Sigma\subset X$ of $\tilde{M}$ in X. The wedge $\tilde{\mathcal{W}}$ is such that $\tilde{\mathcal{W}} \cap \Sigma$ is also a wedge domain in Σ, with edge $\tilde{M}$. Since $\tilde{M}$ is a generic totally real submanifold of its complexification Σ, we can reflect $\tilde{F}|_\Sigma$ across the edges $\tilde{M}$ and $\tilde{M}'$ and apply the edge-of-the-wedge theorem within Σ. This proves that F extends holomorphically across M.

If M and M' are merely smooth, we use an approximate (asymptotic) complexification of the two manifolds and apply the smooth version of the edge-of-the-wedge theorem as in [14] or [25]. The conclusion is that $\tilde{F}$ is smooth on $\tilde{M}$, and hence F is smooth on M. This proves Theorem 3.1.

Reduction of Theorems 1.1 and 1.2 to Theorem 3.1 Let $F: D \to D'$ be a biholomorphic mapping as in Theorem 1.1 or 1.2, mapping the corner manifold M of D homeomorphically to the corner M' of D'. We fix a point $p_0\in M$ and consider our sets and mappings as germs near the points p_0 resp. $q_0=F(p_0)\in M'$ which we take to be the origin in $\mathbf{C}^n$.

Applying Proposition 2.1 to F and F^{-1} we obtain smaller domains $D_\varepsilon \subset D_{\varepsilon_1} \subset D$ and $D'_\delta \subset D'_{\delta_1} \subset D'$ of the form (6), with corner M resp. M', satisfying

$$F(D_\varepsilon)\subset D'_\delta, \quad \text{and} \quad F^{-1}(D'_{\delta 1})\subset D_{\varepsilon_1}.$$

To complete the proof we observe that we can insert between every two domains $D_\varepsilon \subset D_{\varepsilon_1}$ of the form (6) with corner M $(0<\varepsilon_1<\varepsilon)$ a wedge domain $\mathcal{W}=\mathcal{W}(\Gamma, U)$ of the form (10) with edge M, at least in a sufficiently small neighborhood of the origin. It suffices to take a cone Γ that is strictly between the cones (8) of D_ε and D_{ε_1} at 0.

Moreover, since the functions ρ_j defining the domain D (3) are plurisubharmonic, it is easily seen that the cone (8), associated to D at the origin $0 \in M$, contains the Levi cone $C_0(M)$ in its closure and is itself strongly convex. Thus, D is locally near 0 contained in a wedge $\mathscr{W}(\Gamma, U)$ whose cone Γ is strongly convex and contains the Levi cone $C_0(M)$. The same holds for D'.

This shows that the conditions of Theorem 3.1 are satisfied if we choose the wedges such that $\mathscr{W}_1 \subset D_\varepsilon \subset D_{\varepsilon_1} \subset \mathscr{W}_2 \subset D \subset \mathscr{W}$ and $D'_\delta \subset \mathscr{W}_1' \subset \mathscr{W}_2' \subset D'_{\delta_1} \subset D' \subset \mathscr{W}'$. Thus Theorems 1.1 and 1.2 follow immediatelly from Theorem 3.1.

Remark. Notice that the role of Proposition 5.1 in [14] (that depended on overextendability of M resp. M') is replaced here by Proposition 2.1 and Corollary 2.2, applied to F and F^{-1}.

Remark. Recently, Webster [35] found a new 'reflection principle' on generic, Levi non-degenerate, C-R manifolds $M \subset \mathbf{C}^n$. The idea is to lift M to a generic totally real manifold $M^* \subset \mathbf{C}^n \times \mathbf{CP}^{n-1}$, of dimension $2n-1$. The fiber of M^* over a point $p \in M$ is the set of all complex hyperplanes $\Sigma \subset \mathrm{T}_p \mathbf{C}^n$ (considered as points in $\mathbf{CP}^{n-1}$) whose intersection with $\mathrm{T}_p M$ has codimension one in $\mathrm{T}_p M$. (Note that this intersection is either of codimension one or two.) It remains to be seen whether this type of reflection would simplify the proof of our extension theorems.

Proof of Theorem 1.3 For every point $p \in bD$ we denote by $d(p)$ the largest integer such that p is contained in a corner manifold $M \subset bD$ (2) of real codimension d in $\mathbf{C}^n$. Clearly the function $p \in bD \to d(p) \in \mathbf{Z}_+$ is upper semicontiuous. The set

$$M_k = \{p \in bD \colon d(p) = k\} \subset bD$$

for $k = 1, 2, \ldots, n$ is a locally closed, real-analytic manifold of codimension k in $\mathbf{C}^n$, and $\bar{M}_k = \bigcup_{j=k}^{n} M_j$. In particular, M_1 is the set of smooth boundary points of bD.

According to Range [27], F extends to a homeomorphism $F \colon \bar{D} \to \bar{D}'$. We claim that $F(M_k) = M'_k$ for every k, where M'_k is the analogous set in bD'. Theorem 1.3 then follows from Theorem 1.1 by extending F to a neighborhood V_k of every M_k. Clearly $V_1 \cup V_2 \cup \ldots \cup V_n$ is a neighborhood of bD, so we get an extension of F across bD.

In order to reach a contradiction we suppose that there is a point $p_0 \in M_k$ such that $F(p_0) = q_0 \in M'_{k'}$ for some $k' \neq k$. We may take $k > k'$ since the same argument can be applied to F^{-1} at q_0. Choose a neighborhood ω of p_0 in M_k and set

$$k_1 = \min\{d(q) \colon q = F(p), p \in \omega\} < k.$$

Let $p_1 \in \omega$ be a point at which the minimum occurs. Then we have $d(F(p)) = k_1$ for all p in some neighborhood $\omega_1 \subset \omega$ of p_1. Thus, F maps the corner $\omega_1 \subset M_k$ into the corner M'_{k_1} for $k \neq k_1$. This contradicts Proposition 1.4. Thus, F must take M_k to M'_k for every k. Since F is homeomorphic from bD to bD', it follows that $F(M_k) = M'_k$ for every k as claimed. This proves Theorem 1.3.

Acknowledgements. I wish to thank Sidney Webster who raised the question answered by Theorem 1.3. I also thank François Berteloot, Sergei Pinchuk, and Jean-Pierre Rosay for helpful discussions.

References

1. Baouendi, M.S., Jacobowitz, H., Treves, F.: On the analyticity of C-R mappings. Ann. Math. **122**, 365–400 (1985)
2. Baouendi, M.S., Rothschild, L.: Germs of C-R maps between analytic real hypersurfaces. Invent. Math. **93**, 481–500 (1988)
3. Baouendi, M.S., Rothschild, L.: Geometric properties of mappings between hypersurfaces in complex space. J. Differ. Geom. **31**, 473–499 (1990)
4. Bell, S., Catlin, D.: Boundary regularity of proper holomorphic mappings. Duke Math. J. **49**, 385–396 (1982)
5. Bell, S., Catlin, D.: Regularity of C-R mappings. Math. Z. **199**, 357–368 (1988)
6. Bishop, E.: Differentiable manifolds in complex Euclidean spaces. Duke Math. J. **32**, 1–21 (1965)
7. Chirka, E.M.: Regularity of boundaries of analytic sets (in Russian). Mat. Sb., New Ser. **117** (159), 291–336 (1982)
8. Coupét, B.: Construction de disques analytiques et applications. C.R. Acad. Sci. Paris, Ser. I Math. **304**, 427–430 (1987)
9. Coupét, B.: Regularité de fonctions holomorphes sur des wedges. Can. J. Math. **40**, 532–545 (1988)
10. Diederich, K., Fornæss, J.E.: Boundary regularity of proper holomorphic mappings. Invent. Math. **67**, 363–384 (1982)
11. Diederich, K., Fornæss, J.E.: Proper holomorphic mappings between real-analytic pseudoconvex domains in $\mathbf{C}^n$. Math. Ann. **282**, 681–700 (1988)
12. Duchamp, T., Stout, E.L.: Maximum modulus sets. Ann. Inst. Fourier **31**, 37–69 (1981)
13. Fefferman, C.: The Bergman kernel and biholomorphic mappings of pseudoconvex domains. Invent. Math. **26**, 1–65 (1974)
14. Forstneric, F.: Mappings of strongly pseudoconvex Cauchy-Riemann manifolds In: Bedford, E. et al. (eds.) Several complex variables and complex geometry. (Proc. Symp. Pure Math., vol. 52, part 1, 59–92) Providence, RI: Am. Math. Soc. 1991
15. Forstneric, F.: An elementary proof of Fefferman's theorem. Expo. Math. **10**, 135–150 (1992)
16. Forstneric, F.: Mappings of quadric Cauchy-Riemann manifolds. Math. Ann. **292**, 163–180 (1992)
17. Forstneric, F.: Proper holomorphic mappings: A survey (Math. Notes, Princeton, vol. 38) Princeton: Princeton University Press 1993
18. Hill, D.C., Taiani, G.: Families of analytic discs in $\mathbf{C}^n$ with boundaries in a prescribed C-R manifold. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **5**, 327–380 (1978)
19. Khurumov, N.: Boundary regularity of proper holomorphic mappings of strongly pseudoconvex domains (in Russian). Math. Zam. **48**, 149–150 (1990)
20. Lempert, L.: A precise result on the boundary regularity of biholomorphic mappings. Math. Z. **193**, 559–579 (1986)
21. Michel, J., Perotti, A.: $\mathscr{C}^k$-regularity for the $\bar{\partial}$-equation on strongly pseudoconvex domains with piecewise smooth boundareis. Math. Z. **203**, 415–427 (1990)
22. Nirenberg, L., Webster, S., Yang, P.: Local boundary regularity of holomorphic mappings. Commun. Pure Appl. Math. **33**, 305–338 (1980)
23. Pinchuk, S.I.: Holomorphic inequivalence of some classes of domains in $\mathbf{C}^n$ (in Russian). Mat. Sb. **111** (153), 61–86 (1980); English translation in Math. USSR Sb. **39**, no. 1 (1981)
24. Pinchuk, S.I., Chirka, E.M.: On the reflection principle for analytic sets (in Russian). Izv. Akad. Nauk SSSR **52**, 205–216 (1988); English translation in Math. USSR Izv. **32**, no. 1 (1989)
25. Pinchuk, S.I., Khasanov, S.V.: Asymptotically holomorphic functions (in Russian). Mat. Sb. **134** (176), 546–555 (1987); English translation in Math. USSR Sb. **62**, 541–550 (1989)
26. Pinchuk, S.I., Tsyganov, S.I.: Smoothness of C-R mappings of strongly pseudoconvex hypersurfaces (in Russian). Dokl. Akad. Nauk SSSR **53**, 1120–1129 (1989)

27. Range, R.M.: On the topological extension to the boundary of biholomorphic maps in $\mathbf{C}^n$. Trans. Am. Math. Soc. **216**, 203–216 (1976)
28. Range, R.M., Siu, Y.T.: Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strongly pseudoconvex boundaries. Math. Ann. **206**, 325–354 (1973)
29. Sadullaev, A.: A boundary uniqueness theorem in $\mathbf{C}^n$ (in Russian). Mat. Sb., New Ser. **101** (143), no. 4, 568–583 (1976); English translation in Math. USSR Sb. **30**, no. 4 (1976)
30. Tumanov, A.E.: Finite dimensionality of the group of C-R automorphisms of standard C-R manifolds and proper holomorphic mappings of Siegel domains (in Russian). Izv. Akad. Nauk SSSR **52**, 651–659 (1988)
31. Tumanov, A.E.: Extending C-R functions to a wedge from manifolds of finite type (in Russian). Mat. Sb. **136** (178), 128–139 (1988); English translation in Math. USSR Sb. **64**, 129–140 (1989)
32. Tumanov, A.E., Khenkin, G.M.: Local characterization of holomorphic automorphisms of Siegel domains (in Russian). Funkts. Anal. **17**, 49–61 (1983)
33. Webster, S.M.: Holomorphic mappings of domains with generic corners. Proc. Am. Math. Soc. **86**, 236–240 (1982)
34. Webster, S.M.: Analytic discs and the regularity of C-R mappings of real submanifolds in $\mathbf{C}^n$. In: Siv, Y.T. (ed.) Complex analysis of several variables. (Proc. Symp. Pure Math., vol. 41, pp. 199–208) Providence, RI: Am. Math. Soc. 1984
35. Webster, S.M.: The holomorphic contact geometry of a real hypersurface. (Preprint 1991)