Complements of Runge Domains and Holomorphic Hulls

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0. Introduction

A compact subset A of a Stein manifold X is said to be *holomorphically convex* if for every $z \in X \setminus A$ there is a holomorphic function f on X such that $|f(z)| > \sup_A |f|$. The holomorphic hull $\hat{A}$ of a compact set $A \subset X$ is the smallest holomorphically convex set containing A, given by

$$\hat{A} = \{z \in X : |f(z)| \le \sup_A |f| \text{ for all } f \in \mathcal{O}(X)\}.$$

When $X = \mathbf{C}^n$, $\hat{A}$ is the *polynomially* convex hull of A.

Holomorphically convex sets play an extremely important role in complex analysis; we refer the reader to Hörmander [7]. It is usually a difficult problem to decide whether a given set is holomorphically convex, or to determine its holomorphic hull. Therefore, the results that provide obstructions to holomorphic convexity, or that give estimates on the size and shape of the hull, are of interest to complex analysts.

Recently Alexander [1] applied the notion of *linking* to this problem. If a compact set $K \subset \mathbf{C}^n$ is linked by an orientable closed manifold $Y \subset \mathbf{C}^n \setminus K$ of real dimension $q \le n-1$, in the sense that Y is not homologous to zero in $\mathbf{C}^n \setminus K$, then Y must intersect the polynomial hull $\hat{K}$ [1, Thm. 1]. Alexander established a similar result for sets in Stein manifolds. His proof uses differential forms (via de Rham's theorem) and the Poincaré duality. This approach necessitates the use of homology and cohomology with real (or complex) coefficients. Specifically, the assumption that Y is not homologous to zero in $\mathbf{C}^n \setminus K$ means that there exists a differential q-form ω on $\mathbf{C}^n \setminus K$ such that $\int_Y \omega \neq 0$.

In the present paper we obtain more general topological properties of complements of holomorphically convex subsets $A \subset X$ in Stein manifolds X of dimension $n \ge 2$. We show that the inclusion $X \setminus A \hookrightarrow X$ induces isomorphism of low dimensional homology groups (up to dimension $n-2$), with coefficients in an arbitrary abelian coefficient group G. If X is contractible, the groups $H_k(X \setminus A; G)$ vanish in dimensions $1 \le k \le n-1$. When $X = \mathbf{C}^n$, the homotopy groups of $\mathbf{C}^n \setminus A$ in the same dimensions vanish as

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well. The results also hold if $A=\bar{\Omega}$ is the closure of a pseudoconvex *Runge* domain $\Omega \subset\subset X$ with $\mathcal{C}^1$ boundary (or, more generally, if the boundary bA has a collar in X). Recall [7] that a pseudoconvex domain $\Omega \subset X$ is Runge in X if holomorphic functions in Ω can be approximated, uniformly on compacts in Ω, by functions holomorphic in X.

In Section 2 we obtain the analogous result for hulls of compact sets in boundaries of strongly pseudoconvex domains (Theorem 4). We apply these results to the problem of detecting the holomorphic hull of a compact set $K \subset X$ by low-dimensional cycles in $X \setminus K$ that link K, thereby extending the results of Alexander [1]. Our method works in particular when the cycles are defined by closed manifolds, orientable or non-orientable, since we can use homology with coefficients $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$.

In the proof we use strongly plurisubharmonic functions and Morse theory in a similar way as Andreotti and Narasimhan in [4] (see also [3]), where they obtained topological properties of pseudoconvex Runge domains in Stein manifolds. The proof depends on two well-known facts:

(1) Every holomorphically convex set in a Stein manifold can be approximated from outside by sublevel sets of a strongly plurisubharmonic exhaustion function ρ on X with nondegenerate critical points (a Morse function).

(2) The Morse index of a strongly plurisubharmonic Morse function on an n-dimensional complex manifold at any of its critical points is at most n. This was first observed by Lefschetz; see [3].

In Section 1 we state and prove the result for $\mathbf{C}^n$, using the additional information on the behavior of ρ near infinity. We then state the results for arbitrary Stein manifolds (Theorem 2), but the proof is postponed to Section 3. Instead of merely adapting the proof for $\mathbf{C}^n$, we give a proof of Theorem 2 that uses only the homological properties of Runge domains from Andreotti and Narasimhan [4], thereby extending the result to a larger class of sets in Stein manifolds.

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1. Topology of Complements of Holomorphically Convex Sets

THEOREM 1. *If $A \subset \mathbf{C}^n$ ($n \geq 2$) is either a polynomially convex subset of $\mathbf{C}^n$ or the closure $A=\bar{\Omega}$ of a bounded pseudoconvex Runge domain $\Omega \subset \mathbf{C}^n$ with $\mathcal{C}^1$ boundary, then the complement of A is $(n-1)$-connected:*

$$\pi_k(\mathbf{C}^n \setminus A) = 0, \quad 1 \leq k \leq n-1, \tag{1}$$

and for any abelian group G,

$$\mathrm{H}_k(\mathbf{C}^n\setminus A; G)=0, \quad 1\le k\le n-1. \tag{2}$$

REMARK 1. (2) follows immediately from (1). The theorem of Hurewicz [10, p. 398] also gives the isomorphism $\mathrm{H}_n(\mathbf{C}^n\setminus A;\mathbf{Z})=\pi_n(\mathbf{C}^n\setminus A)$. Conversely, if (2) holds for $G=\mathbf{Z}$, and if $\mathbf{C}^n\setminus A$ is simply connected ($\pi_1(\mathbf{C}^n\setminus A)=0$), then (1) follows from Hurewicz's theorem.

REMARK 2. The hypothesis in the second case, namely that A has $\mathcal{C}^1$ boundary, can be replaced by the weaker hypothesis that the boundary bA has a closed neighborhood V in $\mathbf{C}^n$ that is homeomorphic to $bA\times[-1,1]$. Such a neighborhood V is called a *collar* of bA in $\mathbf{C}^n$ [10]. The same remark applies to Theorem 2 below.

Proof of Theorem 1. We first consider the case when A is a compact polynomially convex set in $\mathbf{C}^n$. Given an open neighborhood U of A in $\mathbf{C}^n$, there exist a smooth, strongly plurisubharmonic exhaustion function $\rho:\mathbf{C}^n\to\mathbf{R}$ and a constant $R>0$ satisfying

(i) $\rho<0$ on A and $\rho>0$ on $\mathbf{C}^n\setminus U$;
(ii) $\rho(z)=|z|^2$ for $|z|\ge R$; and
(iii) ρ is a Morse function on $\mathbf{C}^n$, with finitely many critical points, each of them of Morse index at most n.

The construction of such a function is completely standard, and we recall it only for the sake of completeness. First, we choose a strongly plurisubharmonic function ρ' on $\mathbf{C}^n$ satisfying (i) [7, p. 48]. Choose a large $R>0$ such that $\bar{U}\subset B(R/3)$, where $B(R)$ denotes the ball of radius R, centered at the origin. Pick a smooth function $h(t)\ge 0$ on $\mathbf{R}$ that vanishes for $t\le R/3$, is strictly convex and increasing for $t>R/3$, and $h(t)=t^2$ for $t\ge R$. Then $h(|z|)$ is plurisubharmonic on $\mathbf{C}^n$, it is strongly plurisubharmonic outside the closed ball $\bar{B}(R/3)$, and it vanishes on $\bar{U}$. Next we pick a smooth function $0\le\chi(t)\le 1$ on $\mathbf{R}$ such that $\chi(t)=1$ for $t\le R/2$ and $\chi(t)=0$ for $t\ge R$. The function

$$\rho(z)=h(|z|)+\epsilon\chi(|z|)\rho'(z)$$

for a sufficiently small $\epsilon>0$ is strongly plurisubharmonic and satisfies properties (i) and (ii). Finaly, by a small smooth perturbation of ρ, supported on $B(R)$, we may assume that ρ is a Morse function on $\mathbf{C}^n$, thereby satisfying (iii) as well. We may assume in addition that 0 is a regular value of ρ.

For $t\in\mathbf{R}$ we set

$$X_t=\{z\in\mathbf{C}^n:\rho(z)\ge t\}.$$

For $c\ge R^2$ the set X_c is the complement of the ball $B(\sqrt{c})$, and therefore it is homotopically equivalent to the sphere S^{2n-1}. Fix such a c. As t decreases from $t=c$ to $t=0$, the sets X_t increase from X_c to X_0.

We think of $X_t=\{-\rho\le -t\}$ as a sublevel set of the Morse function $-\rho$ whose Morse indices are all $\ge n$. Let $t_1<t_2<\cdots<t_m$ be the critical values

of ρ, with the corresponding Morse indices $2n-m_j \leq n$. Thus, the Morse index corresponding to the critical value $-t_j$ of $-\rho$ is $m_j \geq n$.

By the fundamental result of Morse theory [9], the topological type of X_t changes only at the critical values t_j, and the change at t_j can be described by noting that the set X_t for $t_{j-1}<t<t_j$ is obtained from X_s for $t_j<s<t_{j+1}$ by attaching to X_s a $2n$-cell $(D^{m_j}\times D^{2n-m_j}, \partial D^{m_j}\times D^{2n-m_j})$ by a map h: $\partial D^{m_j}\times D^{2n-m_j}\to \partial X_s$. (Here, D^m denotes the closed ball in $\mathbf{R}^m$.)

The Morse index $k=m_j\geq n$ is the only dimension in which the relative homology group $\mathrm{H}_k(X_t, X_s; \mathbf{Z})$ does not vanish [11, Chap. 5]. If we further choose a number u satisfying $t_{j-2}<u<t_{j-1}$, we have an exact homology sequence of the triple $X_s\subset X_t\subset X_u$ [10, p. 185]:

$$\cdots \to \mathrm{H}_k(X_t, X_s; \mathbf{Z}) \to \mathrm{H}_k(X_u, X_s; \mathbf{Z}) \to \mathrm{H}_k(X_u, X_t; \mathbf{Z}) \to \cdots.$$

When $k\leq n-1$, the outer two terms vanish, and therefore the middle term vanishes as well. Applying this argument finitely many times, we obtain

$$\mathrm{H}_k(X_0, X_c; \mathbf{Z})=0, \quad 0\leq k\leq n-1.$$

Consider now the exact homology sequence of the pair $X_c \hookrightarrow X_0$:

$$\cdots \to \mathrm{H}_k(X_c; \mathbf{Z}) \to \mathrm{H}_k(X_0; \mathbf{Z}) \to \mathrm{H}_k(X_0, X_c; \mathbf{Z}) \to \cdots.$$

When $1\leq k\leq n-1$, the outer two terms vanish, and therefore $\mathrm{H}_k(X_0; \mathbf{Z})=0$ as well. Notice that $\mathbf{C}^n\backslash U\subset X_0\subset \mathbf{C}^n\backslash A$. By passing to the limit as U shrinks down to A we get $\mathrm{H}_k(\mathbf{C}^n\backslash A; \mathbf{Z})=0$ for $1\leq k\leq n-1$. The universal coefficient theorem for homology [10] implies the same result for homology with coefficients in an arbitrary abelian group G.

If we repeat the same argument with the homology groups replaced by the homotopy groups, we obtain (1) (see [11, Chap. 5]). Alternatively, it suffices to notice that the sublevel sets X_t remain simply connected as t decreases through a critical value of ρ, and therefore (1) follows from (2) by the theorem of Hurewicz [10, p. 398].

It remains to deal with the case when $A=\bar{\Omega}$ is the closure of a Range domain with $\mathcal{C}^1$ boundary. There is a closed neighborhood V of bA in $\mathbf{C}^n$ and a homeomorphism $\Phi: V\to bA\times[-1,1]$ such that $\Phi(bA)=ba\times\{0\}$. (In fact, this is the only condition on bA that we need for the proof.) For $-1\leq t\leq 1$ we set

$$A_t=(A\backslash V)\cup\Phi^{-1}(bA\times[-1,t]).$$

With a proper choice of Φ we have $A_s\subset A_t$ when $s<t$.

For each $0<\epsilon<1$ there exists a homeomorphism Θ_ϵ of $\mathbf{C}^n$ onto itself that is the identity outside V and satisfies

$$\Theta_\epsilon(A_{-\epsilon})=A_0=A, \quad \Theta_\epsilon(A)=A_\epsilon.$$

To construct Θ_ϵ we choose a homeomorphism $\psi: [-1,1]\to[-1,1]$ satisfying $\psi(-\epsilon)=0$ and $\psi(0)=\epsilon$; we set $\Psi(x,t)=(x,\psi(t))$ for $(x,t)\in bA\times(-1,1)$ and let $\Theta_\epsilon=\Phi^{-1}\circ\Psi\circ\Phi$ on V. Clearly Θ_ϵ can be extended to $\mathbf{C}^n$ as the identity outside V.

For each $\epsilon > 0$ the set $A_{-\epsilon}$ is compact and contained in $\Omega = \operatorname{Int} A$. Since Ω is pseudoconvex and Runge in $\mathbf{C}^n$, the polynomial hull $K = \hat{A}_{-\epsilon}$ is also compactly contained in Ω [7]. By the first part of the proof, the complement $\mathbf{C}^n \backslash K$ is $(n-1)$-connected. The same is then true for $\Theta_\epsilon(\mathbf{C}^n \backslash K) = \mathbf{C}^n \backslash \Theta_\epsilon(K)$. By construction we have $A \subset \Theta_\epsilon(K) \subset A_\epsilon$. The required result for $\mathbf{C}^n \backslash A$ follows by passing to the limit as $\epsilon \to 0$ (and A_ϵ shrinks down to A). This completes the proof of Theorem 1. □

If X is an arbitrary Stein manifold of dimension $n \geq 2$ and $A \subset X$ is as in Theorem 1, the same proof gives the isomorphism $\mathrm{H}_k(X \backslash A; G) = \mathrm{H}_k(X; G)$ for $0 \leq k \leq n-2$. However, since we no longer have the information on ρ near infinity, we are not able to conclude the same for $k = n-1$. In fact, Example 2 below shows that the result is no longer true for $k = n-1$, unless we impose some homology conditions on X. Here is the general result.

THEOREM 2. *Let X be a Stein manifold of dimension $n \geq 2$, and let $A \subset X$ be either a compact holomorphically convex subset of X, or $A = \bar{\Omega}$, where $\Omega \subset\subset X$ is a pseudoconvex Runge domain with $\mathcal{C}^1$ boundary. Then the inclusion $X \backslash A \hookrightarrow X$ induces isomorphisms*

$$\mathrm{H}_k(X \backslash A; G) = \mathrm{H}_k(X; G), \quad 0 \leq k \leq n-2, \tag{3}$$

for any abelian group G. If $\mathrm{H}_n(X; \mathbf{R}) = 0$ then we also have

$$\mathrm{H}_{n-1}(X \backslash A; \mathbf{R}) = \mathrm{H}_{n-1}(X; \mathbf{R}). \tag{4}$$

If $\mathrm{H}_n(X; \mathbf{Z}) = \mathrm{H}_{n-1}(X; \mathbf{Z}) = 0$ then, for any abelian group G,

$$\mathrm{H}_{n-1}(X \backslash A; G) = \mathrm{H}_{n-1}(X; G).$$

COROLLARY 1. *If X is a contractible Stein manifold of dimension $n \geq 2$, and if $A \subset X$ is as in Theorem 2, then for any abelian group G*

$$\mathrm{H}_k(X \backslash A; G) = 0, \quad 1 \leq k \leq n-1.$$

Theorem 2 is proved in Section 3 below. The next result concerns the relative homology of a pair of holomorphically convex sets.

THEOREM 3. *If $K \subset A$ are compact, holomorphically convex subsets of an n-dimensional Stein manifold X, with K contained in the interior of A, then for any abelian group G*

$$\mathrm{H}_k(X \backslash K, X \backslash A; G) = 0, \quad 0 \leq k \leq n-1.$$

Proof. Given open neighborhoods $K \subset U$ and $A \subset V$, with $\bar{U} \subset \operatorname{Int} A$, there exists a strongly plurisubharmonic Morse function $\rho: X \to \mathbf{R}$ satisfying the following properties:

(i) $\rho < 0$ on K and $\rho > 0$ on $X \backslash U$;
(ii) $\rho < 1$ on A and $\rho > 1$ on $X \backslash V$; and
(iii) 0 and 1 are regular values of ρ.

The construction of ρ is similar to the one in the proof of Theorem 1. We begin by choosing strongly plurisubharmonic functions ϕ and ψ on X, satisfying

(a) $\phi<0$ on K and $\phi>0$ on $X\backslash U$;
(b) $\psi<0$ on A and $\psi>0$ on $X\backslash V$; and
(c) 0 is a regular value of ϕ and ψ.

(See [7, p. 48].) Next we set $\tilde{\psi}=C\psi+1$ for some large constant $C>0$ such that $\tilde{\psi}<0$ on $\bar{U}$. By composing $\tilde{\psi}$ with a suitable function on $\mathbf{R}$ that vanishes for $t\le 0$ and is strictly convex and increasing for $t>0$, we get a plurisubharmonic function $\psi_1\ge 0$ on X that vanishes on $\bar{U}$, is strongly plurisubharmonic where positive, and satisfies property (ii). Finally, let $0\le\chi\le 1$ be a smooth function that equals 1 on the set $\{\psi_1\le 1/3\}$ and has support contained in the set $\{\psi_1\le 2/3\}\subset V$. The function $\rho=\psi_1+\epsilon\chi\phi$ for a sufficiently small $\epsilon>0$ is strongly plurisubharmonic on X, and satisfies the required properties.

Set

$$X_t=\{z\in X:\rho(z)>t\}\quad\text{for } t\in\mathbf{R}.$$

Then $X\backslash U\subset X_0\subset X\backslash K$ and $X\backslash V\subset X_1\subset X\backslash A$. Since the Morse indices of $-\rho$ are all $\ge n$, we have $\mathrm{H}_k(X_0,X_1;\mathbf{Z})=0$ for $0\le k\le n-1$ (see the proof of Theorem 1). Theorem 3 now follows by passing to the limit as U shrinks down to K and V shrinks to A (thus X_0 increases to $X\backslash K$ and X_1 increases to $X\backslash A$). □

EXAMPLE 1. There exist compact, polynomially convex sets $A\subset\mathbf{C}^n$ such that

$$\mathrm{H}_k(\mathbf{C}^n\backslash A;\mathbf{Z})\neq 0,\quad n\le k\le 2n-1.$$

To see this, observe that every compact subset of the totally real subspace $\mathbf{R}^n\subset\mathbf{C}^n$ is polynomially convex since every polynomial on $\mathbf{R}^n$ extends to a holomorphic polynomial on $\mathbf{C}^n$. If $S^m\subset\mathbf{C}^n=\mathbf{R}^{2n}$ is an m-dimensional sphere then $\mathrm{H}_{2n-m-1}(\mathbf{C}^n\backslash S^m;\mathbf{Z})=\mathbf{Z}$, and the other homology groups (in dimensions >0) vanish [10, p. 198]. Thus, if A is a disjoint union of spheres in $\mathbf{R}^n\subset\mathbf{C}^n$ of various dimensions $0\le m\le n-1$, then A is polynomially convex, and the homology groups $\mathrm{H}_k(\mathbf{C}^n\backslash A;\mathbf{Z})$ for $n\le k\le 2n-1$ are nonvanishing.

EXAMPLE 2. The condition $\mathrm{H}_n(X;\mathbf{R})=0$ is necessary in Theorem 2(4). The following example to this effect is due to H. Alexander: Take $X=\mathbf{C}_*^n$, where $\mathbf{C}_*=\mathbf{C}\backslash\{0\}$, and let $A=T^n$ be the standard n-dimensional torus. Then A is holomorphicaly convex in X, $\mathrm{H}_{n-1}(X;\mathbf{R})=\mathrm{H}_{n-1}(A;\mathbf{R})=0$, but $\mathrm{H}_{n-1}(X\backslash A;\mathbf{R})\neq 0$. To see this, let $D_p\subset N_p$ be a small disc neighborhood of a point $p\in A$ in the normal space N_p to A at p. The oriented intersection number of D_p with A (in $\mathbf{C}^n$) equals ± 1, and hence the sphere $S_p^{n-1}=\partial D_p$ is not homologous to zero in $X\backslash A$.

A more general example is obtained as follows. Let A be any closed, orientable, real-analytic manifold of real dimension n such that $\mathrm{H}_{n-1}(A,\mathbf{R})=0$ (for instance, we may take the n-sphere). The complexification $\tilde{A}$ of A is a complex manifold of dimension n that contains A as a maximal totally real submanifold. A has small Stein neighborhoods X in $\tilde{A}$ such that A is holomorphically convex in X, and X is diffeomorphic to a real vector bundle of rank n over A. It follows that $\mathrm{H}_{n-1}(X;\mathbf{R})=\mathrm{H}_{n-1}(A;\mathbf{R})=0$. On the other hand, if the Euler characteristic of A is not equal to ± 1, it can be shown that $\mathrm{H}_{n-1}(X\setminus A;\mathbf{R})\neq 0$.

2. Linking and Holomorphic Hulls

In this section we show that Theorems 1 and 2 immediately imply the results of Alexander [1]. The main result of [1] is how to detect the (nontrivial) holomorphic hull of a compact set K in a Stein manifold X by using closed orientable manifolds $Y\subset X\setminus K$, of real dimension at most $\dim_{\mathbf{C}} X-1$, that link K. Here we do the same in the more general case where Y is a *singular cycle* with coefficients in an arbitrary abelian group G. In practice, the most interesting cases to consider are $G=\mathbf{Z}$ and $G=\mathbf{Z}_2$. The latter group allows us to use closed non-orientable manifolds as well.

We shall use basic notions of singular homology theory, and we refer the reader to Spanier [10] for a systematic treatment. Let $C_q(\Omega;G)$ be the abelian group of singular q-chains in an open set $\Omega\subset X$, with coefficients in an abelian group G. Every such chain is a finite sum $c=\sum\alpha_j f_j$, where $f_j\colon\Delta_q\to\Omega$ is a continuous map from the standard q-dimensional simplex into Ω and $\alpha_j\in G$. The support $\operatorname{supp} c$ of c is the union of images of the maps f_j (with nonzero coefficients) that define c. The boundary ∂c_q of a q-chain in Ω is a $(q-1)$-chain in Ω. If $\partial c_q=0$ then c_q is called a *q-cycle*. The singular homology group $\mathrm{H}_q(\Omega;G)$ is the quotient group of q-cycles modulo the boundaries of $(q+1)$-chains.

The first result is an immediate corollary of Theorems 1 and 2; it corresponds to Theorem 1 of Alexander [1] in the case of cycles defined by closed orientable manifolds $Y\subset X\setminus K$. G is an arbitrary abelian group unless otherwise specified.

COROLLARY 2. *Let K be a compact subset of an n-dimensional Stein manifold X $(n\geq 2)$, and let $c_q\in C_q(X\setminus K;G)$ be a q-cycle in $X\setminus K$ that is homologous to zero in X but not in $X\setminus K$. Assume that at least one of the following conditions holds:*

(a) $0\leq q\leq n-2$;
(b) $q=n-1$, $G=\mathbf{R}$, *and* $\mathrm{H}_n(X;\mathbf{R})=0$;
(c) $q=n-1$ *and* $\mathrm{H}_n(X;\mathbf{Z})=\mathrm{H}_{n-1}(X;\mathbf{Z})=0$.

Then $\operatorname{supp} c_q$ *intersects the hull* $\hat{K}$.

Proof. If c_q is homologous to zero in X, and if its support does not intersect the hull of K, then c_q is homologous to zero in $X\setminus\hat{K}$ by Theorem 1 or 2, and hence it is homologous to zero in the larger set $X\setminus K$. □

Next we consider compact sets in strongly pseudoconvex boundaries.

THEOREM 4. *Suppose that* X *is a Stein manifold of dimension* $n \geq 2$, $D \subset\subset X$ *is a strongly pseudoconvex domain with* $\mathcal{C}^2$ *boundary, and* K *is a compact subset of* bD. *Denote by* $\hat{K}$ *the* $\mathcal{O}(\bar{D})$*-convex hull of* K. *Then the inclusion* $bD \backslash K \hookrightarrow \bar{D} \backslash \hat{K}$ *induces isomorphisms*

$$\mathrm{H}_k(bD\backslash K; G) = \mathrm{H}_k(\bar{D}\backslash \hat{K}; G), \quad 0 \leq k \leq n-2.$$

EXAMPLE 3. Let D be the unit ball in $\mathbf{C}^n$, and let

$$K = bD \cap \mathbf{R}^n.$$

Then K is an $(n-1)$-sphere in bD that is polynomially convex, and we have $\mathrm{H}_{n-1}(bD\backslash K) \neq 0$ while $\mathrm{H}_{n-1}(\bar{D}\backslash\hat{K}) = \mathrm{H}_{n-1}(\bar{D}\backslash K) = 0$. This shows that Theorem 4 fails in dimension $k = n-1$.

COROLLARY 3 (notation as above). *If* $c_q \in C_q(bD\backslash K; G)$ *is a* q*-cycle in* $bD\backslash K$ $(q \leq n-2)$ *that is not homologous to zero in* $bD\backslash K$, *and if* c_{q+1} *is a* $(q+1)$*-chain in* $\bar{D}$ *satisfying* $\partial c_{q+1} = c_q$, *then* $\operatorname{supp} c_{q+1}$ *intersects* $\hat{K}$.

Corollary 3 can be expressed by saying that *any cycle of dimension* $\leq n-2$ *that links* K *in* $bD\backslash K$ *also links* $\hat{K}$ *in* $\bar{D}\backslash\hat{K}$. For cycles defined by smooth orientable manifolds this is Theorem 2 of Alexander [1]. When $n = 2$, Theorem 4 is only applicable for $q = 0$, and it gives the well-known result that *every connected component of* $\bar{D}\backslash\hat{K}$ *contains precisely one connected component of* $bD\backslash K$ (see [1; 2; 5; 8]).

Proof of Theorem 4. There exists a strongly plurisubharmonic function ρ of class $\mathcal{C}^2$ in a neighborhood of $\bar{D}$ such that $D = \{\rho < 0\}$ and $d\rho \neq 0$ near bD. Replacing X by a suitable sublevel set of ρ, we may assume that D is Runge in X and that $\bar{D}$ is holomorphically convex in X. Then $\hat{K} \subset \bar{D}$ is the holomorphically convex hull of K in X. Note that $\hat{K} \cap bD = K$ by strong pseudoconvexity.

Let E_0 be a small tubular neighborhood of bD in X, thought of as a neighborhood of the zero section in the normal bundle to bD in X, with convex fibers. Let $\pi: E_0 \to bD$ be the projection onto the zero section. Choose a small smooth function $\chi \geq 0$ on bD that vanishes precisely on K, and let $E = \{z \in E_0 : |\rho(z)| < \chi(\pi(z))\}$. Clearly $E \subset\subset E_0$ is an open neighborhood of $bD\backslash K$, and with a suitable choice of χ we have $E \cap \hat{K} = \emptyset$. Set $U = (D\backslash\hat{K}) \cup E$ and $V = (X\backslash\bar{D}) \cup E$. Then $U \cup V = X\backslash\hat{K}$ and $U \cap V = E$.

Set $X' = X\backslash\hat{K}$. The definition of E implies that $V = (X\backslash\bar{D}) \cup E$ can be deformed onto $X\backslash\bar{D}$ by a homeomorphism of X'. Therefore $\mathrm{H}_k(X', V) = \mathrm{H}_k(X', X\backslash\bar{D})$ for all k. (We omit the coefficient group G.) This group vanishes in dimensions $0 \leq k \leq n-1$ according to Theorem 3. (To be precise, we use Theorem 3 with $\bar{D}$ replaced by larger sublevel sets $\{\rho \leq \epsilon\}$ and then pass to the limit as $\epsilon > 0$ decreases to zero.)

Next we observe that the set $Y = V\backslash E$ is relatively closed in V, and therefore is excisive for the singular homology of the pair $V \subset X'$ in the sense that

$H_*(X', V) = H_*(X'\backslash Y, V\backslash Y)$ [10, p. 189, Cor. 5]. We have $X'\backslash Y = U$ and $V\backslash Y = E$; hence $0 = H_k(X', V) = H_k(U, E)$ for $0 \le k \le n-1$. From this and the exact homology sequence of the pair $E \subset U$ we get $H_k(E) = H_k(U)$ for $0 \le k \le n-2$. By construction, E is homotopically equivalent to $bD\backslash K$ and U is equivalent to $\bar{D}\backslash \hat{K}$. This gives the required isomorphisms $H_k(bD\backslash K) = H_k(\bar{D}\backslash \hat{K})$ for $0 \le k \le n-2$. Theorem 4 is proved. □

3. Proof of Theorem 2

We begin by recalling some results of Andreotti and Narasimhan [4].

THEOREM 5 (Andreotti and Narasimhan [4]). *Let X be an n-dimensional Stein manifold, and let $\Omega \subset X$ be a pseudoconvex Runge domain. Then:*

(a) $H_k(X; \mathbf{Z}) = 0$ *for* $k > n$, *and* $H_n(X; \mathbf{Z})$ *is torsion free.*
(b) *The natural homomorphism* $H_n(\Omega; \mathbf{Z}) \to H_n(X; \mathbf{Z})$ *is injective; hence* $H^n(X; \mathbf{R}) \to H^n(\Omega; \mathbf{R})$ *is surjective.*
(c) *If* $H_n(X; \mathbf{Z}) = H_{n-1}(X; \mathbf{Z}) = 0$, *then* $H_k(\Omega; \mathbf{Z}) = 0$ *for* $k \ge n$ *and* $H_{n-1}(\Omega; \mathbf{Z})$ *is torsion-free. This holds in particular if X is contractible.*

Proof of Theorem 2. We will prove Theorem 2 under the weaker assumption that $A = \bar{\Omega} \subset X$, where $\Omega \subset\subset X$ is a domain with $\mathcal{C}^1$ boundary that satisfies the homological properties stated in Theorem 5. *We emphasize that no further analytic information on A is needed.* Clearly A and Ω have the same homology and cohomology groups. We shall distinguish three cases.

Case 1: $H_n(X; \mathbf{Z}) = H_{n-1}(X; \mathbf{Z}) = 0$ (e.g., $X = \mathbf{C}^n$, or X contractible). According to Theorem 5(c) we have

$$H_k(A; \mathbf{Z}) = 0, \quad k \ge n;$$

$$H_{k-1}(A; \mathbf{Z}) \text{ is torsion-free.}$$

The set A is a finite CW-complex and therefore its homology and cohomology groups in all dimensions are finitely generated. By [10, p. 244, Cor. 4], the free parts of $H^k(A; \mathbf{Z})$ and $H_k(A; \mathbf{Z})$ are isomorphic, and the torsion part of $H^k(A; \mathbf{Z})$ is isomorphic to the torsion part of $H_{k-1}(A; \mathbf{Z})$. It follows that $H^k(A; \mathbf{Z}) = 0$ for $k \ge n$. The same is then true for every abelian group G by the universal coefficient theorem for cohomology [10, p. 246, Thm. 10]:

$$H^k(A; G) = 0, \quad k \ge n. \tag{5}$$

We now apply the Alexander duality theorem [10, p. 296, Thm. 16] to the compact subset A of the manifold X:

$$H_k(X, X\backslash A; G) = H^{2n-k}(A; G) = 0, \quad 0 \le k \le n. \tag{6}$$

Consider the exact homology sequence of the pair $X\backslash A \subset X$:

$$\cdots \to H_{k+1}(X, X\backslash A; G) \to H_k(X\backslash A; G) \to H_k(X; G) \to H_k(X, X\backslash A; G) \to \cdots.$$

For $0 \leq k \leq n-1$ we have

$$0 \to H_k(X \setminus A; G) \to H_k(X; G) \to 0,$$

which gives the isomorphisms (3) in Theorem 2. If X is contractible, we get $H_k(X \setminus A; G) = 0$ for $1 \leq k \leq n-1$. This settles case 1.

Case 2: X is an arbitrary Stein manifold. By Theorem 5(a), we have $H_k(A; \mathbf{Z}) = 0$ for $k > n$ and $H_n(A; \mathbf{Z})$ is torsion-free. Applying (as before) the universal coefficient theorem for cohomology, we get $H^k(A; G) = 0$ for $k > n$. The same proof as in case 1 gives the isomorphisms of Theorem 2 for $k \leq n-2$.

Case 3: $H_n(X; \mathbf{R}) = 0$. Then $H^n(X; \mathbf{R}) = 0$ and therefore $H^n(A; \mathbf{R}) = 0$ by Theorem 5(b). The Alexander duality theorem (6) (with $G = \mathbf{R}$) implies $H_n(X, X \setminus A; \mathbf{R}) = H^n(A; \mathbf{R}) = 0$. The isomorphism $H_{n-1}(X \setminus A; \mathbf{R}) = H_{n-1}(X; \mathbf{R})$ follows in the same way as in case 1 from the exact homology sequence of the pair $X \setminus A \subset X$ with coefficients $\mathbf{R}$. This concludes the proof of Theorem 2. □

REMARK 3. The fact that pseudoconvex Runge domains (and polynomially convex subsets) $A \subset \mathbf{C}^n$ satisfy $H^k(A; \mathbf{C}) = 0$ for $k \geq n$ is well known and can be found in numerous places in the literature. The stronger result (5), namely that the same holds for cohomology with arbitrary abelian coefficients G, is curiously absent in most texts. For instance, the vanishing of the group $H^n(A; \mathbf{Z}_2)$ for polynomially convex subsets $A \subset \mathbf{C}^n$ implies that *no closed, compact, n-dimensional submanifold of* $\mathbf{C}^n$, *orientable or non-orientable, is polynomially convex.*

REMARK 4. If we restrict ourselves to homology and cohomology with real coefficients, then an alternative proof of all results in this paper can be given by using differential forms and the Poincaré duality theorem. This approach was used by Alexander [1]. We refer the reader to [6] for general results on cohomology with differential forms.

We give an outline of the proof of Theorem 1 for $G = \mathbf{R}$. We want to show that $H_k(X \setminus A; \mathbf{R}) = 0$ for $1 \leq k \leq n-1$, where A is the closure of a Runge domain $\Omega \subset\subset \mathbf{C}^n$ with $\mathcal{C}^1$ boundary. By de Rham's theorem this is equivalent to $\int_Y \alpha = 0$ for every closed orientable submanifold $Y \subset \mathbf{C}^n \setminus A$ of dimension k and for every closed k-form α on $\mathbf{C}^n \setminus A$.

Fix $Y \subset \mathbf{C}^n \setminus A$. The integration of closed k-forms $\alpha \in D^k(\mathbf{C}^n \setminus A)$ over Y defines a linear functional on $H^k(\mathbf{C}^n \setminus A; \mathbf{R})$. By Poincaré duality this functional can be represented by a closed form $\eta = \eta_Y$ of degree $2n-k$, with compact support contained in $\mathbf{C}^n \setminus A$, in the sense that

$$\int_Y \alpha = \int_{\mathbf{C}^n} \alpha \wedge \eta$$

for all closed k-forms α on $\mathbf{C}^n\setminus A$. Such a form η is called a (compact) Poincaré dual of Y.

Since $A=\bar{\Omega}$ has $\mathcal{C}^1$ boundary, it suffices to consider k-forms α defined on $\mathbf{C}^n\setminus A_0$ for some smaller compact set $A_0\subset\Omega$. Choose a smoothly bounded domain $D\subset\subset\Omega$ containing A_0. Since $\mathrm{H}_c^{2n-k}(\mathbf{C}^n;\mathbf{R})=0$ for $k\geq 1$, there exists a compactly supported form ω of degree $2n-k-1$ on $\mathbf{C}^n$ satisfying $d\omega=\eta$. By Stokes' theorem, applied to the form

$$\alpha\wedge\eta=\alpha\wedge d\omega=(-1)^k d(\alpha\wedge\omega)$$

with bounded support on $\mathbf{C}^n\setminus D$, we get

$$\int_Y \alpha=\int_{\mathbf{C}^n}\alpha\wedge\eta=\pm\int_{\mathbf{C}^n\setminus D} d(\alpha\wedge\omega)=\pm\int_{bD}\alpha\wedge\omega.$$

The form ω is closed on Ω since $d\omega=\eta$ is supported on $\mathbf{C}^n\setminus\bar{\Omega}$. Since Ω is Runge in $\mathbf{C}^n$, we have $\mathrm{H}^r(\Omega;\mathbf{R})=0$ for $r\geq n$. If $k\leq n-1$ then $2n-k-1\geq n$, and hence $\omega=d\beta$ for some $(2n-k-2)$-form β on Ω. Since $bD\subset\Omega$, we get

$$\int_Y \alpha=\pm\int_{bD}\alpha\wedge d\beta=\pm\int_{bD} d(\alpha\wedge\beta)=0$$

by Stokes' theorem. This shows that the functional associated to Y is zero, and hence Y is homologous to zero in $\mathbf{C}^n\setminus A$.

REMARK 5. The referee kindly pointed out the recent work by G. Lupacciolu [8], who obtained topological properties of certain classes of compact sets in q-complete complex manifolds. There seems to be no overlap between [8] and the present paper.

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