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Non straightenable complex lines in $\mathbf{C}^2$

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Abhyankar and Moh, in [1, (1.6), p. 151], and M. Suzuki, in [11, §5], proved that if P is a polynomial embedding of $\mathbf{C}$ into $\mathbf{C}^2$, then there exists ψ, a polynomial automorphism of $\mathbf{C}^2$, such that $(\psi\circ P)(\mathbf{C})=\mathbf{C}\times\{0\}$. The corresponding and much easier result has been proved for polynomial embeddings of $\mathbf{C}$ into $\mathbf{C}^n$, for $n\geq 4$ by Z. Jelonek [7] (more generally, Jelonek treats the case of embeddings of $\mathbf{C}^k$ into $\mathbf{C}^n$, for $n\geq 2k+2$). The case of polynomial embeddings of $\mathbf{C}$ into $\mathbf{C}^3$ seems open. See also [8].

The main goal of this paper is to show that the above results do not generalize to holomorphic embeddings of $\mathbf{C}$. Another goal is an interpolation theorem (Proposition 2 below).

Proposition 1. *Let $n>1$. There exists a proper holomorphic embedding $H:\mathbf{C}\to\mathbf{C}^n$ such that for no automorphism ψ of $\mathbf{C}^n$, $(\psi\circ H)(\mathbf{C})=\mathbf{C}\times\{0\}\subset\mathbf{C}^n$.*

Notice however that it has been proved in [4, (4.1)] that for every $R>0$ and $\varepsilon>0$ there exists ψ, an automorphism of $\mathbf{C}^n$, such that $|(\psi\circ H)(\zeta)-(\zeta,0)|\leq\varepsilon$ for every $\zeta\in\mathbf{C}$, $|\zeta|\leq R$. So, compact subsets of the complex line $H(\mathbf{C})$ can be "approximately straightened".

The above proposition has been known for some time, for $n\geq 3$. In [4, (7.8)] this is pointed out as being (non stated but) clear in [10]. For $n=2$, we keep the same approach as in [10]. But we can now take advantage of the ground breaking work by Andersén and Lempert [2], as further developed in [4].

1. Proof of Proposition 1

Proposition 1 is an immediate consequence of the following two propositions:

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Proposition 2. *Let $n>1$. Let $(\alpha_j)_{j\in\mathbf{N}}$ be a discrete sequence in $\mathbf{C}^n$, (i.e. $|\alpha_j|\to+\infty$ as $j\to\infty$). There exists $H:\mathbf{C}\to\mathbf{C}^n$, a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$, such that $\alpha_j\in H(\mathbf{C})$ for every $j\in\mathbf{N}$.*

The case $n>2$ is treated in [10] (where in addition it is shown that, for $n>2$, the preimages of the points α_j can be arbitrarily prescribed), so the result is new only for $n=2$. We wish to point out that the technique in the proof of Proposition 2, without any substantial modification, allows one to embed $\mathbf{C}^k$ into $\mathbf{C}^n$, so that the image of $\mathbf{C}^k$ contains an arbitrary given discrete sequence, if $1\le k<n$.

Proposition 3. (Theorem 4.5 in [9].) *Let $n>1$. There exists a discrete sequence $(\alpha_j)_{j\in\mathbf{N}}$ in $\mathbf{C}^n$ such that for no automorphism ψ of $\mathbf{C}^n$, $\psi(\alpha_j)\in\mathbf{C}\times\{0\}$ (for all j's).*

In [9], such sequences are called "non tame". The existence of non tame sequences is not immediate. Indeed, any sequence is the union of two tame sequences. An interesting, more detailed, study of non tame sequences is to be found in [6]. All that is left is therefore to prove Proposition 2.

2. Proof of Proposition 2

As already said, our work depends on an extension of the work of Andersén–Lempert, as given in [4]. Also, our proof is inspired from [5]. From [4], we shall need only the following:

Lemma 1. *Let K be a polynomially convex compact set in $\mathbf{C}^n$ $(n>1)$. Let p and $q\in\mathbf{C}^n\setminus K$. For every $\varepsilon>0$, there exists ψ, an automorphism of $\mathbf{C}^n$, such that $\psi(p)=q$ and $|\psi(z)-z|\le\varepsilon$ for every $z\in K$. In addition we can fix arbitrarily chosen points $p_1,\dots,p_s$ in K (i.e. $\psi(p_j)=p_j$).*

Proof of Lemma 1. Let $\gamma:[0,1]\to\mathbf{C}^n\setminus K$ be an arc, $\gamma(0)=p$, $\gamma(1)=q$. Apply Theorem 2.1 in [4] to the following situation: In Theorem 2.1 replace K by $K\cup\{p\}$ and consider Ω a sufficiently small neighborhood of $K\cup\{p\}$. Take Φ_t to be the identity on K, and to be $\Phi_t(z)=z+(\gamma(t)-p)$ near p. Fixing finitely many given points is a trivial addition.

Remark. If K is convex, the lemma is very simple to prove with really elementary tools. Without any intent to look for more generality we now simply state the following.

Lemma 2. *Let K be a polynomially convex compact set in $\mathbf{C}^n$. Let H be a proper holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$. Let $R>0$ and $L_R=\{z=H(\zeta):\zeta\in\mathbf{C}, |\zeta|\le R\}$. Then the polynomial hull of $K\cup L_R$ is contained in $K\cup H(\mathbf{C})$.*

Proof of Lemma 2. Let $p \in \mathbf{C}^n$. Assume that $p \notin K \cup H(\mathbf{C})$. Let f be a polynomial such that $f(p)=1$, but $|f|<1$ on K. Let g be an entire function which vanishes identically on $H(\mathbf{C})$, but such that $g(p) \neq 0$. (The existence of such a g follows from Cartan's Theorem A, but in the application we can explicitly exhibit such a g, a polynomial. See the remark at the end of the paper.) Now, for N large enough, we have $|f^N g(p)| > \sup_{K \cup H(\mathbf{C})} |f^N g|$. So, p is not in the polynomial hull of $K \cup L_R$.

We now begin the proof of Proposition 2 itself.

Proof of Proposition 2. We start with the embedding $H_0: \mathbf{C} \to \mathbf{C}^n$, $H_0(\zeta) = (\zeta, 0)$ and $\varrho_0 = 0$. In the j^{th} step of the construction we shall find $\varrho_j > 0$, $\zeta_j \in \mathbf{C}$, and then construct a proper holomorphic embedding $H_j: \mathbf{C} \to \mathbf{C}^n$ such that:

(i) $H_j(\zeta_l) = \alpha_l$, $l \in \{1, \dots, j\}$,

(ii) $|H_j(\zeta)| > |\alpha_j| - 1$ if $|\zeta| \geq \varrho_j$,

(iii) $|H_j(\zeta) - H_{j-1}(\zeta)| \leq \varepsilon_j \leq 2^{-j}$ for $|\zeta| \leq \varrho_j$ with ε_j to be chosen small enough, depending on previous choices,

(iv) $\varrho_j \geq \varrho_{j-1} + 1$.

Once this is done, we set $H = \lim H_j$ (uniform convergence on compact sets). The inequality $|H_j - H_{j-1}| \leq 2^{-j}$ in condition (iii) shows that the sequence of maps H_j does converge, and that the limit H satisfies: $|H(\zeta) - H_j(\zeta)| \leq 1$ for $|\zeta| \leq \varrho_{j+1}$. From (ii) we get that if $\varrho_j \leq |\zeta| \leq \varrho_{j+1}$, then $|H(\zeta)| > |\alpha_j| - 2$. So H is proper. And (i) implies that $H(\zeta_l) = \alpha_l$, for any l.

Finally we have to explain the choice of ε_j, so as to make sure that H is an embedding. Let $R > 0$, and let G be any holomorphic embedding of $\mathbf{C}$ (or of the disk $\{|\zeta| < R\}$) into $\mathbf{C}^n$, and $0 < r < R$. Then, there exists $\eta > 0$ (depending on G, r and R) such that if G' is any holomorphic map from the disk $\{|\zeta| < R\}$ into C^n satisfying $|G - G'| \leq \eta$, on this disk, then the restriction of G' to the smaller disk $\{|\zeta| < r\}$ is an embedding. In the $(j-1)^{\text{st}}$ step of the construction, the radius ϱ_{j-1} and the map H_{j-1} have been chosen. In the j^{th} step the radius ϱ_j will be chosen first, as will be explained below. We then apply the above to $G = H_{j-1}$, $R = \varrho_j$, $r = \varrho_{j-1}$, to get $\eta = \eta_j$. If for every $j \in \mathbf{N}$, $\sum_{l=j}^{+\infty} \varepsilon_l \leq \eta_j$, then the limit map H will be an embedding. Indeed, the restriction of H to any disk $\{|\zeta| < \varrho_{j-1}\}$ will be an embedding, since the inequality $|H - H_{j-1}| \leq \eta_j$ will hold on the disk $\{|\zeta| < \varrho_j\}$. A possible choice of ε_j is therefore: $\varepsilon_j = 2^{-j} \min_{l \leq j}(1, \eta_l)$.

Here is a way to find ϱ_j, ζ_j and construct H_j:

We already have $H_{j-1}(\zeta_l) = \alpha_l$ for $l \in \{1, \dots, j-1\}$. If $\alpha_j = H_{j-1}(\zeta)$ for some $\zeta \in \mathbf{C}$, we just take $\zeta_j = \zeta$, $H_j = H_{j-1}$ and ϱ_j large enough so that (ii) and (iv) hold. Otherwise (the general case), we choose $\varrho_j \geq \varrho_{j-1} + 1$ so large that $|H_{j-1}(\zeta)| > |\alpha_j|$ for every $\zeta \in \mathbf{C}$, $|\zeta| \geq \varrho_j$. Let F be defined by:

$$F = \{z \in \mathbf{C}^n : |z| \leq |\alpha_j| - 1/2\} \cup H_{j-1}\{|\zeta| \leq \varrho_j\}.$$

The polynomial hull of F does not contain α_j since $\alpha_j \notin H_{j-1}(\mathbf{C})$, and according to Lemma 2. Take ζ_j so that $H_{j-1}(\zeta_j)$ does not belong to this hull (it is enough to take $|\zeta_j|$ large enough). By Lemma 1 we can find ψ_j, an automorphism of $\mathbf{C}^n$, fixing $\alpha_1, \dots, \alpha_{j-1}$, as close as we wish to the identity on F and such that $\psi_j(H_{j-1}(\zeta_j))=\alpha_j$. In particular we take ψ_j close enough to the identity on F so that the image of the ball $\{|z|\le|\alpha_j|-\frac{1}{2}\}$ contains the ball $\{|z|\le|\alpha_j|-1\}$. So if $|\zeta|\ge\varrho_j$ then $|H_{j-1}(\zeta)|>|\alpha_j|$, hence $|\psi_j(H_{j-1}(\zeta))|>|\alpha_j|-1$.

We set $H_j=\psi_j\circ H_{j-1}$. Properties (i)–(iv) are immediate to check.

This ends the proof. We just add the following remark with respect to the proof of Lemma 2. One has $H_j=\psi_j\circ\dots\circ\psi_1\circ H_0$, and $H_0(\zeta)=(\zeta,0)$. So if Z_k denotes the k$^{\text{th}}$ coordinate function in $\mathbf{C}^n$, the functions $Z_k\circ\psi_1^{-1}\circ\dots\circ\psi_j^{-1}$, for $k\in\{2,\dots,n\}$, have precisely $H_j(\mathbf{C})$ as their common zero set.

Note. Further examples, related to Proposition 1, have been given by G. Buzzard and J. E. Fornaess ([3]). In particular, they give the example of a complex line embedded in $\mathbf{C}^2$, whose complement is hyperbolic.

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