

The Oka principle for sections of subelliptic submersions

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0 Introduction

In this paper we prove the Oka principle for maps $X \rightarrow Y$ from Stein manifolds X to complex manifolds Y which admit a finite dominating collection of sprays (such manifolds are called subelliptic), as well as the corresponding result for sections of submersions. Our main result, Theorem 1.1, extends the results of Oka [O], Grauert [Gr1, Gr2] and Gromov [G]. We also prove a result on removing intersections of holomorphic maps from Stein manifolds with closed complex subvarieties with subelliptic complements (Sect. 6).

We begin with a brief survey. In 1939 K. Oka [O] proved that a second Cousin problem on a domain of holomorphy is solvable if it is solvable by continuous functions. Oka's result has the following equivalent formulation: *If $h: Z \rightarrow X$ is a principal holomorphic fiber bundle with fiber $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ over a domain of holomorphy (or a Stein manifold) then every continuous section of h is homotopic to a holomorphic section.* In a seminal work of 1957 H. Grauert [Gr1, Gr2] proved Oka's theorem with \mathbb{C}^* replaced by any complex Lie group or complex homogeneous space, with the stronger conclusion that the inclusion $\text{Holo}(X; Z) \hookrightarrow \text{Cont}(X; Z)$ of the space of holomorphic sections into the space of continuous sections is a *weak homotopy equivalence* (it induces isomorphisms of all homotopy groups of the two spaces). This is known as the (parametric) **Oka-Grauert principle**. For related results see [C], [FR] and [HL].

In 1989 M. Gromov [G] introduced the concept of a *dominating spray* and outlined a proof of the *parametric Oka principle for sections of holomorphic submersions $h: Z \rightarrow X$ onto a Stein base X which admit fiber dominating*

sprays over small open subsets of X . (Proofs can be found in [FP1] for fiber bundles and in [FP2, FP3] for submersions).

In this paper we introduce a more flexible and apparently weaker condition which also implies the parametric Oka principle. Recall that a **spray** on a complex manifold Y is a holomorphic map $s: E \rightarrow Y$ from the total space of a holomorphic vector bundle $p: E \rightarrow Y$ such that $s(0_y) = y$ for all $y \in Y$. Y is called **subelliptic** if it admits finitely many sprays $s_j: E_j \rightarrow Y$ such that for any $y \in Y$ the vector subspaces $(ds_j)_{0_y}(E_{j,y}) \subset T_y Y$ together span $T_y Y$ (Definition 2). We prove *the parametric Oka principle for maps from any Stein manifold to any subelliptic manifold, as well as for sections of subelliptic submersions over a Stein base* (Theorem 1.1). This extends Gromov's theorem [G, 4.5 Main Theorem] which assumes the existence of a **dominating spray** $s: E \rightarrow Y$ satisfying $(ds)_{0_y}(E_y) = T_y Y$ for all $y \in Y$ (such manifold Y is called **elliptic**). There is no immediate way of creating dominating sprays from dominating families of sprays unless the bundles E_j are trivial.

Subellipticity is easier to verify than ellipticity and consequently it enables us to extend the Oka principle to a wider class of target manifolds. For instance, if A is a closed complex (=algebraic) subvariety of complex codimension at least two in a complex projective space $\mathbb{C}P^n$ (or in a complex Grassmanian) then its complement is subelliptic and hence the Oka principle holds for maps from Stein manifolds to $\mathbb{C}P^n \setminus A$ (Proposition 1.2). We don't know whether this complement is elliptic in general. (By removing a hyperplane we obtain $\mathbb{C}^n \setminus A$ which is elliptic [G, FP1].) On projective algebraic manifolds subellipticity can be localized: *If each point $y \in Y$ admits a Zariski open neighborhood which is algebraically subelliptic then Y is subelliptic* (Proposition 1.3). No such result is known about ellipticity.

Subellipticity is equivalent to the existence of a *dominating composed spray* (Lemma 2.4). Even though Gromov discussed composed sprays in [G] (see in particular the Sects. 1.3, 1.4.F. and 2.9.A.), this condition has not been formulated before. On a Stein manifold Y subellipticity is equivalent to ellipticity (Lemma 2.2) and both conditions are implied by the validity of the Oka principle for maps $X \rightarrow Y$ from Stein manifolds X with second order interpolation along closed complex submanifolds $X_0 \subset X$ (see [G, 3.2.A] or [FP3, Proposition 1.2]). It is not clear whether the validity of the Oka principle implies subellipticity (or ellipticity) for all complex manifolds.

1 The results

Let $h: Z \rightarrow X$ be a holomorphic submersion onto X . Given a subset $U \subset X$ we write $Z|_U = h^{-1}(U)$. For $z \in Z$ we denote by $VT_z Z$ the kernel of dh_z (which equals the tangent space to the fiber $h^{-1}(h(z))$ at z) and call it the

vertical tangent space of Z at z . If $p: E \rightarrow Z$ is a holomorphic vector bundle we denote by $0_z \in E$ the base point in the fiber $E_z = p^{-1}(z)$. At each point $z \in Z$ we have a natural splitting $T_{0_z}E = T_zZ \oplus E_z$.

Definition 1. [G, sec. 1.1.B] A **spray associated to a holomorphic submersion** $h: Z \rightarrow X$ (an h -spray) is a triple (E, p, s) , where $p: E \rightarrow Z$ is a holomorphic vector bundle and $s: E \rightarrow Z$ is a holomorphic map such that for each $z \in Z$ we have $s(0_z) = z$ and $s(E_z) \subset Z_{h(z)}$. The spray s is **dominating** at the point $z \in Z$ if the derivative $ds: T_{0_z}E \rightarrow T_zZ$ maps E_z surjectively onto $VT_zZ = \ker dh_z$. A **spray on a complex manifold** Y is a spray associated to the trivial submersion $Y \rightarrow \text{point}$.

Definition 2. A holomorphic submersion $h: Z \rightarrow X$ is called **subelliptic** if each point in X has an open neighborhood $U \subset X$ such that $h: Z|_U \rightarrow U$ admits finitely many h -sprays (E_j, p_j, s_j) for $j = 1, \dots, k$ satisfying

$$(ds_1)_{0_z}(E_{1,z}) + (ds_2)_{0_z}(E_{2,z}) \cdots + (ds_k)_{0_z}(E_{k,z}) = VT_zZ \quad (1.1)$$

for each $z \in Z|_U$. A collection of sprays satisfying (1.1) is said to be **dominating** at z . A submersion h is **elliptic** if the above holds with $k = 1$, i.e., if any point $x \in X$ has a neighborhood $U \subset X$ such that $h: Z|_U \rightarrow U$ admits a dominating spray. A complex manifold Y is **elliptic** (resp. **subelliptic**) if the trivial submersion $Y \rightarrow \text{point}$ is such.

Thus every elliptic submersion is also subelliptic. Examples of elliptic manifolds and submersions may be found in [G] (see especially Sects. 0.5.B and 3.4.F) and in [FP1]. The exponential map $\exp: \mathfrak{g} \rightarrow G$ on any complex Lie group G gives a dominating spray $s: E = G \times \mathfrak{g} \rightarrow G$, $s(g, t) = \exp(t)g$.

1.1 Theorem. If $h: Z \rightarrow X$ is a subelliptic submersion onto a Stein manifold X then the inclusion $\iota_h: \Gamma_{\text{holo}}(X; Z) \hookrightarrow \Gamma_{\text{cont}}(X; Z)$ of the space of holomorphic sections of h into the space of continuous sections is a weak homotopy equivalence. (Both spaces are endowed with the topology of uniform convergence on compacts.)

Theorem 1.1 is the main result of this paper. It implies in particular that maps $X \rightarrow Y$ from any Stein manifold X to any subelliptic manifold Y satisfy the parametric Oka principle (since maps $X \rightarrow Y$ correspond to sections of the projection $X \times Y \rightarrow X$). The proof of Theorem 1.1 will show in addition that sections of any subelliptic submersion $h: Z \rightarrow X$ onto a Stein base X satisfy the conclusion of Theorem 1.4 in [FP3] (which is equivalent to the Ell_∞ property introduced by Gromov [G, sec. 3.1.]). This includes uniform approximation of holomorphic sections on compact holomorphically convex subsets of X and interpolation of holomorphic sections on any closed complex subvariety $X_0 \subset X$. The extension in [F2] to multi-valued

sections of ramified mappings $h: Z \rightarrow X$ onto a Stein space X also holds when h is a subelliptic submersion outside its ramification locus.

We now give examples of subelliptic manifolds (for proofs see Sect. 5).

1.2 Proposition. *(a) If Y is a complex Grassman manifold and $A \subset Y$ is a closed complex (=algebraic) subvariety of codimension at least two then $Y \setminus A$ is subelliptic. This holds in particular for $Y = \mathbb{C}P^n$.*

(b) Let $h: Z \rightarrow X$ be a holomorphic fiber bundle whose fiber is $\mathbb{C}P^n$ or a complex Grassmanian. If $A \subset Z$ is a closed complex subvariety whose fiber $A_x = A \cap Z_x$ has codimension at least two in Z_x for any $x \in X$ then the restricted submersion $h: Z \setminus A \rightarrow X$ is subelliptic.

The subvarieties in Proposition 1.2 have codimension at least two. The Oka principle fails in general for maps into complements of complex hypersurfaces or non-algebraic subvarieties of any dimension (see the examples in [FP3]).

Recall that a **projective algebraic manifold** is a closed complex submanifold of a complex projective space and a **quasi-projective manifold** is a Zariski open set in a projective manifold. We may speak of algebraic vector bundles and **algebraic sprays** on such manifolds, and algebraic subellipticity can be localized as follows (compare with Lemma 3.5.B. and 3.5.C. in [G]).

1.3 Proposition. *If Y is a quasi-projective algebraic manifold such that each point $y \in Y$ has a Zariski open neighborhood $U \subset Y$ and algebraic sprays $s_j: E_j \rightarrow Y$ ($j = 1, 2, \dots, k$), defined on algebraic vector bundles $p_j: E_j \rightarrow U$ and satisfying*

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) \cdots + (ds_k)_{0_y}(E_{k,y}) = T_y Y,$$

then Y is subelliptic.

No such localization result is known for ellipticity. Note that the condition in Proposition 1.3 is equivalent to (1.1) when h is the trivial submersion $Y \rightarrow \text{point}$. The ranges of the sprays s_j need not be contained in U .

1.4 Proposition. *If Y is a (quasi-) projective algebraic manifold with an algebraic spray $s: E \rightarrow Y$ which is a submersion of E onto Y then the complement $Y \setminus A$ of every algebraic subvariety of codimension at least two is subelliptic.*

1.5 Corollary. *If G is a complex algebraic Lie group whose exponential map is algebraic then the complement $G \setminus A$ of any algebraic subvariety $A \subset G$ of codimension at least two is subelliptic. This holds in particular if G is nilpotent and simply connected.*

Proof of Corollary 1.5. The exponential map $\exp: \mathfrak{g} \rightarrow G$ is locally biholomorphic and hence the spray $s: E = G \times \mathfrak{g} \rightarrow G$, $s(g, t) = \exp(t)g$, is a submersion of E onto G . If G and \exp are algebraic then s is algebraic and the result follows from Proposition 1.4. This is the case for simply connected nilpotent Lie groups. ♠

Question. Let Y and A be as in Proposition 1.4. Is $Y \setminus A$ always elliptic?

1.6 Proposition. *Let $\pi: \tilde{Y} \rightarrow Y$ be an unramified holomorphic covering map. If Y is subelliptic (resp. elliptic) then so is \tilde{Y} .*

A result of this kind is mentioned in [G, 3.5.B''] (see (**) on p. 883 of [G]). It is not clear whether the converse is true as well, i.e., *does (sub-) ellipticity of \tilde{Y} imply the same property for Y ?* A good test case may be complex tori $T = \mathbb{C}^n / \Gamma$ where $\Gamma \subset \mathbb{C}^n$ is a lattice of real rank $2n$. Denote by $\pi: \mathbb{C}^n \rightarrow T$ the quotient map (which is a universal covering of T). The spray $s: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$, $s(z, t) = z + t$, is Γ -equivariant and hence it passes down to a spray on T . Removing the point $p_0 = \pi(0) \in T$ we obtain a covering map $\pi: \mathbb{C}^n \setminus \Gamma \rightarrow T \setminus \{p_0\}$. It is easily seen that for $n \geq 2$ the lattice Γ is a tame discrete subset of \mathbb{C}^n in the sense of Rosay and Rudin [RR] and hence $\mathbb{C}^n \setminus \Gamma$ admits a dominating spray according to Lemma 7.1 in [FP2]. However, these sprays don't pass to sprays on $T \setminus \{p_0\}$ and it is not clear whether the latter manifold is subelliptic.

Outline of the paper. Section 2 contains some basic results and constructions with sprays. We show that the domination of a family of sprays is equivalent to the domination of the associated *composed spray*. In Sect. 3 we prove a homotopy version of the Oka-Weil approximation theorem for sections of submersions onto a Stein base which admit a finite dominating collection of sprays (Theorem 3.1). In Sect. 4 we prove Theorem 1.1. In Sect. 5 we prove Propositions 1.2, 1.3, 1.5 and 1.6. In Sect. 6 we use the methods developed in Sect. 4 (and in [FP2, F1]) to prove a result on removing intersections of holomorphic maps $X \rightarrow Y$ from Stein manifolds X with any closed complex subvariety $A \subset Y$ whose complement $Y \setminus A$ is subelliptic. Theorem 6.1 contains as special cases the well known theorem of Forster and Ramspott on complete intersections [FR] as well as Theorem 1.3 from [F1]. The appendix contains some remarks on Gromov's paper [G].

2 Subellipticity and composed sprays

In this section we first collect some basic results on subelliptic manifolds and submersions. The constructions in Lemmas 2.1 and 2.3 are due to Gromov [G]. Recall that a finite collection of sprays $s_j: E_j \rightarrow Y$ ($j = 1, 2, \dots, k$) is *dominating* if the subspaces $(ds_j)_{0_y}(E_{j,y}) \subset T_y Y$ together span $T_y Y$ for each $y \in Y$ (Definition 2).

2.1 Lemma. *If $s_j: E_j \rightarrow Y$ ($j = 1, 2, \dots, k$) is a dominating collection of sprays on Y , defined on trivial bundles $E_j \simeq Y \times \mathbb{C}^{m_j}$, then Y admits a dominating spray. The analogous result holds for h -sprays.*

Proof. We may assume that s_j is defined on $Y \times \mathbb{C}^{m_j}$ for each $j = 1, 2, \dots, k$. We define sprays $s^{(j)}: Y \times \mathbb{C}^{m_1+\dots+m_j} \rightarrow Y$ inductively by $s^{(1)} = s_1$ and

$$s^{(j)}(y, e_1, \dots, e_j) = s_j(s^{(j-1)}(y, e_1, \dots, e_{j-1}), e_j), \quad 2 \leq j \leq k.$$

Clearly we have

$$(ds^{(k)})_{0_y}(E_y^{(k)}) = (ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) \cdots + (ds_k)_{0_y}(E_{k,y}).$$

Hence (s_1, \dots, s_k) is a dominating collection of sprays on Y if and only if $s^{(k)}$ is a dominating spray. We call $s^{(k)}$ the *direct sum* of the sprays s_j and write $s^{(k)} = s_1 \oplus \cdots \oplus s_k$. (Observe that this construction is possible only for sprays defined on trivial bundles.) ♠

2.2 Lemma. *Any subelliptic Stein manifold is elliptic. If Z, X are Stein manifolds then any subelliptic submersion $h: Z \rightarrow X$ is elliptic.*

Proof. Assume that Y is Stein and $s_j: E_j \rightarrow Y$ ($j = 1, \dots, k$) is a dominating collection of sprays defined on vector bundles $p_j: E_j \rightarrow Y$. By Cartan’s Theorem A [GR] any holomorphic vector bundle over Y is generated by finitely many global holomorphic sections. This gives for each j a surjective vector bundle map $g_j: Y \times \mathbb{C}^{m_j} \rightarrow E_j$ for some large $m_j \in \mathbb{N}$. Then $\tilde{s}_j = s_j \circ g_j: Y \times \mathbb{C}^{m_j} \rightarrow Y$ is a spray whose vertical derivative at the zero section has the same range as that of s_j . It follows that $\tilde{s}_1 \oplus \cdots \oplus \tilde{s}_k$ (defined in Lemma 2.1) is a dominating spray on Y . A similar proof holds for submersions. ♠

For general sprays one can use the following construction.

Definition 3. (Gromov [G, sec. 1.3]) *Let (E_1, p_1, s_1) and (E_2, p_2, s_2) be h -sprays associated to a holomorphic submersion $h: Z \rightarrow X$. The **composed spray** $(E_1 \# E_2, p_1 \# p_2, s_1 \# s_2) = (E^*, p^*, s^*)$ is defined by*

$$E^* = \{(e_1, e_2) \in E_1 \times E_2: s_1(e_1) = p_2(e_2)\},$$

$$p^*(e_1, e_2) = p_1(e_1), \quad s^*(e_1, e_2) = s_2(e_2).$$

This operation extends to any finite collection of sprays and it includes iterations of sprays (and of composed sprays). The k -th iteration

$(E^{(k)}, p^{(k)}, s^{(k)})$ of (E, p, s) is

$$\begin{aligned} E^{(k)} &= \{e = (e_1, e_2, \dots, e_k) : e_j \in E \text{ for } j = 1, 2, \dots, k, \\ &\quad s(e_j) = p(e_{j+1}) \text{ for } j = 1, 2, \dots, k - 1\}, \\ p^{(k)}(e) &= p(e_1), \quad s^{(k)}(e) = s(e_k). \end{aligned}$$

Composed sprays are not sprays in the sense of Definition 1 since $E_1 \# E_2$ does not have a natural structure of a holomorphic vector bundle over Z . Observe that $E_1 \# E_2$ is the pull-back $s_1^*(E_2)$ of the vector bundle $p_2: E_2 \rightarrow Z$ by the first spray map $s_1: E_1 \rightarrow Z$ (and hence is a holomorphic vector bundle over the total space E_1 of the bundle $p_1: E_1 \rightarrow Z$). However, the composed bundle has a well defined zero section and a partial linear structure on fibers.

2.3 Lemma. *If (E_j, p_j, s_j) for $j = 1, \dots, k$ are h -sprays on Z then the restriction of the composed bundle $E_1 \# \dots \# E_k \rightarrow Z$ to any Stein subset $\Omega \subset Z$ is isomorphic to the direct sum bundle $E_1 \oplus E_2 \oplus \dots \oplus E_k|_\Omega$. Explicitly, there exists a biholomorphic map $\theta: \oplus E_j|_\Omega \rightarrow \# E_j|_\Omega$ which maps fibers to fibers and preserves the zero section. (The set Ω may be either an open Stein subset or a Stein subvariety of Z .)*

Proof. It suffices to prove this for $k = 2$ and apply induction. By construction the composed bundle $E_1 \# E_2$ is the pull-back $s_1^*(E_2)$ of the bundle $p_2: E_2 \rightarrow Z$ by the first spray map $s_1: E_1 \rightarrow Z$. Let $E'_1 = E_1|_\Omega$. Consider the homotopy of holomorphic maps $g_t: E'_1 \rightarrow Z$, $g_t(z, e) = s_1(z, te)$ ($t \in \mathbb{C}$). Since Ω is Stein, the total space E'_1 is also a Stein manifold and hence the pull-back bundles $\pi_t: g_t^*(E_2) \rightarrow E'_1$ for $t \in \mathbb{C}$ are holomorphically isomorphic to each other according to [Gr2]. Since $g_0 = p_1: E'_1 \rightarrow \Omega$ and $g_1 = s_1: E'_1 \rightarrow Z$, we obtain an isomorphism between $E_1 \# E_2|_\Omega \rightarrow E'_1$ and $\pi_1: p_1^*(E_2|_\Omega) \rightarrow E'_1$. The latter bundle carries the structure of a holomorphic vector bundle over Ω isomorphic to $E_1 \oplus E_2|_\Omega$ (with respect to the projection $p_1 \circ \pi_1: p_1^*(E_2|_\Omega) \rightarrow \Omega$). This endows $E_1 \# E_2|_\Omega$ with the structure of the bundle $E_1 \oplus E_2|_\Omega$. ♠

The notion of being ‘dominating’ (Definition 1) carries over in an obvious way to composed sprays by requiring the submersivity of the spray map in the fiber direction along the zero section of the composed bundle. The next lemma explains the relevance of subellipticity.

2.4 Lemma. *Let $h: Z \rightarrow X$ be a holomorphic submersion. A collection of h -sprays (E_j, p_j, s_j) on Z ($j = 1, 2, \dots, k$) is dominating at $z \in Z$ if and only if the composed h -spray $s_1 \# \dots \# s_k: E_1 \# \dots \# E_k \rightarrow Z$ is dominating at z .*

Proof. Observe that $s_1 \# \dots \# s_k(z, 0, \dots, e_j, \dots, 0) = s_j(z, e_j)$ (the vector $e_j \in E_{j,z}$ appears at the j -th spot). Hence the range (in $VT_z Z$) of the

vertical derivative of $s_1 \# \cdots \# s_k$ at the point $0_z \in E_1 \# \cdots \# E_k$ equals the vector sum of the vertical derivatives of the individual sprays s_j at 0_z . The lemma now follows from (1.1) in Definition 2. ♠

2.5 Lemma. *The Cartesian product of any finite family of (sub-) elliptic manifolds is (sub-) elliptic.*

Proof. It suffices to prove the result for the product of two manifolds. Let $Y = Y_1 \times Y_2$ and let $\pi_j: Y \rightarrow Y_j$ ($j = 1, 2$) denote the projection $\pi_j(y_1, y_2) = y_j$. If (E_j, p_j, s_j) is a spray on Y_j for $j = 1, 2$, we get a spray $s = s_1 \oplus s_2$ on the bundle $E = \pi_1^* E_1 \oplus \pi_2^* E_2 \rightarrow Y$ given by

$$s(y_1, y_2, e_1, e_2) = (s_1(y_1, e_1), s_2(y_2, e_2)).$$

If s_1 is dominating on Y_1 and s_2 is dominating on Y_2 then s is dominating on Y . Similarly, if a family of sprays $\{s_i: i = 1, \dots, i_0\}$ is dominating on Y_1 and a family of sprays $\{\sigma_k: k = 1, \dots, k_0\}$ is dominating on Y_2 then the collections $s_i \oplus \sigma_k$ is dominating on Y . ♠

3 The Oka-Weil theorem for subelliptic submersions

3.1 Theorem. *Let $h: Z \rightarrow X$ be a holomorphic submersion of a complex manifold Z onto a Stein manifold X . Assume that there exist h -sprays $s_j: E_j \rightarrow Z$ ($j = 1, 2, \dots, k$) which together dominate at each point $z \in Z$ (i.e., condition (1.1) holds). Let d be a metric on Z inducing the manifold topology. Suppose that K is a compact holomorphically convex set in X , $U \supset K$ is an open set and $f_t: U \rightarrow Z$ ($t \in [0, 1]$) is a homotopy of holomorphic sections such that f_0 extends to a holomorphic section on X . Then for any $\epsilon > 0$ there exists a homotopy of holomorphic sections $\tilde{f}_t: X \rightarrow Z$ such that $\tilde{f}_0 = f_0$ and $d(\tilde{f}_t(x), f_t(x)) < \epsilon$ for $x \in K$ and $t \in [0, 1]$. The analogous result holds for parametrized families of sections.*

Proof. For submersions which admit a global dominating h -spray this is Theorem 4.1 in [FP1], and the general parametric case is Theorem 4.2 in [FP1]. (See also Theorems 2.1 and 2.2 in [FP3] for the Oka-Weil theorem with interpolation on a complex subvariety in X .) To prove Theorem 3.1 we replace the dominating h -spray $s: E \rightarrow Z$ in the proof of Theorem 4.1 in [FP1] by the dominating composed h -spray

$$s = s_1 \# s_2 \# \cdots \# s_k: E = E_1 \# E_2 \cdots \# E_k \rightarrow Z$$

(Definition 3). The main point in the proof of Theorem 4.1 in [FP1, p. 135] is that for some sufficiently large $k \in \mathbb{N}$ (depending only on $\{f_t\}$) there

exists a homotopy of holomorphic sections ξ_t ($t \in [0, 1]$) of the iterated spray bundle $E^{(k)}|_{f_0(V)}$, restricted to $f_0(V) \subset Z$ for a sufficiently small open set $V \supset K$, such that

$$s^{(k)}(\xi_t(f_0(x))) = f_t(x), \quad (x \in V, t \in [0, 1]). \tag{3.1}$$

The same is true in the present situation which can be seen as follows. Since $s: E|_{f_0(V)} \rightarrow Z$ is a submersion on the zero section, there are a number $t_1 > 0$ (depending only on $\{f_t\}$) and a homotopy ξ_t of holomorphic sections of the latter bundle satisfying (3.1) for $k = 1$ and $0 \leq t \leq t_1$. Repeating the argument (and shrinking $V \supset K$ if necessary) we obtain a number $t_2 > t_1$ (depending only on $\{f_t\}$) and a family of sections $\{\xi_t: t_1 \leq t \leq t_2\}$ of $E|_{f_{t_1}(V)}$ such that ξ_{t_1} is the zero section and

$$s(\xi_t(f_{t_1}(x))) = f_t(x), \quad (x \in V, t \in [t_1, t_2]).$$

The two homotopies ξ_t together for $0 \leq t \leq t_2$ define a homotopy of sections of the second iteration $E^{(2)}|_{f_0(V)}$ such that (3.1) holds for $k = 2$ and $t \in [0, t_2]$. In finitely many steps (whose number depends only on $\{f_t\}$) we obtain a family ξ_t satisfying (3.1).

By Lemma 2.3 $E^{(k)} \rightarrow Z$ admits the structure of a holomorphic vector bundle over any Stein subset of Z . Hence the usual Oka-Weil theorem holds for sections of $E^{(k)}|_{f_0(X)} \rightarrow f_0(X)$. This gives a homotopy of holomorphic sections $\tilde{\xi}_t$ of $E^{(k)}|_{f_0(X)}$ for $t \in [0, 1]$ such that $\tilde{\xi}_t$ approximates ξ_t uniformly on $f_0(K)$ for each $t \in [0, 1]$ and $\tilde{\xi}_0$ is the zero section. The homotopy

$$\tilde{f}_t(x) = s^{(k)}(\tilde{\xi}_t(f_0(x))) \quad (x \in X, t \in [0, 1])$$

of sections of $h: Z \rightarrow X$ then satisfies Theorem 3.1. Similarly one proves the parametric version of Theorem 3.1 (see Theorem 4.2 in [FP1]).

4 Proof of Theorem 1.1

We shall follow the proof of Theorem 1.5 in [FP2] (or Theorem 1.4 in [FP3]) and explain the necessary modifications. The basic problem is to deform a continuous section of $h: Z \rightarrow X$ to a holomorphic section. The construction proceeds through a sequence of modifications in which we obtain holomorphic sections over a family of open subsets which increase to X . The proof has two essential ingredients:

- (1) Solution of the *modification problem* explained below, and
- (2) An inductive construction of a sequence of *holomorphic complexes* (in the terminology of [FP2, FP3]) which converges uniformly on compacts in X to a global holomorphic section.

Part (2) (globalization) uses the solution of part 1 and is explained in Sect. 5 of [FP2] or (with slightly less details) in [FP3]. This part does not require any changes whatsoever. The rest of this section is devoted to part 1.

An ordered pair (A, B) of compact subsets of X is said to be a **Cartan pair** in X (Definition 4.1 in [FP2]) if

- (i) $A, B,$ and $A \cup B$ have a basis of Stein neighborhoods,
- (ii) $A \setminus B \cap B \setminus A = \emptyset,$ and
- (iii) the set $C = A \cap B$ is Runge in $B.$ (C may be empty.)

The modification problem. Let (A, B) be a Cartan pair in X and let $a: \tilde{A} \rightarrow Z, b: \tilde{B} \rightarrow Z$ be holomorphic sections of $h: Z \rightarrow X$ in open neighborhoods $\tilde{A} \supset A$ resp. $\tilde{B} \supset B.$ Suppose furthermore that $b_t (t \in [0, 1])$ is a family of holomorphic sections over $\tilde{C} = \tilde{A} \cap \tilde{B}$ such that $b_0 = b|_{\tilde{C}}$ and $b_1 = a|_{\tilde{C}}.$ Assume that $h: Z|_{\tilde{B}} \rightarrow \tilde{B}$ admits a finite dominating family of h -sprays. The goal is to construct a holomorphic section \tilde{a} over a neighborhood of $A \cup B$ which is uniformly close to a on A and is obtained from the initial sections a, b by homotopies over small neighborhoods of A resp. $B.$ (The analogous modification problem must be considered for parametrized families of sections as in [FP1]. However, it will suffice to explain the proof of the basic non-parametric case since the general case follows the same pattern as in [FP1].)

The modification problem will be solved as in [FP1] by performing the following steps.

Step 1 (approximation). Find a family of holomorphic sections $\tilde{b}_t (t \in [0, 1])$ over a neighborhood of B such that $\tilde{b}_0 = b$ and \tilde{b}_t approximates b_t over an open neighborhood of C for each $t \in [0, 1].$

Step 2 (gluing). Replace b by \tilde{b}_1 from Step 1 and ‘glue’ the pair of sections a, b (which are uniformly close over a neighborhood of C) into $\tilde{a}.$

Step 1 has been accomplished by Theorem 3.1 in the present paper (which replaces Theorem 4.1 in [FP1]). Step 2 is accomplished by the following result which replaces Theorems 5.1 and 5.5 in [FP1].

4.1 Theorem. *Let $h: Z \rightarrow X$ be a holomorphic submersion onto a Stein manifold X and let d be a metric on Z compatible with the manifold topology. Let (A, B) be a Cartan pair in $X.$ Assume that $\tilde{B} \supset B$ is an open set in X such that $h: Z|_{\tilde{B}} \rightarrow \tilde{B}$ admits a finite dominating family of h -sprays (Definition 2). Let $a: \tilde{A} \rightarrow Z$ be a holomorphic section in an open set $\tilde{A} \supset A.$ Then for each $\epsilon > 0$ there is a $\delta > 0$ satisfying the following. If $b: \tilde{B} \rightarrow Z$ is a holomorphic section satisfying $d(a(x), b(x)) < \delta$ for $x \in \tilde{A} \cap \tilde{B},$ there exists a homotopy a_t (resp. b_t) of holomorphic sections over a neighborhood A' of A (resp. over a neighborhood B' of B) such that*

$a_0 = a|_{A'}$, $b_0 = b|_{B'}$, $a_1|_{C'} = b_1|_{C'}$ for $C' = A' \cap B'$, and

$$\begin{aligned} d(a_t(x), a(x)) &< \epsilon \quad (x \in A', t \in [0, 1]); \\ d(b_t(x), b(x)) &< \epsilon \quad (x \in C', t \in [0, 1]). \end{aligned}$$

The analogous result holds for parametrized families of sections, i.e., the conclusion of Theorem 5.5 in [FP1] holds in the present context.

Proof of Theorem 4.1. We shall only prove the basic non-parametric case (for the parametric case we refer to Theorem 5.5 in [FP1]). Under the stronger assumption that h admits a dominating spray over \tilde{B} this is Theorem 5.1 in [FP1] which was reduced to Proposition 5.2 in [FP1] (the model case) by using Lemma 5.4 in [FP1]. Unfortunately the proof of this lemma does not hold under the current weaker assumption. The following result is a suitable replacement. We denote by $\mathbb{B}^n(\epsilon)$ the open ball of radius ϵ in \mathbb{C}^n with center at the origin.

4.2 Lemma. *Let U be an open Stein subset of Z and $s_j: U \times \mathbb{B}^n(\epsilon) \rightarrow Z$ ($j = 1, 2$) holomorphic submersions such that $s_j(z, 0) = z$ and $h(s_j(z, t)) = h(z)$ for $z \in U$, $t \in \mathbb{B}^n(\epsilon)$. Let $M_j = \{(z, t) \in U \times \mathbb{C}^n: \partial_t s_j(z, 0) \cdot t = 0\}$. If the bundles $M_1 \rightarrow U$ and $M_2 \rightarrow U$ are isomorphic then for any relatively compact subset $V \subset\subset U$ there exist numbers $\delta > 0, \eta > 0$, with $0 < \eta < \epsilon$, satisfying the following. For any pair of points $z, w \in V$ with $h(z) = h(w)$ and $d(z, w) < \delta$ there exists an injective map $\phi(z, w, \cdot): \mathbb{B}^n(\eta) \rightarrow \mathbb{B}^n(\epsilon)$ which is holomorphic in all variables and satisfies*

$$s_2(w, \phi(z, w, t)) = s_1(z, t), \quad \phi(z, z, 0) = 0.$$

If s_2 is uniformly close to s_1 then we may choose ϕ such that $\phi(z, z, t) \approx t$.

Proof. Since U is Stein, there is a holomorphic splitting $U \times \mathbb{C}^n = M_j \oplus N_j$ for some holomorphic vector subbundle $N_j \subset U \times \mathbb{C}^n$. The differential of s_j at the zero section carries N_j isomorphically onto $VT(Z)|_U$ and hence $N_1 \simeq N_2 \simeq VT(Z)|_U$. Since $M_1 \simeq M_2$ by assumption, there exists an automorphism θ of the trivial bundle $U \times \mathbb{C}^n$ with $\theta(M_1) = M_2$ and $\theta(N_1) = N_2$. Set $\tilde{s}_2 = s_2 \circ \theta: U \times \mathbb{C}^n \rightarrow Z$. The kernel of $d\tilde{s}_2$ at the zero section equals M_1 . If we can find a map ϕ satisfying the conclusion of the lemma for s_1, \tilde{s}_2 then the map ϕ defined by $\theta(w, \tilde{\phi}(z, w, t)) = (w, \phi(z, w, t))$ satisfies it for s_1, s_2 .

Hence we may assume that $M_1 = M_2 = M$ and $N_1 = N_2 = N$. We split the fiber vectors $t = (t', t'') \in M_z \oplus N_z = \mathbb{C}^n$ accordingly (the splitting depends on the base point $z \in U$). The inverse function theorem shows that for each $z \in U$ the restriction $s_1(z, 0', \cdot): N_z \rightarrow Z_{h(z)}$ maps a

neighborhood of $0''$ in N_z biholomorphically onto a neighborhood of z in the fiber $Z_{h(z)}$. The same is true for the map

$$t'' \in N_z \rightarrow s_1(z, t', t'') \in Z_{h(z)} \tag{4.1}$$

for all sufficiently small $t' \in M_z$. If $w \in Z_{h(z)} \cap U$ is chosen sufficiently close to z and if $t' \in M_z$ is sufficiently small then by the same argument the map

$$t'' \in N_z \rightarrow s_2(w, t', \cdot) \in Z_{h(z)} \tag{4.2}$$

maps a neighborhood of $0''$ in N_z biholomorphically onto a neighborhood of w in the fiber $Z_{h(w)} = Z_{h(z)}$. If w is chosen sufficiently close to z , the image of the latter neighborhood also contains the point z and we let $\phi'(z, w, t', \cdot) : N_z \rightarrow N_z$ be the composition of (4.1) with the (unique) local inverse of (4.2). The map

$$(t', t'') \in M_z \oplus N_z \rightarrow \phi(z, w, t', t'') = (t', \phi'(z, w, t', t'')),$$

which is defined for all sufficiently small $t = (t', t'') \in M_z \oplus N_z$, satisfies Lemma 4.2. ♠

We continue with the proof of Theorem 1.1. Write $C = A \cap B$. By Lemma 5.3 in [FP1] there exists a Stein open set $V \subset Z$ containing $a(A)$ and a holomorphic submersion (a local spray) $s : V \times \mathbb{B}^n(\eta) \rightarrow Z$ for some $\eta > 0$ and $n \in \mathbb{N}$ such that $s(z, 0) = z$ and $h(s(z, t)) = h(z)$. (It is important that s is a submersions over a neighborhood of $a(C)$.)

By assumption there exists a dominating family of h -sprays (E_j, p_j, σ_j) ($j = 1, \dots, k$) over $Z|_{\tilde{B}} = h^{-1}(\tilde{B})$. Set $E = E_1 \oplus \dots \oplus E_k$ and $\tilde{E} = E_1 \# \dots \# E_k$ (Sect. 2). Let $\tilde{\sigma} : \tilde{E} \rightarrow Z|_{\tilde{B}}$ denote the composed spray which is dominating by Lemma 2.4. Choose a Stein open set $U \subset Z$ with $a(C) \subset U \subset V \cap (Z|_{\tilde{B}})$. Lemma 2.3 gives a fiber preserving biholomorphic map $\theta : E|_U \rightarrow \tilde{E}|_U$. The map $\tilde{\sigma}\theta : E|_U \rightarrow Z$ is then a dominating h -spray defined on the vector bundle $E|_U$, with values in Z .

Set $v_j(z) = (ds)_{0_z}(\partial/\partial t_j)$ for $z \in V$ and $j = 1, \dots, n$, where s is a local spray on $V \supset a(A)$ as above and $t = (t_1, \dots, t_n) \in \mathbb{C}^n$. There exist holomorphic sections \tilde{v}_j of the bundle $E|_U \rightarrow U$ such that $d(\tilde{\sigma}\theta)_0(\tilde{v}_j) = v_j$ for $j = 1, \dots, n$. (Such sections are unique in any holomorphic subbundle $N \subset E|_U$ which is complementary to the kernel of the differential of $\tilde{\sigma}\theta$ at the zero section.) Let $\tau : U \times \mathbb{C}^n \rightarrow E|_U$ be the map $\tau(z, t_1, \dots, t_n) = \sum_{j=1}^n t_j \tilde{v}_j(z) \in E_z$. The composition $\sigma = \tilde{\sigma}\theta\tau : U \times \mathbb{C}^n \rightarrow Z$ is then a dominating h -spray on U (with values in Z) satisfying $(d\sigma)_{0_z}(\partial/\partial t_j) = v_j(z) = (ds)_{0_z}(\partial/\partial t_j)$ for $z \in U$ and $j = 1, \dots, n$. By Lemma 4.2 there are $\delta, \eta > 0$ and a holomorphic map ϕ satisfying

$$\sigma(w, \phi(z, w, t)) = s(z, t), \quad \phi(z, z, 0) = 0$$

for $t \in \mathbb{B}^n(\eta)$ and $z, w \in U$ with $h(z) = h(w)$ and $d(z, w) < \delta$.

Let $b: \tilde{B} \rightarrow Z$ be a holomorphic section satisfying $d(a(x), b(x)) < \delta$ for $x \in \tilde{A} \cap \tilde{B}$. By decreasing δ we may assume $b(C) \subset U$. The problem is that the spray σ is only defined over U and need not extend holomorphically into any neighborhood of $b(B)$. To complete the proof we shall approximate σ uniformly in a neighborhood of $b(C)$ by a spray σ' which is holomorphic in neighborhood of $b(B)$. We proceed as follows.

Since C is Runge in B and both sets have a basis of Stein neighborhoods in X , there exist open Stein sets $U' \subset W$ in Z such that U' is Runge in W , $b(B) \subset W$, and $b(C) \subset U' \subset U \cap W$. Fix a compact set $K \subset E|_{U'}$ and let $K' \subset U'$ denote its base projection. By Lemma 2.3 and the Oka-Weil theorem there is a fiber preserving holomorphic map $\theta': E|_W \rightarrow \tilde{E}|_W$ which is uniformly close to $\theta: E|_U \rightarrow \tilde{E}|_U$ on K . Also there is for each $j = 1, \dots, n$ a holomorphic section v'_j of $E|_W$ which approximates \tilde{v}_j on $K' \subset U'$. Define $\tau': W \times \mathbb{C}^n \rightarrow E|_W$ by $\tau'(w, t) = \sum_{j=1}^n t_j v'_j(w)$. Then the spray $\sigma' = \tilde{\sigma} \theta' \tau': W \times \mathbb{C}^n \rightarrow Z$ is uniformly close to $\sigma = \tilde{\sigma} \theta \tau$ on the set $L = (\tau')^{-1}(K) \subset U' \times \mathbb{C}^n$. For each $U_1 \subset\subset U'$ we may choose K sufficiently large to insure that $U_1 \times \mathbb{B}^n(3) \subset L$.

Applying Lemma 4.2 to the pair of sprays σ, σ' we obtain (after shrinking $U' \supset b(C)$) a holomorphic map $\xi: U' \times \mathbb{B}^n(2) \rightarrow \mathbb{C}^n$ which is uniformly close to $\xi_0(w, t) = t$ on $U_1 \times \mathbb{B}^n(1)$ and which satisfies $\sigma'(w, \xi(w, t)) = \sigma(w, t)$ and $\xi(w, 0) = 0$ ($w \in U', t \in \mathbb{B}^n(2)$). Then

$$\sigma'(w, \xi(w, \phi(z, w, t))) = \sigma(w, \phi(z, w, t)) = s(z, t)$$

for any $t \in \mathbb{B}^n(\eta)$ and any pair of point $z \in U, w \in U'$ with $h(z) = h(w)$. Choose open sets $A', B' \subset X$ satisfying $A \subset A' \subset\subset \tilde{A}, B \subset B' \subset\subset \tilde{B}, C' = A' \cap B', \overline{a(A')} \subset V, \overline{b(B')} \subset W, \overline{b(C')} \subset U'$. Set

$$\begin{aligned} s_1(x, t) &= s(a(x), t), & (x \in A', t \in \mathbb{B}^n(\eta)); \\ s_2(x, t) &= \sigma'(b(x), t), & (x \in B', t \in \mathbb{C}^n); \\ \psi(x, t) &= \xi(b(x), \phi(a(x), b(x), t)), & (x \in C', t \in \mathbb{B}^n(\eta)). \end{aligned}$$

Then $s_2(x, \psi(x, t)) = s_1(x, t)$ for $x \in C'$ and $t \in \mathbb{B}^n(\eta)$. If b is uniformly close to a on \tilde{C} then (since ξ is close to $\xi_0(w, t) = t$) the map ψ is uniformly close to $\psi_0(x, t) = \phi(a(x), a(x), t)$ ($x \in C', t \in \mathbb{B}^n(\eta)$). Note that $\psi_0(x, 0) = 0$ for all $x \in C'$.

We have thus reduced Theorem 4.1 to Proposition 5.2 in [FP1]. To complete the proof we shrink the sets $A' \supset A, B' \supset B$ and take $\alpha: A' \rightarrow \mathbb{B}^n(\eta), \beta: B' \rightarrow \mathbb{C}^n$ to be holomorphic maps furnished by that proposition, satisfying $\psi(x, \alpha(x)) = \beta(x)$ for $x \in C' = A' \cap B'$. Then

$$s_2(x, \beta(x)) = s_2(x, \psi(x, \alpha(x))) = s_1(x, \alpha(x)), \quad (x \in C')$$

and hence these expressions define a holomorphic section $\tilde{a}: A' \cup B' \rightarrow Z$. Further details can be found in [FP1]. ♠

A remark on [FP3]. The condition $M_1 \simeq M_2$ (i.e., the kernels of ds_1 and ds_2 along the zero section are isomorphic) is necessary for the existence of a map ψ as above. Although this was not stated explicitly in the proof of Theorem 5.1 in [FP3], the construction in [FP1] (see especially Lemma 5.4 in [FP1]) produces a pair of submersions s_1, s_2 for which this condition is satisfied, and one can apply the proof in [FP3] to such a pair. ♠

5 Subelliptic manifolds

In this section we prove Propositions 1.2, 1.3, 1.5 and 1.6.

Proof of Proposition 1.2. In part (a) we shall give a detailed calculation only for $Y = \mathbb{C}\mathbb{P}^n$ and will observe that the same proof applies to complex Grassmanians. Similar arguments apply to part (b) and we omit the details (compare with the proof of Corollary 1.8 in [FP2]).

Given an algebraic subvariety $A \subset \mathbb{C}\mathbb{P}^n$ containing no complex hypersurfaces we wish to construct a finite dominating family of algebraic sprays on $\mathbb{C}\mathbb{P}^n \setminus A$. We begin by choosing a hyperplane $\Lambda \subset \mathbb{C}\mathbb{P}^n$ and homogeneous coordinates $Z = [Z_0: Z_1: \dots : Z_n]$ on $\mathbb{C}\mathbb{P}^n$ such that $\Lambda = \{Z_0 = 0\}$. Set $U_j = \{Z \in \mathbb{C}\mathbb{P}^n: Z_j \neq 0\} \simeq \mathbb{C}^n$ for $j = 0, 1, \dots, n$ (hence $\mathbb{C}\mathbb{P}^n = U_0 \cup \Lambda$). Let $L \rightarrow \mathbb{C}\mathbb{P}^n$ denote the holomorphic line bundle $L = [\Lambda]^{-1}$ where $[\Lambda]$ is the line bundle determined by the divisor of Λ . (The usual notation is $L = \mathcal{O}_{\mathbb{C}\mathbb{P}^n}(-1)$, see [GH].) L admits holomorphic trivializations $\phi_j: L|_{U_j} \rightarrow U_j \times \mathbb{C}$ with transition maps

$$\phi_{ik}(Z, t) = \phi_i \circ \phi_k^{-1}(Z, t) = (Z, tZ_i/Z_k), \quad (Z \in U_{ik} = U_i \cap U_k, t \in \mathbb{C}).$$

Choose a vector $v \in \mathbb{C}^n$ such that the orthogonal projection $\pi: U_0 = \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ with kernel $\mathbb{C}v$ is proper when restricted to $A \cap U_0$. (This is the case precisely when the complex line $\mathbb{C}v$ does not intersect A at any point of Λ . Since A has codimension at least two, it does not contain Λ and hence this holds for almost every v .) Then $A' = \pi(A \cap U_0) \subset \mathbb{C}^{n-1}$ is a proper algebraic subvariety of \mathbb{C}^{n-1} . Let p be any nonzero holomorphic polynomial on \mathbb{C}^{n-1} which vanishes on A' . Then the map $U_0 \times \mathbb{C} \rightarrow U_0$ given by

$$s(z, t) = z + tp(\pi z)v = z + tf(z) \quad (z \in U_0, t \in \mathbb{C})$$

is a spray on U_0 with $\frac{\partial}{\partial t}s(z, 0) = p(\pi z)v = f(z)$. Although s does not extend to a spray $\mathbb{C}\mathbb{P}^n \times \mathbb{C} \rightarrow \mathbb{C}\mathbb{P}^n$ because of singularities in $\Lambda = \{Z_0 = 0\}$, it induces a spray $\tilde{s}: L^{\otimes m} \rightarrow \mathbb{C}\mathbb{P}^n$ where m is the degree of the polynomial p

and $E = L^{\otimes m}$ denotes the m -th tensor power of L . E admits trivializations $\theta_i: E|_{U_i} \rightarrow U_i \times \mathbb{C}$ ($i = 0, 1, \dots, n$) with transition maps

$$\theta_{ik}(Z, t) = (Z, t(Z_i/Z_k)^m), \quad (Z \in U_{ik}, t \in \mathbb{C}).$$

Set $\tilde{s} = s\theta_0: E|_{U_0} \rightarrow \mathbb{C}\mathbb{P}^n$. We claim that \tilde{s} extends to a holomorphic spray $E \rightarrow \mathbb{C}\mathbb{P}^n$. Indeed, writing $Z = [Z_0: Z']$ with $z = Z'/Z_0$, we see that s has the following expression in the homogeneous coordinates $Z \in U_0 \subset \mathbb{C}\mathbb{P}^n$:

$$\begin{aligned} s(Z, t) &= [1: s(Z'/Z_0, t)] = [1: Z'/Z_0 + t f(Z/Z_0)] \\ &= [Z_0: Z' + t Z_0 f(Z/Z_0)]. \end{aligned}$$

Hence we get for $k = 1, \dots, n$ and $Z \in U_0 \cap U_k$

$$\begin{aligned} \tilde{s}\theta_k^{-1}(Z, t) &= s\theta_{0k}(Z, t) = s(Z, t(Z_0/Z_k)^m) \\ &= [Z_0: Z' + t Z_0^{m+1} Z_k^{-m} f(Z/Z_0)]. \end{aligned}$$

By the choice of m the function $Z_0^{m+1} Z_k^{-m} f(Z/Z_0)$ vanishes on $\{Z_0 = 0\} \cap U_k$ and hence $\tilde{s}\theta_k^{-1}$ is holomorphic on U_k .

This shows that $\tilde{s}: E \rightarrow \mathbb{C}\mathbb{P}^n$ is a spray satisfying $\tilde{s}((U_0 \setminus A) \times \mathbb{C}) \subset U_0 \setminus A$ and $\tilde{s}(Z, t) = Z$ for all $Z \in A \cup \Lambda$ and $t \in \mathbb{C}$. For each $Z \in U_0 \setminus A$ we can find finitely many sprays of this kind (corresponding to n linearly independent directions in \mathbb{C}^m) which together dominate at Z and hence at every point in a Zariski open set containing Z . Repeating the construction at other points (and for different choices of the hyperplane Λ) we obtain a finite dominating family of algebraic sprays on $\mathbb{C}\mathbb{P}^n \setminus A$.

The same proof applies to complex Grassmanians $Y = G_{k,n}$ since these can be covered by finitely many Zariski open neighborhoods $U_j \simeq \mathbb{C}^{k(n-k)}$. ♠

Proof of Proposition 1.3. (Compare with Localization Lemma 3.5.B. in [G].) The proof is essentially the same as the proof of Proposition 1.2 (a). We shall repeatedly use the fact that for every closed algebraic subvariety $A \subset Y$ and every point $y \in Y \setminus A$ there exists an algebraic hypersurface $\Lambda \subset Y$ such that $A \subset \Lambda$ but $y \notin \Lambda$.

Fix a point $y_0 \in Y$ and let $U \subset Y$ be a Zariski open neighborhood of y_0 with finitely many algebraic sprays $s_j: E_j \rightarrow Y$ ($j = 1, \dots, k$) which together dominate at y_0 . Replacing U by a smaller Zariski open neighborhood of y_0 we may assume that $\Lambda = Y \setminus U$ is an algebraic hypersurface in Y and the bundle $E_j|_U \rightarrow U$ is algebraically trivial for each j . Composing an algebraic trivialization of $E_j|_U$ with the spray s_j we may therefore assume that s_j is defined on the product bundle $U \times \mathbb{C}^{N_j}$ and has values in Y . To remove the singularities of s_j along Λ we replace the product bundle by $N_j[\Lambda]^{-m_j}$ for a sufficiently large $m_j \in \mathbb{N}$. This gives finitely many sprays

on Y which together dominate at y_0 and hence over a Zariski open neighborhood of y_0 . Finitely many such collections then dominate on Y . ♠

Remark. The idea of using the bundles $E = NL^{\otimes m}$ where $L = [A]^{-1}$ can be found in Sects. 3.5.B. and 3.5.C. of [G]. However, our conclusion is different from the one in [G] where the author claimed the existence of a dominating spray on Y obtained by composing the individual non-dominating sprays (see the Appendix below). The bundles L and $E = NL^{\otimes m}$ used above are strictly negative and do not admit any nontrivial holomorphic sections. Hence there exist no nontrivial vector bundle maps from any trivial bundle to E (since such a map would take a certain constant section to a nontrivial holomorphic section of E). Thus we are unable to replace the collection of sprays obtained above by a single dominating spray using Lemma 2.2. ♠

Proposition 1.5 follows at once from the following more general result.

5.1 Lemma. *Let A be a closed algebraic subvariety of codimension at least two in a (quasi-) projective algebraic manifold Y . Suppose that each point $y_0 \in Y \setminus A$ has a Zariski open neighborhood $U \subset Y$ and an algebraic spray $s: E \rightarrow Y$, defined on a vector bundle $p: E \rightarrow U$, such that s is dominating at y_0 and $s^{-1}(A) \subset E$ contains no hypersurfaces. Then $Y \setminus A$ is subelliptic.*

Indeed, if $s: E \rightarrow Y$ is a submersive algebraic spray then the codimension of $s^{-1}(A)$ in E is the same as the codimension of A in Y , and hence Proposition 1.5 follows.

Proof of Lemma 5.1. After removing an algebraic hypersurface which does not contain y_0 we may assume that $E|_U = U \times \mathbb{C}^N$ and that each fiber $\tilde{A}_y = \{t \in \mathbb{C}^N: s(y, t) \in A\}$ of $\tilde{A} := s^{-1}(A)$ has codimension at least two in \mathbb{C}^N . Note that $0 \notin \tilde{A}_y$ for $y \in U \setminus A$.

Let $t = (t_1, \dots, t_N) \in \mathbb{C}^N$. For each $k = 0, 1, \dots, N$ we set $\mathbb{C}_{(k)} = \mathbb{C}^k \times \{0\}^{N-k}$. Let $\pi_k: U \times \mathbb{C}_{(k)} \rightarrow U \times \mathbb{C}_{(k-1)}$ denote the projection $\pi_k(y, t_1, \dots, t_k) = (y, t_1, \dots, t_{k-1})$. After a linear change of coordinates on \mathbb{C}^N and removing another algebraic hypersurface from U we may assume that

- (i) for each $k = 1, \dots, N$ the set $A_{(k)} = (U \times \mathbb{C}_{(k)}) \cap \tilde{A}$ is a subvariety of $U \times \mathbb{C}_{(k)}$ with fibers of codimension at least two (in particular $A_{(1)} = \emptyset$), and
- (ii) $\pi_k(A_{(k)}) \subset U \times \mathbb{C}_{(k-1)}$ is an algebraic subvariety of $U \times \mathbb{C}_{(k-1)}$ which does not contain the point $(y, 0, \dots, 0)$ for any $y \in U$. (Note that $\pi_k(A_{(k)})$ contains $A_{(k-1)}$ but it may be larger.)

Condition (ii) insures that for each $k = 2, \dots, N$ there exists an algebraic function p_k on $U \times \mathbb{C}_{(k-1)}$ which vanishes on $\pi_k(A_{(k)})$ and satisfies

$p_k(y_0, 0, \dots, 0) \neq 0$. Consider the map $g: U \times \mathbb{C}^N \rightarrow U \times \mathbb{C}^N$,

$$g(y, t) = (y, t_1, p_2(y, t_1)t_2, \dots, p_N(y, t_1, \dots, t_{N-1})t_N).$$

Clearly $g(y, 0) = (y, 0)$, the map $t \rightarrow g(y_0, t)$ is nondegenerate at $t = 0$, and the image of g avoids \tilde{A} . Thus $\sigma = s \circ g: U \times \mathbb{C}^N \rightarrow Y$ is an algebraic spray which is dominating at y_0 and satisfies $\sigma((U \setminus A) \times \mathbb{C}^N) \subset Y \setminus A$. Since $y_0 \in Y \setminus A$ was arbitrary, the subellipticity of $Y \setminus A$ follows from Proposition 1.3. ♠

Proof of Proposition 1.6. Let $s: E \rightarrow Y$ be a dominating spray on Y defined on a vector bundle $p: E \rightarrow Y$. Denote by $\tilde{E} = \pi^*(E) \rightarrow \tilde{Y}$ the pull-back of E by the map $\pi: \tilde{Y} \rightarrow Y$. Explicitly we have

$$\tilde{E} = \{(\tilde{y}, e): \tilde{y} \in \tilde{Y}, e \in E, \pi(\tilde{y}) = p(e)\}.$$

Let $\sigma: \tilde{E} \rightarrow Y$ be defined by $\sigma(\tilde{y}, e) = s(y, e)$ where $y = \pi(\tilde{y}) \in Y$. Fix $\tilde{y} \in \tilde{Y}$. Since the fiber $\tilde{E}_{\tilde{y}}$ is simply connected and π is a holomorphic covering, the map $\sigma(\tilde{y}, \cdot): \tilde{E}_{\tilde{y}} \rightarrow Y$ has a unique holomorphic lifting $\tilde{s}(\tilde{y}, \cdot): \tilde{E}_{\tilde{y}} \rightarrow \tilde{Y}$ (i.e., $\pi(\tilde{s}(\tilde{y}, e)) = \sigma(\tilde{y}, e)$) with $\tilde{s}(\tilde{y}, 0) = \tilde{y}$. Clearly $\tilde{s}: \tilde{E} \rightarrow \tilde{Y}$ is a dominating spray on \tilde{Y} and hence \tilde{Y} is elliptic. A similar argument works for families of sprays, thereby showing that subellipticity of Y implies that of \tilde{Y} .

6 Removing intersections with complex subvarieties

Let X and Y be complex manifolds and $A \subset Y$ a closed complex subvariety. Given a map $f: X \rightarrow Y$ we write $f^{-1}(A) = \{x \in X: f(x) \in A\}$ and call it the *intersection set of f with A* . If A is a hypersurface or, more generally, an effective divisor in Y , there is a well defined pull-back divisor $f^*(A) = \sum m_j V_j$ in X , where each V_j is an irreducible component of the hypersurface $f^{-1}(A)$ and $m_j \in \mathbb{N}$ is its multiplicity. The following questions have been studied by many authors:

To what extent can the preimage $f^{-1}(A)$ resp. $f^*(A)$ be prescribed? How large is the set of all holomorphic maps $f: X \rightarrow Y$ with the given preimage $f^{-1}(A)$ (resp. $f^*(A)$)?

In the simplest case when $X = \mathbb{C}$ and A consists of d points in the Riemann sphere $Y = \mathbb{C}P^1$ the answer changes when passing from $d = 2$ to $d = 3$: One can prescribe the pull-back of any two points in $\mathbb{C}P^1$ by a holomorphic map $f: \mathbb{C} \rightarrow \mathbb{C}P^1$ (and there are infinitely many such maps), but when $d \geq 3$ the pull-back divisor f^*A completely determines the map f . Similar situation occurs when A consists of d hyperplanes in general

position in $Y = \mathbb{C}P^n$: we have flexibility up to $d = n + 1$ (Corollary 6.2 (b)) and rigidity for $d \geq n + 2$ (due to hyperbolicity of $\mathbb{C}P^n \setminus A$).

The following result shows that subellipticity of the complement $Y \setminus A$ implies the validity of the Oka principle for maps $f: X \rightarrow Y$ with the given preimage $f^{-1}(A)$ (or pull-back f^*A when A is a divisor).

6.1 Theorem. *Let A be a closed complex subvariety of a complex manifold Y such that $Y \setminus A$ is subelliptic (Definition 2). If X is a Stein manifold, K is a compact holomorphically convex subset of X and $f: X \rightarrow Y$ is a continuous map which is holomorphic in an open set containing $f^{-1}(A) \cup K$ then for any $r \in \mathbb{N}$ there exist an open set $U \supset f^{-1}(A) \cup K$ and a homotopy $f_t: X \rightarrow Y$ ($t \in [0, 1]$) of continuous maps such that $f_0 = f$, f_t is holomorphic in U and tangent to f to order r along $f_t^{-1}(A) = f^{-1}(A)$ for each $t \in [0, 1]$, and f_1 is holomorphic on X .*

6.2 Corollary. *The conclusion of Theorem 6.1 holds in the following cases:*

- (a) Y is an affine space \mathbb{C}^n , a projective space $\mathbb{C}P^n$ or a complex Grassmanian and $A \subset Y$ is an algebraic subvariety of codimension at least two.
- (b) $Y = \mathbb{C}P^n$ and A consists of at most $n + 1$ complex hyperplanes in general position.
- (c) A complex Lie group acts transitively on $Y \setminus A$.

In any of these cases $Y \setminus A$ is subelliptic by the results in Sect. 1. (Note that (b) is a special case of (c).) Using Theorem 6.1 we shall also prove the following *Oka principle for removing of intersections*.

6.3 Theorem. *Assume that $f: X \rightarrow Y$ is a holomorphic map, A is a complex subvariety of Y and $f^{-1}(A) = X_0 \cup X_1$, where X_0 and X_1 are unions of connected components of $f^{-1}(A)$ and $X_0 \cap X_1 = \emptyset$. Assume that X is Stein and the manifolds Y and $Y \setminus A$ are subelliptic. If there exists a homotopy $\tilde{f}_t: X \rightarrow Y$ ($t \in [0, 1]$) of continuous maps satisfying $\tilde{f}_0 = f$, $\tilde{f}_1^{-1}(A) = X_0$, and $\tilde{f}_t|_U = f_t|_U$ for some open set $U \supset X_0$ and for all $t \in [0, 1]$, then for each $r \in \mathbb{N}$ there exists a homotopy of holomorphic maps $f_t: X \rightarrow Y$ such that $f = f_0$, $f_1^{-1}(A) = X_0$, and for each $t \in [0, 1]$ the map f_t agrees to order r with f along X_0 (which is a union of connected components of $f_t^{-1}(A)$).*

When $A = \{0\} \subset Y = \mathbb{C}^d$, Theorem 6.3 coincides with the main result of [FR] on *holomorphic complete intersections*. When $Y = \mathbb{C}^d$ and $Y \setminus A$ is elliptic this is Theorem 1.3 in [F1]. Theorem 6.3 applies if Y is any of the manifolds \mathbb{C}^n , $\mathbb{C}P^n$ or a complex Grassmanian (these are complex homogeneous and therefore elliptic) and $A \subset Y$ is an algebraic subvariety of codimension at least two ($Y \setminus A$ is then subelliptic by Proposition 1.2).

Example. For each $n \geq 1$ there exists a discrete set $A \subset \mathbb{C}^n$ for which Theorem 6.3 fails. To see this, we choose a discrete set $D \subset \mathbb{C}^n$ which is *unavoidable* in the sense that every entire map $F: \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus D$ has rank $< n$ at each point (see [RR]). Choose a point $p \in \mathbb{C}^n \setminus D$ and set $A = D \cup \{p\}$. Take $X = \mathbb{C}^n$, $f = Id: \mathbb{C}^n \rightarrow \mathbb{C}^n$, $X_0 = \{p\}$ and $X_1 = D$. Then the conditions of Theorem 6.3 are satisfied but the conclusion fails (since the rank condition for holomorphic maps $F: \mathbb{C}^n \rightarrow \mathbb{C}^n \setminus D$ implies that $F^{-1}(p)$ contains no isolated points and hence $X_0 = \{p\}$ cannot be a connected component of $F^{-1}(p)$ for any such F). ♠

Theorem 6.1 is a special case of the following result.

6.4 Theorem. *Let $h: Z \rightarrow X$ be a holomorphic submersion onto a Stein manifold X . Suppose that $Z_0 \subset Z$ is a closed complex subvariety of Z , $f: X \rightarrow Z$ is a continuous section and $X_0 = \{x \in X: f(x) \in Z_0\}$. Assume that f is holomorphic in an open neighborhood of $X_0 \cup K$ where K is a compact holomorphically convex subset of X . If the restricted submersion $h: Z \setminus Z_0 \rightarrow X$ is subelliptic over $X \setminus X_0$ then for each $r \in \mathbb{N}$ there is a homotopy $f_t: X \rightarrow Z$ of continuous sections of h such that $f_0 = f$, f_1 is holomorphic on X , and for each $t \in [0, 1]$ the section f_t is holomorphic in a neighborhood of $X_0 \cup K$, tangent to f to order r along X_0 and satisfies $\{x \in X: f_t(x) \in Z_0\} = X_0$. The analogous result holds for families of sections.*

Indeed we obtain Theorem 6.1 by taking $h: Z = X \times Y \rightarrow X$ to be the projection $h(x, y) = x$ and $Z_0 = X \times A$. Then $h: Z \setminus Z_0 = X \times (Y \setminus A) \rightarrow X$ is a subelliptic submersion when the fiber $Y \setminus A$ is subelliptic.

Proof of Theorem 6.4. We shall follow the proof of Theorem 1.4 in [FP3] with some modifications which we shall explain. (The proof is essentially the same as the proof of Theorem 1.1 in Sect. 4 above, except for the additional interpolation condition on the subvariety X_0 .) We inductively construct a sequence of deformations of the given section which are holomorphic on increasingly large open sets in X containing X_0 (and exhausting X) while at the same time paying attention not to introduce any additional intersection points of the section with the subvariety Z_0 (other than X_0 where the section is kept fixed through the entire process). To insure that no additional intersections with Z_0 appear in small neighborhoods of X_0 we keep the sections tangent to f to a very high order along X_0 (this will be measured by a suitable coherent sheaf of ideals on X). Away from X_0 we shall perform the modification procedure using the restricted submersion $h: Z \setminus Z_0 \rightarrow X$, thereby insuring that the sections remain in $Z \setminus Z_0$ over $X \setminus X_0$.

Definition 4. *Let $\mathcal{S} \subset \mathcal{O}_X$ be a coherent analytic sheaf of ideals on X and $X_0 = \{x \in X: \mathcal{S}_x \neq \mathcal{O}_{X,x}\}$. We say that local holomorphic sections f_0 and f_1 of $h: Z \rightarrow X$ at a point $x \in X_0$ are \mathcal{S} -tangent at x (denoted $\delta_x(f_0, f_1) \in$*

\mathcal{S}_x) if there exists a local holomorphic chart ϕ on Z at $z = f_0(x)$ such that the germ at x of every component of the map $\phi f_0 - \phi f_1$ belongs to \mathcal{S}_x . If f_0 and f_1 are holomorphic in an open set $U \supset X_0$ and \mathcal{S} -tangent at each $x \in X_0$, we say that f_0 and f_1 are \mathcal{S} -tangent and write $\delta(f_0, f_1) \in \mathcal{S}$.

\mathcal{S} -tangency is clearly independent of the choice of local charts on Z .

We now define a sheaf of ideals $\mathcal{R} \subset \mathcal{O}_X$ which measures the order of contact of a section $f: X \rightarrow Z$ with a subvariety $Z_0 \subset Z$ along $X_0 = \{x \in X: f(x) \in Z_0\}$. Assume that f is holomorphic in a neighborhood of X_0 . Fix $x \in X_0$ and let $z = f(x) \in Z_0$. Let g_1, \dots, g_k be holomorphic functions in a neighborhood $V \subset Z$ of z which generate the sheaf of ideals of $Z_0 \cap V$ at each point of V . Let \mathcal{R}_V denote the sheaf of ideals in $\mathcal{O}_X|_V$ generated by the functions $g_j \circ f$, $1 \leq j \leq k$. (For $x \in V \setminus X_0$ we have $\mathcal{R}_x = \mathcal{O}_x$.) It is easily seen that \mathcal{R}_V does not depend on the choice of the local generators g_j and hence we obtain a coherent analytic sheaf of ideals $\mathcal{R} \subset \mathcal{O}_X$ with support X_0 .

Fix an integer $r \in \mathbb{N}$ and let $\mathcal{S} = \mathcal{R} \cdot \mathcal{J}^r$, where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals of X_0 . Let d be a metric on Z . The following lemma was proved in [Fo1, Sect. 3].

6.5 Lemma. *Let $f: X \rightarrow Z$ be a continuous section of $h: Z \rightarrow X$ which is holomorphic in an open set U containing $X_0 = \{x \in X: f(x) \in Z_0\}$. If $g: U \rightarrow Z$ is a holomorphic section of $h: Z|_U \rightarrow U$ satisfying $\delta(f, g) \in \mathcal{S}$ then there is an open set $V \supset X_0$ such that $\{x \in V: g(x) \in Z_0\} = X_0$. Furthermore if $K \subset\subset K'$ are compact sets in U then there is an $\epsilon > 0$ such that for any g as above satisfying $d(f(x), g(x)) < \epsilon$ ($x \in K'$) we have $\{x \in K: g(x) \in Z_0\} = X_0 \cap K$.*

Everything is now ready for us to explain the modification problem which is the main ingredient in the proof of Theorem 6.4.

The modification problem. (Assumptions as in Theorem 6.4.) Let $B \subset X \setminus X_0$ be a compact set such that (K, B) is a Cartan pair in X (Sect. 4) and $K \cup B$ is holomorphically convex in X . Let $A = X_0 \cup K$ and let $a: \tilde{A} \rightarrow Z$, $b: \tilde{B} \rightarrow Z \setminus Z_0$ be holomorphic sections of h in open neighborhoods $\tilde{A} \supset A$ resp. $\tilde{B} \supset B$ such that $\{x \in \tilde{A}: a(x) \in Z_0\} = X_0$. Suppose furthermore that $b_t: \tilde{C} \rightarrow Z \setminus Z_0$ ($t \in [0, 1]$) is a family of holomorphic sections over $\tilde{C} = \tilde{A} \cap \tilde{B} \subset X \setminus X_0$ such that $b_0 = b|_{\tilde{C}}$ and $b_1 = a|_{\tilde{C}}$. From these data we must construct a homotopy $(\tilde{a}_t, \tilde{b}_t)$ ($t \in [0, 1]$) of holomorphic sections in smaller open neighborhoods $A' \supset A$ resp. $B' \supset B$ satisfying the following properties:

- (i) $\tilde{a}_0 = a|_{A'}$, $\tilde{b}_0 = b|_{B'}$,
- (ii) $\tilde{a}_1 = \tilde{b}_1$ on $A' \cap B'$, and hence this pair defines a holomorphic section \tilde{a} on $A' \cup B'$,

- (iii) for each $t \in [0, 1]$ we have $\delta(\tilde{a}_t, a) \in \mathcal{S}$, $\{x \in A': \tilde{a}_t(x) \in Z_0\} = X_0$,
and $\tilde{a}_t|_K$ is uniformly close to $a|_K$, and
(iv) $\tilde{b}_t(x) \in Z \setminus Z_0$ for all $x \in B'$ and $t \in [0, 1]$.

The final section \tilde{a} is holomorphic on $A' \cup B' \supset X_0 \cup K \cup B$, it is \mathcal{S} -tangent to a along X_0 and its graph intersects Z_0 precisely over X_0 as required. This will complete the induction step. To prove the parametric version of Theorems 6.4 and 6.1 one must also consider the analogous modification problem for continuous families of sections $\{(a_p, b_p): p \in P\}$ on a neighborhood of A resp. B , where P is a compact Hausdorff space. This extension presents no additional difficulties and we refer to [FP1] for the details.

To solve the above modification problem we proceed as in Sect. 4 above. By Theorem 3.1 (the Oka-Weil theorem) we can deform b through a homotopy of holomorphic sections of $Z \setminus Z_0$ over a neighborhood of B to another section (still denoted b) which approximates a uniformly on a neighborhood of $C = A \cap B$. The remaining problem is to patch a and b . This will be done as in Sect. 4, but with a couple of modifications which we now explain. As in Sect. 4 we find open neighborhoods $A' \supset A$, $B' \supset B$ of A resp. B and

- (a) a local h -spray $s_1: A' \times \mathbb{B}^n(\eta) \rightarrow Z$ with $s_1(x, 0) = a(x)$ for $x \in A'$,
(b) a global h -spray $s_2: B' \times \mathbb{C}^m \rightarrow Z \setminus Z_0$ with $s_2(x, 0) = b(x)$ for $x \in B'$,
and
(c) a transition map $\psi: C' \times \mathbb{B}^n(\eta) \rightarrow \mathbb{C}^m$ satisfying

$$s_2(x, \psi(x, t)) = s_1(x, t), \quad (x \in C' = A' \cap B', t \in \mathbb{B}^n(\eta)).$$

The only addition is that we build \mathcal{S} -tangency into the construction of the local spray s_1 as follows. First we choose a preliminary local dominating h -spray $s'_1: A' \times \mathbb{B}^k(\eta') \rightarrow Z$ with $s'_1(x, 0) = a(x)$. By Cartan's Theorem A [GR] there exist finitely many global sections h_1, \dots, h_m of the sheaf \mathcal{S} on X such that $X_0 = \{x \in X: h_j(x) = 0, 1 \leq j \leq m\}$. (The h_j 's need not generate \mathcal{S} .) Define $\tau: A' \times (\mathbb{C}^k)^m \rightarrow \mathbb{C}^k$ by $\tau(x, t_1, \dots, t_m) = \sum_{j=1}^m h_j(x)t_j$, where $t_j \in \mathbb{C}^k$ for each j . Set $n = mk$ and $t = (t_1, \dots, t_m) \in \mathbb{C}^m$. Then for suitably small $\eta > 0$ and $A' \supset A$ the map

$$s_1(x, t) = s'_1(x, \tau(x, t)), \quad (x \in A', t \in \mathbb{B}^n(\eta))$$

is a local h -spray which is dominating on $A' \setminus X_0$, and any section of the form $a_\alpha(x) = s_1(x, \alpha(x))$ (where $\alpha: A' \rightarrow \mathbb{B}^n(\eta)$ is a holomorphic map) is \mathcal{S} -tangent to a along X_0 . If $\eta > 0$ and $A' \supset A$ are chosen sufficiently small then Lemma 6.5 insures that the graph of a_α intersects Z_0 precisely over X_0 .

After shrinking the sets $A' \supset A$ and $B' \supset B$ we obtain by Proposition 4.1 in [FP3] a pair of holomorphic maps $\alpha: A' \rightarrow \mathbb{B}^n(\eta)$, $\beta: B' \rightarrow \mathbb{C}^m$

such that $\psi(x, \alpha(x)) = \beta(x)$ for $x \in C' = A' \cap B'$. The holomorphic homotopies

$$\tilde{a}_t(x) = s_1(x, t\alpha(x)) \quad (x \in A'), \quad \tilde{b}_t(x) = s_2(x, t\beta(x)) \quad (x \in B')$$

for $t \in [0, 1]$ then solve the modification problem. In fact for $t = 1$ and $x \in C'$ we have

$$\tilde{b}_1(x) = s_2(x, \beta(x)) = s_2(x, \psi(x, \alpha(x))) = s_1(x, \alpha(x)) = \tilde{a}_1(x),$$

and the other requirements are easily verified.

Using the solution of this modification problem we prove Theorem 6.4 by following the globalization scheme in the proof of Theorem 1.4 in [FP3]. More precisely, we follow the second approach in [FP3, pp. 65-66] whose main advantage is that the patching of (families of) sections is performed only on sets $C = A \cap B \subset X \setminus X_0$ and hence no special condition on the submersion h is required over X_0 . (In [F2] it is shown that we may even allow h to have ramification points, provided that these project by h into the subvariety X_0 .) With the same tools one can obtain the extension of Theorem 6.4 to continuous families of sections $\{f_p: p \in P\}$ with the parameter in a compact Hausdorff space P (see [FP2, FP3]). ♠

Proof of Theorem 6.3. We replace maps $X \rightarrow Y$ by sections of $Z = X \times Y \rightarrow X$ without changing the notation. By hypothesis there is a continuous homotopy $\tilde{f}_t: X \rightarrow Z$ ($t \in [0, 1]$), with $\tilde{f}_0 = f$, which is fixed near X_0 and satisfies $\{x \in X: \tilde{f}_1(x) \in Z_0\} = X_0$. We now apply Theorem 6.4, with \tilde{f}_1 as the initial section, to obtain a homotopy $\tilde{f}_t: X \rightarrow Z$ ($t \in [1, 2]$), where the final section \tilde{f}_2 is holomorphic on X and satisfies $\{x \in X: \tilde{f}_2(x) \in Z_0\} = X_0$. We rescale the parameter interval $[0, 2]$ back to $[0, 1]$. Since $Z \rightarrow X$ is assumed to be subelliptic over $X \setminus X_0$, we can apply Theorem 6.4 to the homotopy $\{\tilde{f}_t: t \in [0, 1]\}$, with the sheaf \mathcal{S} defined above (for some fixed $r \in \mathbb{N}$), to obtain a two-parameter homotopy $h_{t,s}: X \rightarrow Z$ ($t, s \in [0, 1]$) of continuous sections which are holomorphic in a neighborhood $V \supset X_0$ (independent of t, s) and satisfy:

- (i) $h_{t,0} = \tilde{f}_t$ for all $t \in [0, 1]$,
- (ii) $h_{0,s} = \tilde{f}_0 = f_0$ and $h_{1,s} = \tilde{f}_1$ for all $s \in [0, 1]$,
- (iii) $\delta(h_{t,0}, h_{t,s}) \in \mathcal{S}$ for all $s, t \in [0, 1]$, and
- (iv) the map $f_t := h_{t,1}$ is holomorphic on X for each $t \in [0, 1]$.

It follows from (iii) and Lemma 6.5 that there is a neighborhood $U \subset V$ of X_0 such that $\{x \in U: h_{t,s}(x) \in Z_0\} = X_0$ for all $s, t \in [0, 1]$. The homotopy $\{f_t: t \in [0, 1]\}$ defined by (iv) above then satisfies the conclusion of Theorem 6.3. ♠

Appendix: Remarks on the paper [G]

We wish to point out certain inconsistencies in Sect. 3 of the paper [G], in particular those concerning the notion Ell_∞ .

1. In the holomorphic category the property Ell_∞ for a complex manifold Y means the validity of a certain strong form of the Oka principle for maps from all Stein manifolds to Y (Sect. 3.1. in [G]). However, in [G, Sect. 3.5] an algebraic manifold Y is said to be *algebraically Ell_∞* if it admits a dominating algebraic spray $s: E \rightarrow Y$, defined on an algebraic vector bundle $E \rightarrow Y$. The conclusion of Localization Lemma 3.5.B. in [G] is that a manifold Y satisfying the hypothesis of that lemma is algebraically Ell_∞ . However, the proof offered there only gives finitely many algebraic sprays $s_i: E_i \rightarrow Y$ which together dominate at each point $y \in Y$ (thus showing that Y is subelliptic in our sense) and concludes with the sentence: *Then a composition of finitely many such s_i ...gives us the desired dominating spray over Y .* Since the bundles E_i are not necessarily trivial (in the proof of Proposition 1.2 they are in fact negative and do not admit any nontrivial holomorphic sections), the sprays s_i cannot be pulled back to nondegenerate sprays on trivial bundles over Y which would be necessary in order to use Lemma 2.4.

2. A similar remark applies to Sect. 3.5.C. in [G] which claims the implication (4.5) for any Zariski closed subset A of codimension at least two in an algebraic manifold Y . Being unable to prove this we offer Proposition 1.5 (and Lemma 5.1) above as a replacement.

3. Another inconsistency can be found in [G, Sect. 3.2.A']. The question considered in [G, Sect. 3.2] is whether the Ell_∞ property of a complex manifold Y implies the existence of a dominating spray on Y . In [G, sec. 3.2.A'] it is claimed that this is the case if Y is a projective variety which admits a 'sufficiently negative' vector bundle $E \rightarrow Y$ whose rank N is large compared to $\dim Y$. The idea in [G] is to first construct a 'local spray' s_0 on Y , defined in a small tubular neighborhood of the zero section $Y_0 \subset E$ in E , and subsequently Runge approximate s_0 by a global holomorphic map $s: E \rightarrow Y$ (which is then a dominating spray on Y). Here the author refers to the assumed axiom Ell_∞ which pertains to maps from Stein manifolds to Y . The problem is that the manifold E is not Stein (it contains the compact complex submanifold Y_0). When the bundle $E \rightarrow Y$ is negative, E is holomorphically convex and admits an exhaustion function which is zero on Y_0 and strongly plurisubharmonic on $E \setminus Y_0$. In order to make the above conclusion valid one would have to change the axiom Ell_∞ so that it would pertain to maps from proper modifications of Stein spaces to the given manifold Y . Although it seems likely that subellipticity of Y implies a suitable version of the Oka principle for maps $X \rightarrow Y$ from such manifolds X , no such result has been proved yet.

4. Section 3 in [G] contains several further results which we have not been able to understand. For instance, in Sect. 3.5.C. of [G] one finds the following statement: *If a Zariski closed subset $Y_0 \subset Y$ in an algebraic manifold Y satisfies $\text{codim} Y_0 \geq 2$ then $(\text{Ell}_\infty \text{ for } Y) \Rightarrow (\text{Ell}_\infty \text{ for } Y' = Y \setminus Y_0)$.* Here Ell_∞ property for Y means the existence of a dominating algebraic spray on Y according to the first sentence in [G, 3.5.A]. We are unable to justify the argument which is supposed to bring a spray on Y in ‘general position’ with respect to the subvariety Y_0 (see the discussion in Sect. 3.5.C’ in [G], and compare with our Proposition 1.5 and Lemma 5.1.).

A remark on [FP3]. We take this occasion to point out the following omission in the hypothesis of Theorem 5.2 in [FP3]: *The sets A and B in the statement of that theorem must have a basis of open Stein neighborhoods in X .* This hypothesis, which was assumed in the closely related Theorem 5.1 in [FP3] but was omitted in Theorem 5.2, is satisfied in all applications of Theorem 5.2 in [FP3].

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