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A Carleman type theorem for proper holomorphic embeddings

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1. Introduction

We denote by $\mathbf{C}$ the field of complex numbers and by $\mathbf{R}$ the field of real numbers. To motivate our main result we recall the Carleman approximation theorem [4], [11]: *For each continuous function* $\lambda: \mathbf{R} \to \mathbf{C}$ *and positive continuous function* $\eta: \mathbf{R} \to (0, \infty)$ *there exists an entire function* f *on* $\mathbf{C}$ *such that* $|f(t) - \lambda(t)| < \eta(t)$ *for all* $t \in \mathbf{R}$. If λ is smooth, we can also approximate its derivatives by those of f. A more general result was proved by Arakelian [2] (see [14] for a simple proof).

Let $\mathbf{C}^n$ be the complex Euclidean space of dimension n. Our main result is an extension of Carleman's theorem to *proper holomorphic embeddings* of $\mathbf{C}$ into $\mathbf{C}^n$ for $n > 1$:

1.1. Theorem. *Let* $n > 1$ *and* $r \geq 0$ *be integers. Given a proper embedding* $\lambda: \mathbf{R} \hookrightarrow \mathbf{C}^n$ *of class* C^r *and a continuous positive function* $\eta: \mathbf{R} \to (0, \infty)$, *there exists a proper holomorphic embedding* $f: \mathbf{C} \hookrightarrow \mathbf{C}^n$ *such that*

$$|f^{(s)}(t) - \lambda^{(s)}(t)| < \eta(t), \quad t \in \mathbf{R}, \ 0 \leq s \leq r.$$

If in addition $T = \{t_j\} \subset \mathbf{R}$ *is discrete, there exists* f *as above such that*

$$f^{(s)}(t) = \lambda^{(s)}(t), \quad t \in T, \ 0 \leq s \leq r.$$

Definition. Two proper holomorphic embeddings $f, g: \mathbf{C} \hookrightarrow \mathbf{C}^n$ are said to be $\operatorname{Aut} \mathbf{C}^n$-equivalent if $\Phi \circ f = g$ for some holomorphic automorphism Φ of $\mathbf{C}^n$.

1.2. Corollary. *For each* $n > 1$ *the set of* $\operatorname{Aut} \mathbf{C}^n$*-equivalence classes of proper holomorphic embeddings* $\mathbf{C} \hookrightarrow \mathbf{C}^n$ *is uncountable.*

For $n \geq 3$ the corollary is due to Rosay and Rudin [16]. The corollary follows from Theorem 1.1 and a result of Rosay and Rudin [15] to the effect that for each

$n>1$ there exist uncountably many discrete sets in $\mathbf{C}^n$ which are pairwise inequivalent under the group of holomorphic automorphisms of $\mathbf{C}^n$. Theorem 1.1 provides for each discrete set $E=\{e_k:k=1,2,3,...\}\subset\mathbf{C}^n$ a proper holomorphic embedding $f_E:\mathbf{C}\hookrightarrow\mathbf{C}^n$ such that $f_E(k)=e_k$ for all $k=1,2,3,...$. (For $n\geq 3$ such embeddings were constructed in [16].) Clearly the embeddings corresponding to inequivalent discrete sets are inequivalent.

In this context we recall that the first construction of proper holomorphic embeddings $\mathbf{C}\hookrightarrow\mathbf{C}^2$ which are inequivalent to the standard embedding $\zeta\mapsto(\zeta,0)$ by automorphisms of $\mathbf{C}^2$ can be found in [8]. On the other hand, it is well known that all polynomial embeddings $\mathbf{C}\hookrightarrow\mathbf{C}^2$ are equivalent to the standard embedding by polynomial automorphisms of $\mathbf{C}^2$ [1], [18].

Remark 1. We emphasize that, in Theorem 1.1, one cannot expect in general to *extend* λ to a holomorphic embedding of $\mathbf{C}$ into $\mathbf{C}^n$. If λ is real-analytic, it will extend holomorphically to some open set in $\mathbf{C}$, but in general not to all of $\mathbf{C}$; and even if λ extends to all of $\mathbf{C}$, the (unique!) extension need not be a *proper* map into $\mathbf{C}^n$. So the best we can do in general is to approximate λ by a proper holomorphic embedding as in Theorem 1.1.

Remark 2. If the embedding $\lambda:\mathbf{R}\hookrightarrow\mathbf{C}^n$ is of class $\mathcal{C}^\infty$, our method can be modified so that we approximate to increasingly high order on complements of compact subsets of $\mathbf{R}$. Another possible extension is to approximate a proper smooth embedding by a proper holomorphic embedding on a set of disjoint lines or other real curves in $\mathbf{C}$. We shall not go into details of this.

The original motivation for Theorem 1.1 was the question, communicated to us by R. Narasimhan, as to whether there exist proper holomorphic embeddings $f:\mathbf{C}\hookrightarrow\mathbf{C}^2$ such that $f(\mathbf{C})$ is a nontrivial knot in $\mathbf{C}^2$, i.e., $\mathbf{C}^2\backslash f(\mathbf{C})$ is not homeomorphic to $\mathbf{C}^2\backslash(\mathbf{C}\times\{0\})$, the complement of the embedding $\zeta\mapsto(\zeta,0)$. Unfortunately we have not been able to construct such embeddings with the aid of Theorem 1.1 because real one-dimensional curves in $\mathbf{C}^2\cong\mathbf{R}^4$ are always unknotted.

In order to place Theorem 1.1 in context we recall some recent developments on embedding Stein manifolds in $\mathbf{C}^n$ from [3], [7], [8], [9]. (For Stein manifolds and other topics in several complex variables mentioned here we refer the reader to Hörmander [12].) In those papers it was shown that a Stein manifold M which admits a proper holomorphic embedding in $\mathbf{C}^n$ for some $n>1$ also admits an embedding $f:M\hookrightarrow\mathbf{C}^n$ whose image $f(M)\subset\mathbf{C}^n$ contains a given discrete subset $E\subset\mathbf{C}^n$ [7, Theorem 5.1]. (Recall that any Stein manifold M embeds in $\mathbf{C}^n$ for $n>\frac{1}{2}(3\dim M+1)$ according to Eliashberg and Gromov [5].) With methods of the present paper one can show moreover that for each pair of discrete sets $A=$

$\{a_j\}_{j=1}^{\infty}\subset M$ and $E=\{e_j\}_{j=1}^{\infty}\subset \mathbf{C}^n$ there exists a proper holomorphic embedding $f: M\hookrightarrow \mathbf{C}^n$ such that $f(a_j)=e_j$ for $j=1,2,3,\dots$.

In light of this, a natural question is whether one can replace discrete sets in M by certain positive dimensional submanifolds $N\subset M$, i.e., when is it possible to approximate a smooth proper embedding $\lambda: N\to \mathbf{C}^n$ by the restrictions to N of proper holomorphic embeddings $f: M\hookrightarrow \mathbf{C}^n$? For compact, totally real, holomorphically convex submanifolds $N\subset M$ the answer is affirmative and it follows immediately from the approximation theorems in [6] and [10]. The case when N is noncompact is much harder. Our main result in this paper provides an affirmative answer in the simplest such case when $M=\mathbf{C}$ and $N=\mathbf{R}\times\{i0\}\subset \mathbf{C}$. It seems likely that the result remains valid when N is a properly embedded real line in any Stein manifold M. The details of our construction are considerable, even in this simplest case, and the full scope of the method remains to be seen.

The paper is organized as follows. In Section 2 we introduce the notation and give an outline of the proof of Theorem 1.1. In Section 3 we collect some technical lemmas needed in the proof. The details of the proof of Theorem 1.1 are given in Section 4.

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2. Outline of proof

Since the proof of Theorem 1.1 is somewhat intricate, we give in this section an outline of the proof. We also recall a technical result from [10] (Proposition 2.1 below) which will be used in the proof.

We begin by explaining the notation. We denote by Δ_ϱ the closed disc in $\mathbf{C}$ of radius ϱ and center 0, by $\mathbf{B}$ the open unit ball in $\mathbf{C}^n$ with center 0, and by $R\mathbf{B}$ the ball of radius R. For a set $A\subset \mathbf{C}^n$ and $\varrho>0$, let $A+\varrho\overline{\mathbf{B}}=\{a+z: a\in A,\ |z|\le\varrho\}$. We identify $\mathbf{C}$ and $\mathbf{R}$ with their images in $\mathbf{C}^n$ under the embedding $\zeta\mapsto(\zeta,0,\dots,0)$. For $1\le j\le n$ we denote by π_j the coordinate projection $\pi_j(z_1,\dots,z_n)=z_j$.

In the proof we shall use special automorphisms of $\mathbf{C}^n$ of the form

$$\Psi(z)=z+f(\pi z)v,\quad z\in \mathbf{C}^n,$$

where $v\in \mathbf{C}^n$, $\pi: \mathbf{C}^n\to \mathbf{C}^k$ is a linear map for some $k<n$ with $\pi v=0$ (in most cases $k=1$), and f is an entire function on $\mathbf{C}^k$. An automorphism of this form is called a *shear*; clearly $\Psi^{-1}(z)=z-f(\pi z)v$.

One of the main technical tools in our construction is the following result from [10]. The case $r=0$ was obtained earlier in [9]. This result can also be obtained by methods in [6].

2.1. Proposition. *Let $K\subset\subset\mathbf{C}^n$ ($n\geq 2$) be a compact, polynomially convex set, and let $C\subset\mathbf{C}^n$ be a smooth embedded arc of class $\mathcal{C}^\infty$ which is attached to K in a single point of K. Given a $\mathcal{C}^\infty$ diffeomorphism $F\colon K\cup C\to K\cup C'\subset\mathbf{C}^n$ such that F is the identity on $(K\cup C)\cap U$ for some open neighborhood U of K, and given numbers $r\geq 0$, $\varepsilon>0$, there exist a neighborhood W of K and an automorphism $\Phi\in\operatorname{Aut}\mathbf{C}^n$ satisfying*

$$\|\Phi-\mathrm{Id}\,\|_{\mathcal{C}^r(W)}<\varepsilon,\quad \|\Phi-F\|_{\mathcal{C}^r(C)}<\varepsilon.$$

(Here Id *denotes the identity map.) Moreover, for each finite subset $Z\subset K\cup C$ we can choose Φ such that it agrees with the identity to order r at each point $z\in Z\cap K$ and $\Phi|_C$ agrees with F to order r at each point of $Z\cap C$.*

The same result holds with any finite number of disjoint arcs attached to K.

We now give the outline of the proof of Theorem 1.1. We wish to approximate a given proper embedding $\lambda\colon\mathbf{R}\hookrightarrow\mathbf{C}^n$ by the restriction to $\mathbf{R}$ of a proper holomorphic embedding $f\colon\mathbf{C}\hookrightarrow\mathbf{C}^n$. By standard results we may assume that λ is C^∞ and that any $\mathcal{C}^r$ map of $\mathbf{R}$ into $\mathbf{C}^n$ which has $\mathcal{C}^r$ distance less than $\eta(t)$ from λ (as in Theorem 1.1) is a proper embedding.

We start with the standard embedding $\gamma_0(t)=(t,0,\dots,0)$ and identify $\mathbf{R}$ with $\gamma_0(\mathbf{R})$. We inductively define automorphisms of $\mathbf{C}^n$ of the form $f_k=\Phi_k\circ\ldots\circ\Phi_1\circ\Psi_1\circ\ldots\circ\Psi_k$, where each Φ_j and Ψ_j is an automorphism of $\mathbf{C}^n$ chosen so that $f_k|_{\gamma_0(\mathbf{R})}$ approximates λ on larger and larger compact sets. Moreover, we construct the sequence f_k such that the limit $f=\lim_{k\to\infty}f_k$ exists on an open set $D\subset\mathbf{C}^n$ containing $\mathbf{C}\times\{0\}$, and f is a biholomorphic map of D onto $\mathbf{C}^n$. The restriction of f to the z_1-coordinate axis is then a proper holomorphic embedding of $\mathbf{C}=\mathbf{C}\times\{0\}$ into $\mathbf{C}^n$ satisfying the required properties.

The inductive correction proceeds as follows. We assume that there is an interval $I_k\subset\mathbf{R}$ such that f_k approximates λ on I_k in the sense of Theorem 1.1 and that both $\lambda(t)$ and $f_k(t)$ lie outside some closed ball B_k for $t\notin I_k$. We want to produce an interval I_{k+1} and a ball B_{k+1}, each of which has radius at least one greater than the corresponding set at the kth stage, and a map f_{k+1} which gives the desired approximation on I_{k+1}.

We do this by applying a version of Proposition 2.1 to get an automorphism Φ_{k+1} which is close to the identity map on the sets $f_k(I_k)$, B_k, and $f_k(\Delta_{k+1})$, and such that $\Phi_{k+1}\circ f_k$ approximates λ on I_{k+1}. In order to apply Proposition 2.1, we define a polynomially convex set K_k which includes B_k, $f_k(\Delta_{k+1})$, and most of $f_k(I_k)$, and we also define a smooth, proper embedding $\lambda_k\colon\mathbf{R}\to\mathbf{C}^n$ which agrees with λ on $\mathbf{R}\backslash I_k$, agrees with f_k on most of I_k, and is $\mathcal{C}^r$-near λ everywhere. We can then apply Proposition 2.1 via Lemma 3.4 to get Φ_{k+1} so that $\Phi_{k+1}\circ f_k$ approximates λ on some larger interval I_{k+1} and is near the identity on K_k (this is required for convergence).

The problem now is that the point $\Phi_{k+1}\circ f_k(t)$ may come very close to the previous ball B_k for some $t\in\mathbf{R}\backslash I_{k+1}$. Unless we control this distance from below, the limit map may not be a proper embedding. Hence we precompose $\Phi_{k+1}\circ f_k$ with a shear of the form $\Psi_{k+1}(z)=z+\mu_{k+1}(z_1)\nu_{k+1}$, for some μ_{k+1} holomorphic in one variable and some vector ν_{k+1} with $\pi_1\nu_{k+1}=0$. By Lemma 3.2, we can choose Ψ_{k+1} near the identity on $\Delta_{k+1}\cup I_{k+1}$ and such that $\Phi_{k+1}\circ f_k\circ\Psi_{k+1}(\mathbf{R}\backslash I_{k+1})$ avoids some larger ball B_{k+1}. Except for technicalities, this finishes the induction.

The proof is completed by showing that $f=\lim_{k\to\infty} f_k$ exists and gives a proper holomorphic embedding of $\mathbf{C}=\mathbf{C}\times\{0\}$ to $\mathbf{C}^n$. This is so because the limit $\Psi=\lim_{k\to\infty}\Psi_1\circ\ldots\circ\Psi_k$ exists and is an automorphism of $\mathbf{C}^n$, while the limit $\Phi=\lim_{k\to\infty}\Phi_k\circ\ldots\circ\Phi_1$ exists on an open set $\Omega\subset\mathbf{C}^n$ containing $\Psi(\mathbf{C}\times\{0\})$, and $\Phi:\Omega\to\mathbf{C}^n$ is a biholomorphic map *onto* $\mathbf{C}^n$. Thus $f=\Phi\circ\Psi$ is a biholomorphic map from $D=\Psi^{-1}(\Omega)$ onto $\mathbf{C}^n$ whose restriction to $\mathbf{C}\times\{0\}\subset D$ provides the desired proper holomorphic embedding into $\mathbf{C}^n$. The approximation properties of f are clear from the inductive step.

3. Some lemmas

The following is standard, e.g., [13, Proposition 2.15.4].

3.1. Lemma. *Let $\lambda:\mathbf{R}\hookrightarrow\mathbf{C}^n$ be a $\mathcal{C}^\infty$ proper embedding. Then there exists a continuous $\eta:\mathbf{R}\to(0,\infty)$ such that if $\gamma:\mathbf{R}\to\mathbf{C}^n$ with $|\gamma^{(s)}(t)-\lambda^{(s)}(t)|<\eta(t)$ for all $t\in\mathbf{R}$, $s=0,1$, then γ is a proper embedding.*

Recall that a compact set $A\subset\subset\mathbf{C}^n$ is polynomially convex if for each $z\in\mathbf{C}^n\backslash A$ there is a holomorphic polynomial P on $\mathbf{C}^n$ such that $|P(z)|>\max\{|P(w)|:w\in A\}$. We refer the reader to [12] for properties of such sets.

3.2. Lemma. *Let $A\subset\mathbf{C}^n$ be compact and polynomially convex and $\varrho>0$. Let $I\subset\mathbf{R}$ be an interval whose endpoints lie in $\mathbf{C}^n\backslash(A\cup\Delta_\varrho)$, and let $r,\varepsilon>0$. Then there exists an automorphism $\Psi(z)=z+g(z_1)e_2$ of $\mathbf{C}^n$ such that*

(i) $|\Psi(z)-z|<\varepsilon$ *for* $z\in\Delta_\varrho$,

(ii) $\|\Psi|_\mathbf{R}(t)-t\|_{\mathcal{C}^r(I)}<\varepsilon$, *and*

(iii) $\Psi(t)\notin A$ *for* $t\in\overline{\mathbf{R}\backslash I}$.

If $Z\subset I$ is finite, we can choose Ψ as above so that $g^{(s)}(t)=0$ for $t\in Z$, $0\le s\le r$.

Proof. Let $\mu_1<\mu_2$ denote the endpoints of I in $\mathbf{R}$, and let $\Gamma_j=\{(\mu_j,\zeta,0,\ldots,0):\zeta\in\mathbf{C}\}$ for $j=1,2$. Let $R>\max\{|\mu_1|,|\mu_2|\}+1$ such that $A\subset R\mathbf{B}$. Consider the set $E_j=A\cap\Gamma_j$. Since A is polynomially convex, E_j is polynomially convex in Γ_j and hence $\Gamma_j\backslash E_j$ is connected. Since the endpoints of I lie in $\Gamma_j\backslash E_j$, there exists a smooth curve $\gamma_j\colon[0,1]\to\Gamma_j\backslash E_j$ with $\gamma_j(0)=(\mu_j,0,\dots,0)$ and $|\pi_2\gamma_j(1)|>R+1$ for $j=1,2$.

Since A is compact, there exists $\delta>0$ such that $\gamma_j([0,1])+3\delta\overline{\mathbf{B}}\subset\mathbf{C}^n\backslash A$. Let $\pi_2(z)=z_2$. Let $K=\{x+iy\in\mathbf{C}:\mu_1-\frac{1}{2}\delta\le x\le\mu_2+\frac{1}{2}\delta,\ |y|\le\varrho+1\}$. Define a function $h\colon K\cup[-R,R]\to\mathbf{C}$ by

$$h(t)=\begin{cases}\pi_2\gamma_1(1), & \text{if } t\in[-R,\mu_1-2\delta];\\ \pi_2\gamma_1((\mu_1-\delta-t)/\delta), & \text{if } t\in[\mu_1-2\delta,\mu_1-\delta];\\ \pi_2\gamma_2((t-\mu_2-\delta)/\delta), & \text{if } t\in[\mu_2+\delta,\mu_2+2\delta];\\ \pi_2\gamma_2(1), & \text{if } t\in[\mu_2+2\delta,R];\\ 0, & \text{otherwise.}\end{cases}$$

Choose η, $0<\eta<\min\{\varepsilon,\delta\}$. By Mergelyan's theorem [17, Theorem 20.5] there is an entire function g on $\mathbf{C}$ such that $|h(z)-g(z)|<\eta$ for $z\in K\cup[-R,R]$. The shear $\Psi(z)=z+g(z_1)e_2$ then satisfies (i) and (iii). Since $I\subset\operatorname{Int}K$, Cauchy's estimates imply that it also satisfies (ii) provided that $\eta>0$ is chosen sufficiently small. The last condition on g is a trivial addition to Mergelyan's theorem. □

3.3. Lemma. *Let $\lambda\colon\mathbf{R}\hookrightarrow\mathbf{C}^n$ be a proper, $\mathcal{C}^\infty$ embedding, $K\subset\mathbf{C}^n$ compact, $\varepsilon>0$, and $r\in\mathbf{Z}_+$. Let $Z\subset\mathbf{R}$ be finite, and suppose $\lambda(t)\in\mathbf{C}=\mathbf{C}\times\{0\}^{n-1}$ for each $t\in Z$. Then there exists a shear $\Psi(z)=z+h(z_1)v$ for some $v\in\mathbf{C}^n$ with $\pi_1v=0$ such that*

(i) $\Psi(\mathbf{C})\cap\lambda(\mathbf{R})=\lambda(Z)$,
(ii) $|\Psi(z)-z|<\varepsilon$ *for* $z\in K$, *and*
(iii) $\Psi(z)=z+O(|z-\lambda(t)|^{r+1})$ *as* $z\to\lambda(t)$, *for all* $t\in Z$.

Proof. Let $Z=\{t_j\}_{j=1}^s$. For $\zeta\in\mathbf{C}$ let $h(\zeta)=\Pi_{1\le j\le s}(\zeta-\pi_1\lambda(t_j))^{r+1}$. Consider the map $\Phi\colon\mathbf{C}\times\mathbf{C}^{n-1}\to\mathbf{C}^n$ given by

$$\Phi(z_1,\alpha_2,\dots,\alpha_n)=(z_1,0,\dots,0)+h(z_1)(0,\alpha_2,\dots,\alpha_n).$$

Clearly Φ is an automorphism of $(\mathbf{C}\backslash\lambda(Z))\times\mathbf{C}^{n-1}$. Let $\Delta_{R,j}$ denote the closed disc of radius R in $\mathbf{C}$ with center $\pi_1\lambda(t_j)$ for $j=1,2,\dots,s$. Choose $R>0$ such that the discs $\Delta_{R,j}$ for $1\le j\le s$ are pairwise disjoint. Choose a ϱ, $0<\varrho<R$, such that ϱ^2 is a regular value of $\mu_j(t)=|\pi_1\lambda(t)-\pi_1\lambda(t_j)|^2$ $(t\in\mathbf{R})$ for each $j=1,2,\dots,s$.

Let $M_\varrho=\mathbf{C}\backslash\bigcup_{1\le j\le s}\operatorname{int}\Delta_{\varrho,j}$. Let $\Phi_\varrho=\Phi|_{M_\varrho\times\mathbf{C}^{n-1}}$ and $\partial\Phi_\varrho=\Phi|_{\partial M_\varrho\times\mathbf{C}^{n-1}}$. A simple check shows that Φ_ϱ and $\partial\Phi_\varrho$ are transverse to $\lambda(\mathbf{R})$. Hence by the transversality theorem, there exists a set $A_\varrho\subset\mathbf{C}^{n-1}$ of full measure such that for each $\alpha=(\alpha_2,\dots,\alpha_n)\in A_\varrho$, $\Phi(M_\varrho,\alpha)=\{\Phi(z_1,\alpha):z_1\in M_\varrho\}$ and $\lambda(\mathbf{R})$ are transverse, hence disjoint by dimension considerations.

Let $A=\bigcap_{j=1}^{\infty} A_{1/j}$. Then $A\subset\mathbf{C}^{n-1}$ has full measure, and for each $\alpha\in A$ we see that $\Phi(\mathbf{C}\setminus\lambda(Z),\alpha)$ and $\lambda(\mathbf{R})$ are disjoint. Finally, choose $\alpha\in A$ such that $|h(z_1)\alpha|<\varepsilon$ for $z_1\in\pi_1(K)$, and let $\Psi(z)=z+h(z_1)\alpha$. Then $\Psi(z_1,0,\dots,0)=\Phi(z_1,\alpha_2,\dots,\alpha_n)$, and Ψ satisfies the conclusions of the lemma. □

3.4. Lemma. *Let $\lambda:\mathbf{R}\hookrightarrow\mathbf{C}^n$ be a $\mathcal{C}^\infty$ embedding, $f:\mathbf{C}\hookrightarrow\mathbf{C}^n$ a proper holomorphic embedding, and $I\subset\mathbf{R}$ a closed interval with $f|_I=\lambda|_I$. Let $K\subset\mathbf{C}^n$ be compact and polynomially convex, $a,r,\varepsilon>0$, and $T\subset\mathbf{R}$ discrete. Suppose that $\lambda(t),f(t)\notin K$ for $t\in\overline{\mathbf{R}\setminus I}$. Then there exists $\Phi\in\operatorname{Aut}\mathbf{C}^n$ such that if $g=\Phi\circ f$, then*

(i) $|g^{(s)}(t)-\lambda^{(s)}(t)|<\varepsilon$ *for* $t\in[-a,a]$, $0\le s\le r$,

(ii) $g^{(s)}(t)=\lambda^{(s)}(t)$ *for* $t\in T\cap[-a,a]$, $0\le s\le r$, *and*

(iii) $|\Phi(z)-z|<\varepsilon$ *for* $z\in K$.

Proof. We may assume that $I\subset(-a,a)$. Let I_1, I_2 be the two connected components of $\{\zeta\in I:f(\zeta)\in\mathbf{C}^n\setminus K\}$ containing the respective endpoints of I, and let $I_0=I\setminus(I_1\cup I_2)$. Let A be the polynomial hull of $K\cup f(I_0)$. Then A is the union of $K\cup f(I_0)$ and the bounded connected components of $f(\mathbf{C})\setminus(K\cup f(I_0))$. Note that $f(I_1)$ and $f(I_2)$ lie in $f(\mathbf{C})\setminus A$ since $f(t)\notin K$ for all $t\in\overline{\mathbf{R}\setminus I}$.

Let $L=A\cup f([-a,a])$. Then $C=\overline{L\setminus A}$ is the union of two embedded arcs, each containing an endpoint of $f([-a,a])$. Define F on L by $F(z)=z$ if $z\in A$, and $F(z)=\lambda f^{-1}(z)$ if $z\in f([-a,a])$. Then F is a $\mathcal{C}^\infty$ diffeomorphism of L which extends as the identity map on $(A\cup C)\cap U$ for some neighborhood U of A. Apply Proposition 2.1 to get $\Phi\in\operatorname{Aut}\mathbf{C}^n$ such that $|\Phi(z)-z|<\varepsilon$ for $z\in K$ and such that $g=\Phi\circ f$ satisfies (i) and (ii). □

4. Proof of Theorem 1.1

Choose a smooth cutoff function χ on $\mathbf{R}$ such that $\chi(t)=1$ for $|t|$ small and $\operatorname{supp}\chi\subset(-1,1)$. Define the constant $C=C_r>1$ such that $\|\chi h\|_{\mathcal{C}^r}\le C\|h\|_{\mathcal{C}^r}$ for each $h\in\mathcal{C}^r(\mathbf{R})$. We fix such C for the entire proof.

By approximation we may assume that $\lambda:\mathbf{R}\hookrightarrow\mathbf{C}^n$ in Theorem 1.1 is a proper $\mathcal{C}^\infty$ embedding. Decreasing η if necessary we may also assume that η satisfies Lemma 3.1 for λ and $\eta(t)<\frac{1}{2}$ for all $t\in\mathbf{R}$.

We use an inductive procedure to obtain a sequence of proper holomorphic embeddings $f_k:\mathbf{C}\hookrightarrow\mathbf{C}^n$ such that $f=\lim_{k\to\infty}f_k$ exists on $\mathbf{C}$ and satisfies Theorem 1.1. Each f_k will be a restriction to $\mathbf{C}=\mathbf{C}\times\{0\}^{n-1}$ of a holomorphic automorphism of $\mathbf{C}^n$. The next map f_{k+1} will be of the form $f_{k+1}=\Phi_{k+1}\circ f_k\circ\Psi_{k+1}$ for suitably chosen $\Phi_{k+1},\Psi_{k+1}\in\operatorname{Aut}\mathbf{C}^n$.

We will describe the case $k=1$ after the inductive step is given. Recall that Δ_k is the closed disc in $\mathbf{C}=\mathbf{C}\times\{0\}^{n-1}$ with center 0 and radius k, and $\mathbf{B}$ is the unit ball in $\mathbf{C}^n$. For the induction at step k, suppose we have the following:

(a) closed balls $B_j=R_j\overline{\mathbf{B}}\subset\mathbf{C}^n$ with $R_j\geq\max\{j+1,R_{j-1}+1\}$, $j=1,\dots,k$,

(b) automorphisms $\Phi_1,\dots,\Phi_k$ of $\mathbf{C}^n$ with $|\Phi_j(z)-z|<2^{-j}$ for $z\in B_{j-1}$, $j=2,\dots,k$,

(c) numbers $\varepsilon_j>0$ such that $\varepsilon_1<2^{-1}$ and $\varepsilon_j<\frac{1}{2}\varepsilon_{j-1}<2^{-j}$ for $j=2,\dots,k$,

(d) automorphisms $\Psi_1,\dots,\Psi_k$ of $\mathbf{C}^n$ of the form $\Psi_j(z)=z+g_j(z_1)e_2+h_j(z_1)v_j$, where $\pi_1(v_j)=0$ and $|\Psi_j(z)-z|<\varepsilon_j$ for $|z|\leq j$,

(e) closed intervals $I_j=[-a_j,a_j]$, $j=1,\dots,k$, with $a_j>\max\{a_{j-1}+2,j+2\}$, and

(f) numbers $0<\delta_j<C^{-1}\inf\{\eta(t):t\in I_j\}$, $j=1,\dots,k$,

such that the automorphism

$$f_k=\Phi_k\circ\dots\Phi_1\circ\Psi_1\circ\dots\circ\Psi_k\in\operatorname{Aut}\mathbf{C}^n$$

(whose restriction to $\mathbf{C}=\mathbf{C}\times\{0\}^{n-1}$ provides an embedding $\mathbf{C}\hookrightarrow\mathbf{C}^n$) satisfies:

(1_k) $f_k(\Delta_j+\varepsilon_k\overline{\mathbf{B}})\subset\operatorname{Int}B_j$ for $j=1,\dots,k$,

(2_k) $|f_k^{(s)}(t)-\lambda^{(s)}(t)|<\eta(t)$ for $t\in I_k$ and $0\leq s\leq r$,

(3_k) $|f_k^{(s)}(t)-\lambda^{(s)}(t)|<\delta_k$ for $t\in I_k\backslash(-a_k+1,a_k-1)$ and $0\leq s\leq r$,

(4_k) $f_k^{(s)}(t)=\lambda^{(s)}(t)$ for $t\in T\cap I_k$, $0\leq s\leq r$,

(5_k) $f_k(\mathbf{C})\cap\lambda(\mathbf{R})=\lambda(T\cap I_k)$,

(6_k) $|\lambda(t)|>R_k+1$ for $|t|\geq a_k-1$,

(7_k) $|f_k(t)|>R_k$ for $|t|\geq a_k-1$.

We will now show how to obtain these hypotheses at step $k+1$. Let I_k^1 and I_k^2 be the two connected components of the set $\{\zeta\in I_k\backslash\Delta_{k+1}:|f_k(\zeta)|>R_k\}$ containing the respective endpoints of the interval I_k. Let $I_k^0=I_k\backslash(I_k^1\cup I_k^2)$ be the middle interval. By (7_k) we have $I_k^0\subset(-a_k+1,a_k-1)$.

Let K_k be the polynomial hull of the set $B_k\cup f_k(\Delta_{k+1}\cup I_k^0)$. Since $f_k(\mathbf{C})$ is a complex submanifold of $\mathbf{C}^n$ and B_k is polynomially convex, it is seen easily that K_k is contained in $B_k\cup f_k(\mathbf{C})$, and it is the union of $B_k\cup f_k(\Delta_{k+1}\cup I_k^0)$ and the bounded connected components of the complement $f_k(\mathbf{C})\backslash(B_k\cup f_k(\Delta_{k+1}\cup I_k^0))$ (see Lemma 5.4 in [7]). Note that (7_k) and (e) imply that $f_k(\mathbf{R}\backslash(-a_k+1,a_k-1))\subset\mathbf{C}^n\backslash K_k$.

Choose $R_{k+1}>R_k+1$ such that $K_k\subset(R_{k+1}-1)\mathbf{B}$, and let $B_{k+1}=R_{k+1}\overline{\mathbf{B}}$. Choose $a_{k+1}>a_k+2$ to get (6_{k+1}). We now want to approximate λ on the larger interval $I_{k+1}=[-a_{k+1},a_{k+1}]$ by the image of the next embedding $\mathbf{C}\hookrightarrow\mathbf{C}^n$ (to be constructed). In order to apply Lemma 3.4 we first approximate λ as follows:

4.1. Lemma. *There exists a proper C^∞ embedding $\lambda_k:\mathbf{R}\hookrightarrow\mathbf{C}^n$ satisfying*

(i) $\lambda_k=f_k$ *on* $[-a_k+1,a_k-1]$,

(ii) $\lambda_k=\lambda$ *on* $\mathbf{R}\backslash I_k$,
(iii) $|\lambda_k^{(s)}(t)-\lambda^{(s)}(t)|<\eta(t)$ *for* $t\in I_k\backslash(-a_k+1,a_k-1)$, $0\le s\le r$,
(iv) $\lambda_k^{(s)}(t)=\lambda^{(s)}(t)$ *for* $t\in T$, $0\le s\le r$, *and*
(v) $\lambda_k(t)\notin K_k$ *when* $|t|\ge a_k-1$.

Proof. We define the cutoff function χ_k on $\mathbf{R}$ using χ, so that $\chi_k=0$ on $\mathbf{R}\backslash I_k$, $\chi_k=1$ on $[-a_k+1,a_k-1]$, and $\|\chi_k h\|_{\mathcal{C}^r}<C\|h\|_{\mathcal{C}^r}$ as before. Let

$$\hat\lambda_k(t)=f_k(t)\chi_k(t)+\lambda(t)(1-\chi_k(t)),\quad t\in\mathbf{R}.$$

By Lemma 3.1, (3_k), (4_k), and choice of η and δ_k, we see that (i)–(iv) are satisfied for $\hat\lambda_k$ in place of λ_k.

To obtain (v) we use a transversality argument to perturb $\hat\lambda_k$ on the set $I_k\backslash(-a_k+1,a_k-1)$. First note that if $|t|>a_k$, then $|\hat\lambda_k(t)|=|\lambda(t)|>R_k+1$ by (6_k), so $\hat\lambda_k(t)\notin B_k$. Also, by ($5_k$), we see that $\hat\lambda_k(t)\notin f_k(\mathbf{C})$, so $\hat\lambda_k(t)\notin K_k$. Next, if $t\in T\cap(I_k\backslash(-a_k+1,a_k-1))$, then by ($4_k$), ($7_k$), and (e) we see that $\hat\lambda_k(t)=f_k(t)\notin K_k$. Hence there exists a neighborhood V of $T\cap(I_k\backslash(-a_k+1,a_k-1))$ such that $\hat\lambda_k(\overline{V})\cap K_k=\emptyset$.

Thus we need only perturb $\hat\lambda_k$ on $I_k\backslash(V\cup(-a_k+1,a_k-1))$ to get (v). Note that if $t\in I_k\backslash(-a_k+1,a_k-1)$, then from ($6_k$) and ($2_k$) we see that $|\hat\lambda_k(t)|>R_k+\frac{1}{2}$, so $\hat\lambda_k(t)\notin B_k$. Finally, a simple transversality argument implies that we can make an arbitrarily small $\mathcal{C}^\infty$ perturbation of $\hat\lambda_k$ to avoid $f_k(\mathbf{C})$, and hence we get λ_k with $\lambda_k=\hat\lambda_k$ outside $I_k\backslash(V\cup(-a_k+1,a_k-1))$ and λ_k satisfying (i)–(v). □

Now we can use Lemma 3.4 to approximate λ_k, hence to approximate λ. Set

$$\delta_{k+1}=\min\{\eta(t):t\in I_{k+1}\}/2C,$$
$$\sigma_{k+1}=\min\{\eta(t)-|\lambda_k^{(s)}(t)-\lambda^{(s)}(t)|:t\in I_{k+1},\ 0\le s\le r\}>0.$$

Choose $\varepsilon>0$ so small that

$$\varepsilon<\min\{2^{-(k+1)},\delta_{k+1},\sigma_{k+1}\},\quad f_k(\Delta_j+\varepsilon_k\overline{\mathbf{B}})+\varepsilon\overline{\mathbf{B}}\subset\operatorname{Int}B_j,\ 1\le j\le k.$$

Apply Lemma 3.4 with $\lambda=\lambda_k$, $f=f_k$, $I=[-a_k+1,a_k-1]$, $K=K_k$, $a=a_{k+1}$, r and T unchanged, and ε as above. This provides $\Phi_{k+1}\in\operatorname{Aut}\mathbf{C}^n$ and $G=\Phi_{k+1}\circ f_k\in\operatorname{Aut}\mathbf{C}^n$ satisfying

$$\begin{cases}|\Phi_{k+1}(z)-z|<\varepsilon & \text{for } z\in K_k\text{, hence on } B_k;\\ |G^{(s)}(t)-\lambda_k^{(s)}(t)|<\varepsilon & \text{for } t\in I_{k+1},\ 0\le s\le r;\\ G^{(s)}(t)=\lambda_k^{(s)}(t) & \text{for } t\in T\cap I_{k+1},\ 0\le s\le r.\end{cases}$$

In particular, (2_{k+1})–(4_{k+1}) hold with G in place of f_{k+1}.

Since $f_k(\Delta_{k+1}) \subset K_k \subset (R_{k+1}-1)\mathbf{B}$, we can choose $\varepsilon'_{k+1} < \varepsilon_k$ small enough that (1_{k+1}) holds with G in place of f_{k+1} and ε'_{k+1} in place of ε_{k+1}, and such that if $\psi \in \operatorname{Aut} \mathbf{C}^n$ with $\|\psi(t)-t\|_{C^r(I_{k+1})} < \varepsilon'_{k+1}$, then (2_{k+1}) and (3_{k+1}) hold with $G\circ\psi$ in place of f_{k+1}. Let $\varepsilon_{k+1}=\frac{1}{2}\varepsilon'_{k+1}$. Then with G in place of f_{k+1}, we have (1_{k+1})–(4_{k+1}), (6_{k+1}), and $G(-a_{k+1}), G(a_{k+1}) \notin B_{k+1}$ by (6_{k+1}) and (2_{k+1}).

Next we want to obtain (7_{k+1}). We do this using Lemma 3.2 to change the embedding so that the image of $\mathbf{R}\backslash I_{k+1}$ misses B_{k+1} while leaving the embedding essentially unchanged on $\Delta_{k+1} \cup I_{k+1}$. Apply Lemma 3.2 with $A=G^{-1}(B_{k+1})$, $\varrho=k+1$, $I=I_{k+1}$, r unchanged, $Z=T\cap I_{k+1}$ and $\varepsilon=\frac{1}{2}\varepsilon_{k+1}$. This gives a shear

$$\psi_{k+1}(z) = z + g_{k+1}(z_1)e_2$$

with

$$\begin{cases} |\psi_{k+1}(z)-z| < \frac{1}{2}\varepsilon_{k+1}, & z\in\Delta_{k+1}; \\ \|\psi_{k+1}|_{\mathbf{R}}(t)-t\|_{C^r(I_{k+1})} < \frac{1}{2}\varepsilon_{k+1}; & \\ g_{k+1}^{(s)}(t)=0, & t\in T\cap I_{k+1},\ 0\le s\le r; \\ \psi_{k+1}(t)\notin G^{-1}(B_{k+1}), & t\in\overline{\mathbf{R}\backslash I_{k+1}}. \end{cases}$$

Let $H=G\circ\psi_{k+1}$. Then with H in place of f_{k+1}, we have (1_{k+1})–(4_{k+1}), (6_{k+1}), and (7_{k+1}).

For the final correction, we use Lemma 3.3 to obtain (5_{k+1}) while maintaining the other properties. Let $R>a_{k+1}$ be such that $A=G^{-1}(B_{k+1})\subset R\mathbf{B}$. Let $\delta>0$ be such that

$$\psi_{k+1}([-R,R]\backslash(-a_{k+1},a_{k+1})+\delta\overline{\mathbf{B}})\cap A=\emptyset,$$

and such that if $\theta\in\operatorname{Aut}\mathbf{C}^n$, with $|\theta(z)-z|<\delta$ on $R\overline{\mathbf{B}}$, then

$$\|\psi_{k+1}\circ\theta|_{\mathbf{R}}(t)-t\|_{C^r(I_{k+1})} < \varepsilon_{k+1}. \tag{1}$$

Apply Lemma 3.3 with λ replaced by $H^{-1}\circ\lambda$, $K=R\overline{\mathbf{B}}$, r unchanged, $Z=T\cap I_{k+1}$, and $\varepsilon=\min\{\delta, \frac{1}{2}\varepsilon_{k+1}\}$. This gives a shear $\theta_{k+1}(z)=z+h_{k+1}(z_1)v_{k+1}$ with $\pi_1 v_{k+1}=0$ such that

$$\begin{cases} |\theta_{k+1}(z)-z| < \min\{\delta, \frac{1}{2}\varepsilon_{k+1}\}, & z\in\Delta_{k+1}; \\ \theta_{k+1}(\mathbf{C})\cap H^{-1}\lambda(\mathbf{R}) = H^{-1}\lambda(T\cap I_{k+1}); & \\ h_{k+1}^{(s)}(t)=0, & t\in T\cap I_{k+1},\ 0\le s\le r, \end{cases}$$

and such that (1) holds with $\theta=\theta_{k+1}$. Also, by the choice of R and δ,

$$\psi_{k+1}\circ\theta_{k+1}(\overline{\mathbf{R}\backslash I_{k+1}})\cap A=\emptyset.$$

Taking $\Psi_{k+1}=\psi_{k+1}\circ\theta_{k+1}$ and

$$f_{k+1}=H\circ\theta_{k+1}=\Phi_{k+1}\circ f_k\circ\Psi_{k+1}$$

we obtain (5_{k+1}) and preserve the remaining hypotheses. Hence we obtain (1_{k+1})–(7_{k+1}). Note that $(k+1)\mathbf{B}\subset B_{k+1}$ so we also obtain (a)–(f), thus finishing the inductive step.

The case $k=1$ is similar to the general step. First apply Proposition 2.1 with $K=\emptyset$, $C=[-3,3]\subset\mathbf{C}$, $F=\lambda$, $\varepsilon=C^{-1}\inf\{\eta(t):t\in[-3,3]\}$, and $Z=T\cap[-3,3]$ to get $\phi_1^1\in\operatorname{Aut}\mathbf{C}^n$ satisfying the conclusions of that proposition. Choose $R_1\geq 2$ such that $\phi_1^1(\Delta_1)\subset(R_1-1)\mathbf{B}$, choose $a_1>4$ to get (6_1), and let $I_1=[-a_1,a_1]$. Choose δ_1 to satisfy (f) for $j=1$.

Define a proper $\mathcal{C}^\infty$ embedding λ_0 as in Lemma 4.1 so that (i)–(v) are satisfied with λ_0 in place of λ_k, ϕ_1^1 in place of f_k, 3 in place of a_k, $[-3,3]$ in place of I_k, and $\phi_1^1(\Delta_1)$ in place of K_k. Apply Lemma 3.4 with $\lambda=\lambda_0$, $f=\phi_1^1$, $I=[-2,2]$, $K=\phi_1^1(\Delta_1)$, $a=a_1$, $\varepsilon=\delta_1$, and T and r unchanged. This gives $\phi_1^2\in\operatorname{Aut}\mathbf{C}^n$ such that

$$|\phi_1^2(z)-z|<\delta_1\leq\tfrac{1}{2},\quad z\in\phi_1^1(\Delta_1),$$

and such that $\Phi_1=\phi_1^2\phi_1^1$ satisfies

$$\begin{cases} \|\Phi_1-\lambda_0\|_{\mathcal{C}^r(I_1)}<\varepsilon; \\ \Phi_1^{(s)}(t)=\lambda_0^{(s)}(t), \qquad t\in T\cap I_1,\ 0\leq s\leq r. \end{cases}$$

As before, we can apply Lemmas 3.2 and 3.3 to obtain $\varepsilon_1>0$ and a shear Ψ_1 such that the hypotheses (1_1)–(7_1) hold for $f_1=\Phi_1\circ\Psi_1$, and (a)–(f) hold for $k=1$. This completes the base case.

To finish the proof of Theorem 1.1, note that

$$\Psi_1\circ\ldots\circ\Psi_k(z)=z+\sum_{j=1}^{k}(g_j(z_1)e_2+h_j(z_1)v_j)$$

and that (c) implies

$$|g_j(z_1)e_2+h_j(z_1)v_j|<2^{-j},\quad |z_1|<j.$$

Hence this sum converges uniformly on compacts to a shear $\Psi(z)=z+G(z_1)$ for some holomorphic map $G:\mathbf{C}\to\{0\}\times\mathbf{C}^{n-1}$.

By Proposition 4.2 in [7], the composition $\Phi_k\circ\ldots\circ\Phi_1$ converges locally uniformly to a biholomorphic map from a domain Ω onto $\mathbf{C}^n$, and

$$\Omega=\bigcup_{k=1}^{\infty}(\Phi_k\circ\ldots\circ\Phi_1)^{-1}(B_{k-1}).$$

We claim that $\Psi(\mathbf{C}\times\{0\})\subset\Omega$. Let $k>1$. By (1_k) we have

$$\Psi_1\circ\ldots\circ\Psi_k(\Delta_{k-1}+\varepsilon_k\overline{\mathbf{B}})\subset(\Phi_k\circ\ldots\circ\Phi_1)^{-1}(B_{k-1}).$$

Since $|\Psi_j(z)-z|<\varepsilon_j$ on Δ_{k-1} for $j\geq k$, and $\sum_{j=k+1}^{\infty}\varepsilon_j<\varepsilon_k$ by (c), we see that

$$\lim_{m\to\infty}\Psi_{k+1}\circ\ldots\circ\Psi_m(z)\in\Delta_{k-1}+\varepsilon_k\overline{\mathbf{B}}$$

for each $z\in\Delta_{k-1}$. Hence

$$\Psi(\Delta_{k-1})\subset(\Phi_k\circ\ldots\circ\Phi_1)^{-1}(B_{k-1})\subset\Omega,\quad k>1,$$

so the claim holds. In particular, $\Phi\circ\Psi:\mathbf{C}\to\mathbf{C}^n$ is a proper holomorphic embedding.

Finally, using the conditions (1_k)–(7_k), we see that $\Phi\circ\Psi:\mathbf{C}\hookrightarrow\mathbf{C}^n$ is a proper holomorphic embedding with the desired properties. □

References

1. Abhyankar, S. S. and Moh, T. T., Embeddings of the line in the plane, *J. Reine Angew. Math.* **276** (1975), 148–166.
2. Arakelian, N. U., Uniform approximation on closed sets by entire functions, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1187–1206 (Russian).
3. Buzzard, G. and Fornæss, J. E., An embedding of $\mathbf{C}$ into $\mathbf{C}^2$ with hyperbolic complement, *Math. Ann.* **306** (1996), 539–546.
4. Carleman, T., Sur un théorème de Weierstrass, *Ark. Mat. Astr. Fys.* **20**:3 (1927).
5. Eliashberg, Y. and Gromov, M., Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2+1$, *Ann. of Math.* **136** (1992), 123–135.
6. Forstneric, F., Approximation by automorphisms on smooth submanifolds of $\mathbf{C}^n$, *Math. Ann.* **300** (1994), 719–738.
7. Forstneric, F., Interpolation by holomorphic automorphisms and embeddings in $\mathbf{C}^n$, Preprint, 1996.
8. Forstneric, F., Globevnik, J. and Rosay, J.-P., Non straightenable complex lines in $\mathbf{C}^2$, *Ark. Mat.* **34** (1996), 97–101.
9. Forstneric, F., Globevnik, J. and Stensønes, B., Embedding holomorphic discs through discrete sets, *Math. Ann.* **305** (1995), 559–569.
10. Forstneric, F. and Løw, E., Global holomorphic equivalence of certain smooth submanifolds in $\mathbf{C}^n$, to appear in *Indiana Univ. Math. J.* (1997).
11. Gaier, D., *Lectures on Carleman Approximation*, Birkhäuser, Boston, Mass., 1987.
12. Hörmander, L., *An Introduction to Complex Analysis in Several Variables,* 3rd ed., North-Holland, Amsterdam, 1990.
13. Narasimhan, R., *Analysis on Real and Complex Manifolds,* 3rd printing, North-Holland, Amsterdam, 1985.

14. ROSAY, J.-P. and RUDIN, W., Arakelian's approximation theorem, *Amer. Math. Monthly* **96** (1989), 432–434.
15. ROSAY, J.-P. and RUDIN, W., Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$, *Trans. Amer. Math. Soc.* **310** (1988), 47–86.
16. ROSAY, J.-P. and RUDIN, W., Holomorphic embeddings of $\mathbf{C}$ in $\mathbf{C}^n$, in *Several complex variables*: Proceedings of the Mittag-Leffler Institute, 1987–88 (Fornæss, J. E., ed.), pp. 563–569. *Math. Notes* **38**, Princeton Univ. Press, Princeton, N. J., 1993.
17. RUDIN, W., *Real and Complex Analysis*, 3rd ed., McGraw Hill, New York, 1987.
18. SUZUKI, M., Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbf{C}^2$, *J. Math. Soc. Japan* **26** (1974), 241–257.

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