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Embedding holomorphic discs through discrete sets

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1. Introduction and the main result

We denote by D the open unit disc in C: $D = \{\zeta \in C: |\zeta| < 1\}$. A domain $\Omega \subset C^n$ is said to be *Runge* in C^n if every holomorphic function in Ω can be approximated, uniformly on compacts in Ω, by restrictions to Ω of holomorphic polynomials on C^n. We refer the reader to Hörmander [12] for general results concerning pseudoconvex Runge domains. Our main result is

Theorem. *Given a connected pseudoconvex Runge domain $\Omega \subset C^n$ $(n \geqq 2)$ and a discrete subset $Z = \{z_j: j \in Z_+\} \subset \Omega$, there exists a proper holomorphic embedding $f: D \to \Omega$ of the disc into Ω whose image $f(D)$ contains Z.*

The most interesting case is $n = 2$. For $n \geqq 3$ this was proved by a different method in [16] for $\Omega = C^n$ and in [9] for all convex domains $\Omega \subset C^n$. In the case when $\Omega = C^n$ $(n \geqq 3)$ one can in addition prescribe a discrete set $\{\zeta_j\} \subset C$ and require that $f(\zeta_j) = z_j$ for all j [16].

For $n = 2$ the methods of [9] and [16] only give proper holomorphic *immersions* of the disc through a given discrete set in Ω. The main problem of course is that, in dimension two, one cannot remove self-intersections of complex curves by small deformations.

We do not know whether our Theorem holds for non-Runge pseudoconvex domains in C^n; our methods do not seem to extend to this case. In this direction it was proved in [6] that for every *finite* subset Z in an arbitrary connected pseudoconvex domain $\Omega \subset C^n$ $(n > 1)$ there exist proper holomorphic mappings $f: D \to \Omega$ of the disc into Ω such that $Z \subset f(D)$. It is likely that a refinement of the construction in [6] gives proper holomorphic immersions $f: D \to \Omega$ whose image contains a given discrete subset of Ω. An example

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in [6] shows that the Theorem does not extend to arbitrary non-pseudoconvex domains.

In this paper we develop an essentially new approach which works in every dimension $n \geqq 2$. Our methods were inspired in part by the earlier work of two of the authors [10]. The main idea is to construct a sequence of proper holomorphic embeddings $f_k : \mathbf{C} \to \mathbf{C}^n$ such that the points $z_0, \ldots, z_k$ are contained in the same connected component of $f_k(\mathbf{C}) \cap \Omega$. The next map f_{k+1} is of the form $f_{k+1} = \Phi_k \circ f_k$, where Φ_k is a suitably chosen holomorphic automorphism of $\mathbf{C}^n$ which is very close to the identity on a prescribed polynomially convex set $K_k \subset\subset \Omega$, it fixes the points $z_0, \ldots, z_k$, and it moves the image variety outside K_k so that $f_{k+1}(\mathbf{C})$ contains the next point $z_{k+1} \in Z$ as well.

A similar but technically simpler construction was used in [7] to construct proper holomorphic embeddings $f : \mathbf{C} \to \mathbf{C}^2$ of the complex line into $\mathbf{C}^2$ whose image $f(\mathbf{C})$ contains a given discrete set $Z \subset \mathbf{C}^2$. The additional difficulty in the present paper is to keep the points $z_0, \ldots, z_k$ in the same connected component of $f_k(\mathbf{C}) \cap \Omega$ at each step. To achieve this we need the following tool which, we hope, will be of independent interest. It is an extension of some earlier results of Rosay and one of the authors [8] (and whose proof is based in part on the ground breaking work by Andérsen [3] and Andersén and Lempert [4]).

Proposition (Combing hair by holomorphic automorphisms). *Let $K \subset\subset \mathbf{C}^n$ ($n \geqq 2$) be a compact polynomially convex set and let $C \subset \mathbf{C}^n$ be a smooth embedded arc of class $\mathcal{C}^r$, $r \geqq 3$, (a diffeomorphic image of $[0,1] \subset \mathbf{R}$) which is attached to K in a single point of K. Given a homeomorphism $F : K \cup C \to K \cup C' \subset \mathbf{C}^n$ such that F is the identity on $(K \cup C) \cap U$ for some open neighborhood $U \subset \mathbf{C}^n$ of K, there exists for each $\varepsilon > 0$ a global holomorphic automorphism Φ of $\mathbf{C}^n$ satisfying $|\Phi(z) - F(z)| < \varepsilon$ for $z \in K \cup C$. Moreover, for each pair of finite subsets $A \subset C$, $B \subset \mathbf{C}^n \backslash C$, there is a Φ as above such that $\Phi|_A = F|_A$ and Φ fixes B pointwise.*

Remarks. 1. The same result holds for any finite number of disjoint hair attached to K. In other words, *one can comb hair on a polynomially convex head in $\mathbf{C}^n$ ($n \geqq 2$) approximately by global holomorphic automorphisms of $\mathbf{C}^n$.*

2. Similar results for arcs without the presence of the polynomially convex set K have been proved by Rosay [14] and Forstneric [5]. Our proof of the proposition can easily be refined to show that, if F as above is a $\mathcal{C}^r$ diffeomorphism on C and $r \geqq 3$, then an approximating sequence $\Phi_j \in \mathrm{Aut}\,\mathbf{C}^n$ can be chosen such that $\Phi_j|_C$ converges to $F|_C$ in $\mathcal{C}^{r-3}(C)$. This result is not sharp, the loss of derivatives being due to the fact that we are using Hörmander's L^2-method for solving a $\bar\partial$ equation in certain thin tubes. By using a more precise method such as the one in [5] or in [13] one expects no loss of derivatives as soon as $r \geqq 2$.

The proposition is proved in Sect. 2. Granted the proposition we shall now prove the Theorem. We will need the following Lemma (which is not original).

Lemma. *If $\Omega \subset \boldsymbol{C}^n$ is a pseudoconvex Runge domain and if $f: \boldsymbol{C} \to \boldsymbol{C}^n$ is a proper holomorphic embedding, then each connected component of $f(\boldsymbol{C}) \cap \Omega$ is simply connected and hence biholomorphic either to the disc or to $\boldsymbol{C}$.*

Proof. Set $\Lambda = f(\boldsymbol{C})$. Let A be a connected component of $\Lambda \cap \Omega$ and let $U = \{\zeta \in \boldsymbol{C}: f(\zeta) \in A\}$. We must show that U is simply connected. If not, choose a point ζ_0 in a bounded component of $\boldsymbol{C}\backslash U$. Define a holomorphic function $F: A \to \boldsymbol{C}$ by $F(f(\zeta)) = 1/(\zeta - \zeta_0)$. By Cartan's theorem A [12] F extends to a holomorphic function in Ω. Since Ω is Runge in $\boldsymbol{C}^n$, F can be approximated by holomorphic polynomials P_j. Then $P_j \circ f$ is a sequence of entire functions on $\boldsymbol{C}$ which converges to the function $\zeta \to 1/(\zeta - \zeta_0)$ uniformly on compacts in U. Clearly this is a contradiction. □

Proof of the Theorem. We shall first consider the case $\Omega \neq \boldsymbol{C}^n$. Choose a smooth plurisubharmonic exhaustion function $\rho: \Omega \to \boldsymbol{R}$ such that $\rho|_Z$ is one to one. We shall repeatedly use the fact that for each $R \in \boldsymbol{R}$ the set

$$\Omega(R) = \{z \in \Omega: \rho(z) \leqq R\}$$

is polynomially convex in $\boldsymbol{C}^n$ (see Hörmander [12]). By reordering the points in Z we may assume $\rho(z_j) < \rho(z_{j+1})$ for each $j \in \boldsymbol{Z}_+$. Choose numbers R_j ($j = 0, 1, 2, \ldots$) such that

$$\rho(z_j) < R_j < \rho(z_{j+1})$$

and set $K_j = \Omega(R_j)$. Then $\{K_j\}$ is an increasing sequence of compact polynomially convex sets such that $\bigcup_{0 \leqq j < \infty} K_j = \Omega$. We also have $K_j \cap Z = \{z_0, z_1, \ldots, z_j\}$ and $(K_{j+1}\backslash K_j) \cap Z = \{z_{j+1}\}$.

The initial step. Fix a point $a \in \boldsymbol{C}^n\backslash\Omega$. Let $\zeta_0 = 0 \in \boldsymbol{C}$. Choose a proper holomorphic embedding $f_0: \boldsymbol{C} \to \boldsymbol{C}^n$ such that $f(0) = z_0$, $f(1) = a$, and $\Lambda_0 = f_0(\boldsymbol{C})$ does not contain the point z_1. Set $L_{-1} = \emptyset$, $\Delta_{-1} = \emptyset$, and $V_{-1} = \emptyset$.

The inductive step. Let $k \in \boldsymbol{Z}_+$. Suppose we already have a proper holomorphic embedding $f_k: \boldsymbol{C} \to \boldsymbol{C}^n$ with image $\Lambda_k = f_k(\boldsymbol{C})$, a set of points $\{\zeta_0, \zeta_1, \ldots, \zeta_k\} \subset \boldsymbol{C}\backslash\{1\}$, a number M_{k-1}, and a smoothly bounded, simply connected domain $\Delta_{k-1} \subset\subset \boldsymbol{C}\backslash\{1\}$ such that

(i) $f_k(\zeta_j) = z_j$ for $0 \leqq j \leqq k$,
(ii) $f_k(1) = a$,
(iii) $z_{k+1} \notin \Lambda_k$, and
(iv) the set $\{\zeta_0, \zeta_1, \ldots, \zeta_k\} \cup \overline{\Delta_{k-1}}$ is contained in one connected component U_k^0 of the set

$$U_k = \{\zeta \in \boldsymbol{C}: f_k(\zeta) \in \Omega\}.$$

Note that $\rho \circ f_k$ is an exhaustion function on U_k. Choose a number

$$M_k \geqq \max(R_k, M_{k-1}) + 1 \tag{1}$$

which is a regular value of $\rho \circ f_k|_{U_k}$ and such that $\{\zeta_0, \zeta_1, \dots, \zeta_k\} \cup \overline{\Delta_{k-1}}$ is contained in one connected component Δ_k of the set

$$V_k = \{\zeta \in U_k : \rho \circ f_k(\zeta) < M_k\} \subset\subset U_k. \tag{2}$$

V_k consists of finitely many smoothly bounded components $\Delta_k = \Delta_k^0, \Delta_k^1, \dots, \Delta_k^{j_k}$ which are simply connected and have disjoint closures.

Set

$$L_k = K_k \cup \big(A_k \cap \Omega(M_k)\big) = K_k \cup \overline{f_k(V_k)}.$$

We claim that L_k is polynomially convex. Suppose that a point $z \in C^n \backslash L_k$ belongs to the polynomial hull $\hat{L}_k$. Then $z \in \Omega(M_k)$ since the set $\Omega(M_k)$ is polynomially convex and it contains L_k. Thus $z \notin A_k \cup K_k$. By Cartan's theorem A [12] there exists a holomorphic function g on C^n such that $g(z) = 1$ and $g = 0$ on A_k. Since K_k is polynomially convex, there exists another holomorphic function h on C^n such that $h(z) = 1$ and $\sup_{K_k} |h| < 1$. The holomorphic function $G = gh^N$ for sufficiently large $N > 0$ satisfies $G(z) = 1$ and $\sup_{L_k} |G| < 1$. This contradiction shows that $L_k = \hat{L}_k$ as claimed.

We now choose a smooth arc $\lambda_k \subset U_k^0 \backslash \Delta_k$ which is attached to $\overline{\Delta}_k$ in a single point and which does not intersect any other set $\overline{\Delta_k^j}$ for $1 \leqq j \leqq j_k$. Denote the other endpoint of λ_k by ζ_{k+1}. Hence $C_k = f_k(\lambda_k)$ is an arc in $A_k \cap \Omega$ with one endpoint $f_k(\zeta_{k+1})$ which is attached to $f(\overline{\Delta}_k) \subset L_k$ in the other endpoint.

By the Proposition, applied to the polynomially convex set $L_k \cup C_k$, we can find for any given $\varepsilon_k > 0$ a holomorphic automorphism Φ_k of C^n satisfying

(a) $|\Phi_k(z) - z| < \varepsilon_k$ for $z \in L_k$,
(b) $\Phi_k\big(f_k(\zeta_{k+1})\big) = z_{k+1}$,
(c) $\Phi_k(z_j) = z_j$ for $0 \leqq j \leqq k$ and $\Phi_k(a) = a$,
(d) $\Phi_k(C_k) \subset \Omega$, and
(e) $\Phi_k(A_k)$ does not contain z_{k+2}.

Set

$$f_{k+1} = \Phi_k \circ f_k : C \to C^n, \quad A_{k+1} = f_{k+1}(C).$$

Clearly f_{k+1} is a proper holomorphic embedding of C into C^n which satisfies the properties (i)–(iv) above, with k replaced by $k+1$.

Remark. It is important to observe that the point z_{k+1} belongs to the same connected component of $A_{k+1} \cap \Omega$ as the points $z_0, \dots, z_k$ since the automorphism Φ_k maps the arc C_k (connecting $f_k(\zeta_{k+1})$ to $f_k(\overline{\Delta}_k)$) into Ω. If $\Omega = C^n$, we do not have to worry about this, and consequently one can replace our Proposition with a much simpler result to the effect that, given a polynomially convex set $K \subset\subset C^n$, one can move a point $p \in C^n \backslash K$ to a point $q \in C^n \backslash K$ by an automorphism of C^n which is arbitrary close to the identity on K. This approach was used in [7] to construct proper holomorphic embeddings of the complex line C into C^2 passing through a given discrete set $Z \subset C^2$.

Completion of the proof. Set

$$V = \bigcup_{0 \leq k < \infty} V_k, \qquad \Delta = \bigcup_{0 \leq k < \infty} \Delta_k,$$

where V_k is given by (2) and $\Delta_k = \Delta_k^0$ is the connected component of V_k as above. Then $\{\zeta_j : j \in \boldsymbol{Z}_+\} \subset \Delta \subset V \subset \boldsymbol{C}\backslash\{1\}$. By property (a) of Φ_k we have

$$|f_{k+1} - f_k| < \varepsilon_k \quad \text{on} \quad V_k. \tag{3}$$

In each step of the construction we choose the number $\varepsilon_k > 0$ so that the following hold:

1.) $\varepsilon_k \leqq \varepsilon_{k-1}/2$ for each $k \geqq 1$,
2.) $2\varepsilon_k < d(L_k, \boldsymbol{C}^n \backslash \Omega(M_k + 1))$,
3.) $2\varepsilon_k < d(K_{k-1}, \boldsymbol{C}^n \backslash K_k)$.

Here $d(K, L)$ denote the Euclidean distance between the sets K and L. Since $M_{k+1} \geqq M_k + 1$ by (1), properties 1.) and 2.) of the sequence ε_k insure that

$$V_k \subset\subset V_{k+1}, \quad k \in \boldsymbol{Z}_+, \tag{4}$$

and the sequence $f_k : \boldsymbol{C} \to \boldsymbol{C}^n$ converges to a limit map

$$f = \lim_{k \to \infty} f_k : V \to \boldsymbol{C}^n$$

uniformly on compacts $\overline{V}_k \subset\subset V$. On V_k we have

$$|f - f_k| \leqq \sum_{j=k}^{\infty} |f_{j+1} - f_j| < \sum_{j=k}^{\infty} \varepsilon_j < 2\varepsilon_k.$$

Hence $f(V_k) \subset \Omega$ and therefore $f(V) \subset \Omega$.

Since $f_k : \boldsymbol{C} \to \boldsymbol{C}^n$ is an embedding for each k, we can insure by choosing $\varepsilon_k > 0$ sufficiently small that any holomorphic map $g : \boldsymbol{C} \to \boldsymbol{C}^n$ satisfying $|f_k - g| < 2\varepsilon_k$ on V_k is an embedding on the smaller set $V_{k-1} \subset\subset V_k$. Thus the limit map $f : V \to \Omega$ is an injective immersion into Ω.

We claim that $f : V \to D$ is also *proper*. Since $|\Phi_k(z) - z| < \varepsilon_k$ for $z \in K_k$ according to (a), the conditions 1.) and 3.) on ε_k imply that no point from $\boldsymbol{C}^n \backslash K_k$ will enter the smaller set K_{k-1} after k-th step of the construction. Since $f_k(\boldsymbol{C} \backslash V_k) \subset \boldsymbol{C}^n \backslash K_k$, it follows that

$$f(V \backslash V_k) \subset \Omega \backslash K_{k-1}, \quad k \in \boldsymbol{Z}_+,$$

and hence f is proper as claimed.

To summarize, we have constructed a proper holomorphic embedding $f : V \to \Omega$. Since Δ_k is a connected component of V_k for each k, (4) implies that Δ is a connected component of V. Hence the restriction $f : \Delta \to \Omega$ is a proper holomorphic embedding of Δ into Ω. Property (c) of Φ_k implies $f(\zeta_k) = f_k(\zeta_k) = z_k$ for each $k \in \boldsymbol{Z}$, and hence $f(\Delta)$ contains the given discrete set Z. Since Δ is an increasing union of connected and simply connected domains $\Delta_k \subset \boldsymbol{C} \backslash \{1\}$, Δ is itself a simply connected domain in $\boldsymbol{C} \backslash \{1\}$ and hence biholomorphic to the unit disc D. Thus the map $f: \Delta \to \Omega$ satisfies our Theorem.

This proves the Theorem when Ω is a proper subdomain in C^n. In the remaining case $\Omega = C^n$ we can either apply the previous proof to a Fatou-Bieberbach Runge domain $\Omega \subset C^n$ (which is biholomorphically equivalent to C^n), or else we construct the sequence f_k as above such that $f_k(1)$ diverges to infinity (so $1 \notin V$).

2. Combing hair by holomorphic automorphisms

In this section we prove the Proposition stated in Sect. 1. The last requirement concerning the behavior of Φ on a finite set is a trivial addition since one can move a finite set of points for a small distance in any direction by a finite composition of shear automorphisms which are close to the identity on a chosen compact subset (see [15]). Hence it suffices to prove the first part of the proposition.

By approximation we may (and shall) assume that $F: C \to C'$ is a C^r diffeomorphism onto another embedded C^r arc, and F is the identity near K. Shrinking the neighborhood U of K if necessary we may assume that $C \cap \overline{U} = C' \cap \overline{U}$. We extend F as the identity on $\overline{U}$. Choose a one parameter family of C^r diffeomorphisms $F_t: \overline{U} \cup C \to \overline{U} \cup C_t \subset C^n$, smooth with respect to $0 \leqq t \leqq 1$, such that the t-derivative dF_t/dt is also of class C^r and the following hold:

(i) F_0 is the identity on $\overline{U} \cup C$,
(ii) $F_1 = F$, and
(iii) $F_t|_{\overline{U}}$ is the identity for each $0 \leqq t \leqq 1$.

Let $C_t = F_t(C)$. Observe that $C_t \cap \overline{U} = C \cap \overline{U}$ for all $t \in [0,1]$. Let $X_t: U \cup C_t \to C^n$ be the velocity vector field of F_t, defined by the equation

$$\frac{d}{dt}F_t(z) = X_t\big(F_t(z)\big), \quad z \in U \cup C,\ 0 \leqq t \leqq 1.$$

Then X_t is of class C^r in both variables (z,t) and $X_t|_U = 0$ for each t. Thus $F_t(z)$ $(z \in U \cup C)$ is the flow of the time dependent vector field X_t.

To simplify the analysis we include the parameter t as an additional complex variable. Define the following subsets in C^{n+1}:

$$S = \bigcup_{0 \leqq t \leqq 1} C_t \times \{t\}, \quad L_0 = K \times [0,1], \quad L = L_0 \cup S.$$

Since the set $K \cup C_t \subset C^n$ is polynomially convex for each $t \in [0,1]$ (Stolzenberg [17] and Alexander [2]), the sets L_0 and L are polynomially convex in C^{n+1}.

Let $U' \subset\subset C$ be a neighborhood of the segment $[0,1] \subset R \subset C$, and let $U_0 = U \times U' \subset\subset C^{n+1}$ be the corresponding neighborhood of L_0. We define a mapping $X: U_0 \cup S \to C^n$ by

$$X(z,t) = X_t(z), \quad z \in C_t,\ 0 \leqq t \leqq 1,$$

and $X|_{U_0} = 0$.

Note that $S \subset C^{n+1}$ is a totally real submanifold of class C^r. Since X is of class, C^r on S and zero on U_0, X extends to a map $X: C^{n+1} \to C^n$ of class C^r, with compact support, such that

$$\bar{\partial} X(\zeta) = o\left(d(\zeta, S)^{r-1}\right)$$

(see Lemma 4.3 in [11]). Here, $\bar{\partial}$ is taken with respect to the variables $\zeta = (z,t) \in C^{n+1}$, and $d(\zeta, S)$ denotes the Euclidean distance from ζ to S.

For each compact set $K \subset R^m$ we denote $K(\varepsilon) = \{z \in R^m : d(z,K) < \varepsilon\}$.

The Proposition follows immediatelly from the following three lemmas.

Lemma 1. (*Notation as above*) *Assume* $r \geqq 3$. *There is an* $\varepsilon_0 > 0$ *and a continuous function* $\eta: R_+ \to R_+$, $\eta(t) > 0$ *for* $t > 0$, $\eta(0) = 0$, *such that for each* $0 < \varepsilon \leqq \varepsilon_0$ *there exists an entire holomorphic mapping* $Y_\varepsilon: C^{n+1} \to C^n$ *satisfying*

$$\|X - Y_\varepsilon\|_{L^\infty(L(\varepsilon))} \leqq \eta(\varepsilon)\varepsilon. \tag{5}$$

(*The function* η *depends on the dimension* n, *on the set* L, *and on the vector field* X.)

Recall [1] that each Lipschitz time dependent vector field X_t on R^n has a local flow ϕ_t satisfying

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x)), \quad \phi_0(x) = x.$$

Lemma 2. *Let* X_t *and* Y_t $(0 \leqq t \leqq 1)$ *be time dependent Lipschitz vector fields on* R^n *with local flows* ϕ_t *resp.* ψ_t. *Assume that the flow* $\phi_t(x)$ *is defined for all* $x \in K \subset\subset R^n$ *and* $0 \leqq t \leqq 1$. *Set* $K_t = \phi_t(K)$, *and let*

$$A(\varepsilon) = \sup\{|X_t(x) - Y_t(x)| : x \in K_t(\varepsilon),\ 0 \leqq t \leqq 1\},$$
$$B = \sup\{|X_t(x) - X_t(y)| : x, y \in K_t(1),\ 0 \leqq t \leqq 1\}.$$

If $A(\varepsilon)e^B \leqq \varepsilon \leqq 1$, *then the flow* $\psi_t(x)$ *of* Y_t *is defined for all* $x \in K$ *and* $0 \leqq t \leqq 1$, *and*

$$|\phi_t(x) - \psi_t(x)| \leqq A(\varepsilon)e^{Bt}, \quad x \in K,\ 0 \leqq t \leqq 1.$$

In particular we have $\psi_t(x) \in K_t(\varepsilon)$ *for* $x \in K$ *and* $0 \leqq t \leqq 1$.

Lemma 3. *Let* Y_t *be an entire vector field on* C^n *for each* $0 \leqq t \leqq 1$, *of class* C^1 *in* $(z,t) \in C^n \times [0,1]$. *Let* Ω *be an open subset of* C^n. *Assume that the differential equation* $dR/dt = Y_t(R(t))$ *can be integrated for* $0 \leqq t \leqq 1$ *with arbitrary initial condition* $R(0) = z \in \Omega$. *Set* $G_t(z) = R(t)$ *as above. Then* G_t $(0 \leqq t \leqq 1)$ *is a biholomorphic map from* Ω *into* C^n *that can be approximated, uniformly on compact sets in* Ω, *by holomorphic automorphisms of* C^n.

Lemma 3 is proved in [8] (Lemma 1.4), using results of Andersén [3] and Andersén and Lempert [4]. Although it is stated there only for time independent fields, the proof applies to time dependent entire fields as well. Lemmas 1 and 2 are proved below.

Granted these lemmas we can complete the proof of the Proposition as follows. Fix an $\varepsilon > 0$ for which Lemma 1 holds. Using Lemma 1 we approximate X by an entire map $Y: C^{n+1} \to C^n$ such that the estimate (5) is satisfied. If ε is sufficiently small, Lemma 2 shows that the flow $G_t(z)$ $(z \in K \cup C)$ of the time dependent holomorphic vector field $Y_t = Y(\cdot, t): C^n \to C^n$ exists and remains in the ε-neighborhood $K_t(\varepsilon)$ of $K_t = K \cup C_t \subset C^n$ for all $0 \leqq t \leqq 1$, and we have

$$|F_t(z) - G_t(z)| < \varepsilon, \qquad z \in K \cup C, \ 0 \leqq t \leqq 1.$$

Applying Lemma 3 for $t = 1$ we get a $\Phi \in \mathrm{Aut}C^n$ such that $\|\Phi - G_1\|_{L^\infty(K\cup C)} < \varepsilon$. Hence $\|\Phi - F\|_{L^\infty(K \cup C)} < 2\varepsilon$. □

Proof of Lemma 1. This is essentially proved in the paper by Hörmander and Wermer [11] (proof of Theorem 4.1, pp. 15–16), except that the solution is obtained only in a small neighborhood of L (since L is only assumed to be holomorphically convex). We shall indicate the necessary modifications to get a globally defined solution when L is polynomially convex, with estimates near L. We need the following lemma which should be compared with Theorem 3.1 in [11]. We denote the variables on C^{n+1} by z.

Lemma 4. *(Notation as above) There exists a continuous plurisubharmonic exhausting function $\rho \geqq 0$ on C^{n+1} such that*
(a) $\rho^{-1}(0) = L = L_0 \cup S$,
(b) $\rho(z) \leqq d(z,L)^2$ for z near L, and
(c) $\rho(z) = d(z,S)^2$ for z near $S \backslash U_0$.

We postpone the proof of Lemma 4 for a moment and continue with the proof of Lemma 1. We adopt the notation of [11]. For $\varepsilon > 0$ set

$$\omega_\varepsilon = \{z \in C^{n+1}: \rho(z) < \varepsilon^2\}.$$

Choose $\varepsilon_0 > 0$ such that $\omega_{\varepsilon_0} \subset L_{\varepsilon_0} \cup U_0$ and Lemma 4 holds for $z \in \omega_{\varepsilon_0}$. For $0 < \varepsilon \leqq \varepsilon_0$ we then have

$$L(\varepsilon) \subset \omega_\varepsilon \subset L(\varepsilon) \cup U_0, \quad \omega_\varepsilon \backslash U_0 = L(\varepsilon) \backslash U_0 = S(\varepsilon) \backslash U_0.$$

Recall that $f = \bar{\partial} X$ satisfies the estimate,

$$|f(z)| = o(d(z,L)^{r-1})$$

and $f|_{U_0} = 0$. Let $v = \dim S (= 2$ in our case). We get an estimate

$$\int_{\omega_{3\varepsilon}} |f|^2 dV = o(\varepsilon^{2(r-1)}) O(\varepsilon^{2n}) = o(\varepsilon^{2(r+n-1)}) \tag{6}$$

as $\varepsilon \to 0$, the extra term $O(\varepsilon^{2n})$ coming from the volume of the tube $S(3\varepsilon)$. Fix an ε, $0 < \varepsilon \leqq \varepsilon_0/3$. Let $\phi_\varepsilon = h_\varepsilon \circ \rho$, where $h_\varepsilon: \boldsymbol{R} \to \boldsymbol{R}_+$ is a convex increasing function such that $h_\varepsilon(t) = 0$ for $t \leqq 2\varepsilon$, and h_ε is increasing so fast on $t > 2\varepsilon$ that $\int_{\boldsymbol{C}^{n+1}\setminus\omega_{3\varepsilon}} |f|^2 e^{-\phi_\varepsilon} dV$ is no larger than the integral in (6). Thus

$$\int_{\boldsymbol{C}^{n+1}} |f|^2 e^{-\phi_\varepsilon} dV = o(\varepsilon^{2(r+n-1)}).$$

According to Theorem 4.4.2 in [12] there is a solution of the equation $\bar{\partial} w_\varepsilon = f$ defined on all of $\boldsymbol{C}^{n+1}$ such that

$$\int_{\boldsymbol{C}^{n+1}} |w_\varepsilon|^2 e^{-\phi_\varepsilon} \frac{dV}{(1+|z|^2)^2} = o(\varepsilon^{2(r+n-1)}).$$

Since $\phi_\varepsilon = 0$ on the set $\omega_{2\varepsilon} \subset\subset \boldsymbol{C}^{n+1}$, we get

$$||w_\varepsilon||_{L^2(\omega_{2\varepsilon})} = o(\varepsilon^{r+n-1}).$$

Since $L(2\varepsilon) \subset \omega_{2\varepsilon}$, the Cauchy estimates (see Lemma 4.4 in [11]) allow us to pass from the L^2 estimate on $\omega_{2\varepsilon}$ to a sup norm estimate on $L(\varepsilon)$:

$$||w_\varepsilon||_{L^\infty(L(\varepsilon))} = O(\varepsilon^{-(n+1)}) o(\varepsilon^{r+n-1}) = o(\varepsilon^{r-2}).$$

The mapping $Y_\varepsilon = X - w_\varepsilon: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^n$ is holomorphic on all of $\boldsymbol{C}^{n+1}$ and it satisfies Lemma 1 for sufficiently small $\varepsilon > 0$, provided that $r \geqq 3$. This proves Lemma 1, granted that Lemma 4 holds. □

Proof of Lemma 4. Since L is polynomially convex, there exists a smooth plurisubharmonic exhaustion function $\rho_1 \geqq 0$ on $\boldsymbol{C}^{n+1}$ such that $\rho_1^{-1}(0) = L$ and ρ_1 is strongly plurisubharmonic outside L [12]. Since ρ_1 vanishes to second order on L, we may assume (replacing ρ_1 by $c\rho_1$ for a small $c > 0$ if necessary) that $\rho_1(z) \leqq d(z,L)^2$. Thus (a) and (b) hold for ρ_1.

We now modify ρ_1 near $S\setminus U_0$ in order to satisfy (c) there. The function $\rho_2(z) = d(z,S)^2$ is strongly plurisubharmonic on a sufficiently small tubular neighborhood $V \subset \boldsymbol{C}^{n+1}$ of $S\setminus U_0$. Choose a smooth real function χ, with compact support contained in $U_0 \cap V$, and such that $\chi|_{bV\cap S} > 0$. If $\delta > 0$ is sufficiently small, the function $\rho_2' = \rho_2 - \delta\chi$ is still strongly plurisubharmonic in V, and $\rho_2' = \rho_2$ near $S\setminus U_0$. Near the set $S \cap bV \subset\subset U_0$ we have $\rho_2' < 0 \leqq \rho_1$. Hence the function

$$\rho_3 = \max(\rho_1, \rho_2')$$

is well defined, continuous and plurisubharmonic in a smaller neighborhood $W \subset U_0 \cup V$ of L. We have $\rho_3 = \rho_1$ near L_0, $\rho_3 = \rho_2$ in $W\setminus U_0$, and $\rho_3^{-1}(0) = L$.

It remains to extend ρ_3 to an exhausting plurisubharmonic function ρ on $\boldsymbol{C}^{n+1}$ such that $\rho = \rho_3$ near L. This can be achieved by taking $\rho = \max\{\rho_3, C(\rho_1 - \eta)\}$ for a suitably large constant $C > 0$ and a small constant $\eta > 0$. This proves Lemma 4. □

Proof of Lemma 2. Fix $x \in K$ and set $f(t) = |\phi_t(x) - \psi_t(x)|$. This function is defined for $0 \leqq t \leqq t_0$ for some $t_0 > 0$. We have

$$\begin{aligned} f(t) &= \left|\int_0^t (X_s(\phi_s(x)) - Y_s(\psi_s(x)))\, ds\right| \\ &\leqq \left|\int_0^t (X_s(\phi_s(x)) - X_s(\psi_s(x)))\, ds\right| + \left|\int_0^t (X_s(\psi_s(x)) - Y_s(\psi_s(x)))\, ds\right| \\ &\leqq B\int_0^t f(s)ds + A(\varepsilon). \end{aligned}$$

The Gronwall's inequality [1, p. 63] implies

$$f(t) \leqq A(\varepsilon)e^{Bt}$$

for all $0 \leqq t \leqq 1$ where the flow $\psi_t(x)$ is defined. Since $A(\varepsilon)e^B \leqq \varepsilon$ by hypothesis, the above inequality shows that $\psi_t(x) \in K_t(\varepsilon)$ where it is defined. Hence $\psi_t(x)$ is defined for all $x \in K$ and all $0 \leqq t \leqq 1$. This proves Lemma 2. □

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