

STRONGLY PSEUDOCONVEX HANDLEBODIES

FRANC FORSTNERIČ AND JERNEJ KOZAK

ABSTRACT. We construct strongly pseudoconvex handlebodies in \mathbb{C}^n whose center is a quadratic strongly pseudoconvex domain with an attached flat Lagrangian disc or plane.

1. Introduction

Let \mathbb{C}^n be the complex n -dimensional Euclidean space with coordinates $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$. Let J denote the standard almost complex structure operator on $T\mathbb{C}^n$: $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$, $J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$. A \mathcal{C}^1 submanifold $M \subset \mathbb{C}^n$ is *totally real* at $p \in M$ if $T_p M \cap JT_p M = \{0\}$, that is, the tangent space $T_p M \subset T_p \mathbb{C}^n$ contains no complex line. A \mathcal{C}^2 function $\rho: U \subset \mathbb{C}^n \rightarrow \mathbb{R}$ is *strongly plurisubharmonic* on U if

$$\mathcal{L}_\rho(z; v) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) v_j \bar{v}_k > 0 \quad (z \in U, v \in \mathbb{C}^n \setminus \{0\}).$$

$\mathcal{L}_\rho(z; v)$ is called the *Levi form* of ρ at z in the direction of the vector v .

Assume that $D \subset \mathbb{C}^n$ is a closed, smoothly bounded, strongly pseudoconvex domain. Thus $D = \{\rho \leq 0\}$ where ρ is a strongly plurisubharmonic function in an open set $U \supset D$, with $d\rho \neq 0$ on $bD = \{\rho = 0\}$. Let $M \subset \mathbb{C}^n$ be a smooth totally real submanifold with boundary $bM = S \cup S'$, where each of the sets S , S' is a union of connected components of bM (S' may be empty). Assume furthermore that

$$M \cap D = S \subset bD, \quad T_p(S) \subset T_p^{\mathbb{C}}(bD) := T_p(bD) \cap JT_p(bD) \quad (p \in S).$$

Such M will be called a *totally real handle* attached to D along the *Legendrian (complex tangential)* submanifold $S \subset bD$. (Some authors reserve the word ‘handle’ for the case when M is diffeomorphic to the

Received March 4, 2003.

2000 Mathematics Subject Classification: 32A07, 32T15.

Key words and phrases: Levi form, pseudoconvex domains, handlebodies.

closed ball in some \mathbb{R}^k and $bM = S^{k-1}$.) We consider the following problem.

THE HANDLEBODY PROBLEM. Given a (small) open set $U \supset M$, find a closed strongly pseudoconvex domain $K \subset \mathbb{C}^n$ satisfying $D \cup M \subset K \subset D \cup U$ which admits a strong deformation retraction onto $E := D \cup M$.

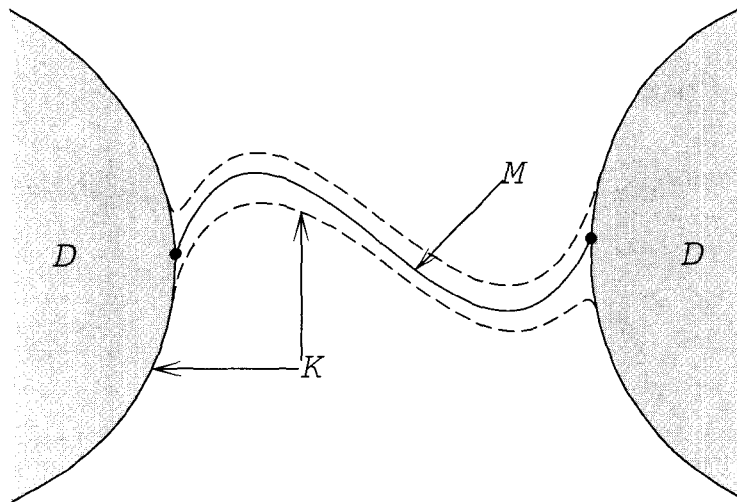


FIGURE 1. A handlebody K with center $E = D \cup M$

Such K will be called a *strongly pseudoconvex handlebody with center E* (Figure 1). The existence of a strong deformation retraction of K onto E implies that K is homotopically equivalent to E .

It is well known that any totally real submanifold M in \mathbb{C}^n (or in any complex manifold) has a basis of strongly pseudoconvex tubular neighborhoods. (If $M \subset \mathbb{C}^n$ is compact and of class \mathcal{C}^2 , we may take neighborhoods defined by the Euclidean distance to M .) Hence the above problem is nontrivial only along the attaching submanifold $S = D \cap M \subset bD$. If S fails to be Legendrian in bD at some point $p \in S$ then $D \cup M$ may have a nontrivial local envelope of holomorphy at p , containing small analytic discs with boundaries in $bD \cup M$ (this follows from the results in [1]), and in such case there exist no small pseudoconvex neighborhoods. Local envelope may also appear at points $p \in M$ for which $T_p M$ contains a nontrivial complex subspace; see [2]. This justifies the above hypotheses on M and S .

The simpler problem concerning the existence of a basis of (strongly) pseudoconvex neighborhoods of $E = D \cup M$, without insisting on the existence of a deformation retraction onto E , has been considered by several authors; see e.g. Stolzenberg [17], Hörmander and Wermer [14], Fornæss and Stout [5], [6], Chirka and Smirnov [16], and Rosay [15]. However, in many problems one actually needs strongly pseudoconvex handlebodies which have ‘the same shape’ as $D \cup M$.

An important general construction of handlebodies was given in 1990 by Eliashberg (Lemma 3.4.3 in [4]). Write the coordinates on \mathbb{C}^n in the form $z = x + iy$, with $x, y \in \mathbb{R}^n$. Set $|x|^2 = x_1^2 + \cdots + x_n^2$, $|y|^2 = y_1^2 + \cdots + y_n^2$. Let

$$D_\lambda = \{x + iy \in \mathbb{C}^n : |y|^2 \geq 1 + \lambda|x|^2\}, \quad M = \{iy : |y| \leq 1\}.$$

Thus M is the unit ball in the Lagrangian subspace $i\mathbb{R}^n \subset \mathbb{C}^n$, attached to the quadric domain D_λ along the $(n-1)$ -sphere $S = bM = \{iy : |y| = 1\} \subset bD_\lambda$ which is Legendrian in bD_λ . Note that D_λ is strongly pseudoconvex precisely when $\lambda > 1$. In this situation, Lemma 3.4.3 in [4] gives for each open set $U \supset M$ a strongly pseudoconvex handlebody $K = \{|y| \geq \varphi(|x|)\}$, for a suitably chosen function φ , such that $K \subset D_\lambda \cup U$ and the center of K equals $E_\lambda = D_\lambda \cup M$. In [4] this was used in the construction of Stein manifolds with prescribed homotopy type (see also Gompf [9] and Chapter 11 in [10]).

Five years later, in 1995, B. Boonstra [3] (Ph. D. dissertation, unpublished) constructed handlebodies whose center is the union of an ellipsoid with a Lagrangian plane in \mathbb{C}^n . He also constructed handlebodies in more general situations by the ‘osculation and patching’ technique. Even though Boonstra cited Eliashberg’s paper [4], his construction seems independent since the details are somewhat different.

The content of this paper is as follows. In Section 2 we obtain a differential condition on a function f which gives the necessary and sufficient condition for (strong) pseudoconvexity of the domain $D_+ = \{x + iy \in \mathbb{C}^n : |y| \geq f(|x|)\}$ (resp. of $D_- = \{|y| \leq f(|x|)\}$) along the hypersurface $\Sigma = \{|y| = f(|x|)\}$ (Proposition 2.1 and Corollary 2.2). A sufficient condition for strong pseudoconvexity of such domains was obtained earlier by Eliashberg; see (*, **) on p. 39 of [4]. Our derivation of these conditions is different from the one in [4] and is somewhat similar to the one in [3].

In Section 3 we prove Proposition 3.1 which is the same as Lemma 3.4.3 in [4]. Our proof, based on the differential conditions from Section 2, is similar to the original proof in [4], but differs from it in certain

details. The extension to handles of lower dimension is immediate; see Lemma 3.1.1 in [4].

Proposition 3.3 in the same section gives an explicit construction of strongly pseudoconvex handlebodies whose center is the union of $D = \{x + iy \in \mathbb{C}^n : |y|^2 \leq \lambda|x|^2 + 1\}$ (with $\lambda < 1$) and the Lagrangian plane $i\mathbb{R}^n$. Note that D is strongly pseudoconvex precisely when $\lambda < 1$; it is an unbounded hyperboloid when $0 < \lambda < 1$, a tube when $\lambda = 0$, and a bounded ellipsoid when $\lambda < 0$. Boonstra [3] found explicit handlebodies for $\lambda < 0$ and gave an indirect construction for $0 \leq \lambda < 1$. We give an explicit construction for all values $\lambda < 1$.

In Sect. 4 we construct monotone families of strongly pseudoconvex handlebodies whose center is the union of a sublevel set of a general quadratic strongly plurisubharmonic function $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$ and an attached disc $M \subset \mathbb{R}^k \subset \mathbb{C}^n$ that passes through the critical point $0 \in \mathbb{C}^n$ of ρ . Unlike the handlebodies constructed by Eliashberg (or in Section 3 above), these handlebodies are not ‘thin’ everywhere around M , but only in a smaller neighborhood of the origin. Indeed these handlebodies are sublevel sets of a certain noncritical strongly plurisubharmonic function. The construction is independent from the one in [4] (and from the rest of this paper) and is much simpler. A crucial use of this result was made in [7] (Lemma 6.7) in the construction of holomorphic submersions of Stein manifolds to complex Euclidean spaces.

Using standard bumping and patching techniques for strongly plurisubharmonic functions one may adapt the construction of handlebodies in [4] (and in this paper) to more general handle attachments, assuming of course that the boundary of the handle is Legendrian in the boundary of the domain D . A particularly simple case is when the handle M is real analytic along bM ; in such case M can be locally flattened near any point $p \in bM$ by a local biholomorphic change of coordinates, and the resulting domain can be osculated along $bM \subset D$ by a quadratic model of the type considered in [4] or in this paper. (This was used for instance in [15], but with the weaker conclusion that $D \cup M$ admits a Stein neighborhood basis. Certain cases have been treated by Boonstra [3], but his work remains unpublished.)

The case of smooth (but non real-analytic) handles can possibly be handled by using coordinate changes near points $p \in bM$ which are $\bar{\partial}$ -flat on M . Such coordinate changes clearly preserve strong pseudoconvexity of bD locally near p . However, to see that the model handlebodies remain strongly pseudoconvex under such coordinate changes, one must estimate the terms in their Levi form coming from the non-holomorphic

terms in the coordinate change. We are not aware of any published work in this direction.

Professor Eliashberg informed us in a private communication (May 14, 2003) that a solution of the handlebody problem for handles M of different topological type (i.e., non-disc type) follows from the case of disc-handles. Indeed, taking any Morse function on M which is constant on bM , one decomposes M into a union of disc-handles and then successively applies the disc-handle lemma. (The details do not seem to exist in print.) We wish to thank him for this remark.

2. Pseudoconvexity of spherical domains

Let $z = (z_1, \dots, z_n) = x + iy \in \mathbb{C}^n$, with $z_k = x_k + iy_k$ for $k = 1, \dots, n$. Set $|x|^2 = x_1^2 + \dots + x_n^2$, $|y|^2 = y_1^2 + \dots + y_n^2$. Let U be a nonempty open set in \mathbb{R}^n which is invariant under the action of the orthogonal group $O(n)$ (i.e., $x \in U$ and $|x'| = |x|$ implies $x' \in U$). Set $I = \{|x|^2 : x \in U\} \subset \mathbb{R}_+$. Assume that $\theta : I \rightarrow (0, +\infty)$ is a positive function of class \mathcal{C}^2 .

PROPOSITION 2.1. *Let $n > 1$. The domain*

$$(1) \quad D_- = \{x + iy \in \mathbb{C}^n : x \in U, |y|^2 < \theta(|x|^2)\}$$

is strongly pseudoconvex along the hypersurface $\Sigma = \{|y|^2 = \theta(|x|^2)\}$ if and only if θ satisfies the following differential inequalities on I :

$$(2) \quad \theta' < 1, \quad 2|x|^2\theta\theta'' < (1 - \theta')(|x|^2\theta'^2 + \theta).$$

(θ and its derivatives are calculated at $|x|^2$). The domain

$$(3) \quad D_+ = \{x + iy \in \mathbb{C}^n : x \in U, |y|^2 > \theta(|x|^2)\}$$

is strongly pseudoconvex along Σ if and only if the reverse inequalities hold in (2). If θ solves the differential equation

$$(4) \quad 2|x|^2\theta\theta'' = (1 - \theta') \cdot (|x|^2\theta'^2 + \theta)$$

then $\theta' < 1$ implies that D_- is weakly pseudoconvex along Σ while $\theta' > 1$ implies that D_+ is weakly pseudoconvex along Σ .

Proof. Set $\rho(x + iy) = |y|^2 - \theta(|x|^2)$. A calculation gives for $1 \leq j \neq k \leq n$

$$\begin{aligned} -\rho_{z_k} &= x_k\theta' + iy_k, \\ -2\rho_{z_k\bar{z}_k} &= 2x_k^2\theta'' + \theta' - 1, \\ -2\rho_{z_j\bar{z}_k} &= 2x_jx_k\theta'', \end{aligned}$$

where θ and its derivatives are evaluated at $|x|^2$. The calculation of the Levi form of $\Sigma = \{\rho = 0\}$ can be simplified by observing that ρ is invariant under the action of the real orthogonal group $O(n)$ on \mathbb{C}^n by $A(x + iy) = Ax + iAy$ for $A \in O(n)$. Fix a point $p = r + is \in \Sigma$ ($r, s \in \mathbb{R}^n$). After an orthogonal rotation we may assume that $p = (x_1 + iy_1, i\tilde{y}_2, \dots, i\tilde{y}_n)$, with $x_1 = |r| \geq 0$. Applying another orthogonal map which restricts to the identity on $\mathbb{C} \times \{0\}^{n-1}$ we may further assume that $p = (x_1 + iy_1, iy_2, 0, \dots, 0)$, where $y_1^2 + y_2^2 = |s|^2 = \theta(x_1^2)$. At this point we have

$$\rho_{z_1}(p) = -x_1\theta' - iy_1, \quad \rho_{z_2}(p) = -iy_2, \quad \rho_{z_k}(p) = 0 \text{ for } k = 3, \dots, n.$$

Hence the complex tangent space $T_p^{\mathbb{C}}\Sigma = \{v \in \mathbb{C}^n : \sum_{k=1}^n \frac{\partial \rho}{\partial z_k}(p)v_k = 0\}$ consists of all $v \in \mathbb{C}^n$ satisfying $v_1 = -\lambda iy_2$, $v_2 = \lambda(x_1\theta' + iy_1)$ for arbitrary choices of $\lambda \in \mathbb{C}$ and $v'' = (v_3, \dots, v_n) \in \mathbb{C}^{n-2}$. We also have

$$\begin{aligned} 2\rho_{z_1\bar{z}_1}(p) &= 1 - \theta' - 2x_1^2\theta'', \\ 2\rho_{z_k\bar{z}_k}(p) &= 1 - \theta' \quad (k = 2, \dots, n), \\ 2\rho_{z_j\bar{z}_k}(p) &= 0 \quad (1 \leq j \neq k \leq n). \end{aligned}$$

For $v \in T_p^{\mathbb{C}}\Sigma$, we thus get (noting that $y_1^2 + y_2^2 = \theta(x_1^2)$)

$$\begin{aligned} (5) \quad 2\mathcal{L}_\rho(p; v) &= (1 - \theta' - 2x_1^2\theta'')|\lambda|^2y_2^2 + (1 - \theta')|\lambda|^2(x_1^2\theta'^2 + y_1^2) + (1 - \theta')|v''|^2 \\ &= |\lambda|^2 \left(-2x_1^2y_2^2\theta'' + (1 - \theta')(x_1^2\theta'^2 + \theta) \right) + (1 - \theta')|v''|^2, \end{aligned}$$

where θ and its derivatives are evaluated at $x_1^2 = |r|^2$. Thus $\mathcal{L}_\rho(p; v) > 0$ for all choices of $\lambda \in \mathbb{C}$ and $v'' \in \mathbb{C}^{n-2}$ with $|\lambda|^2 + |v''|^2 > 0$ if and only if

$$\theta' < 1, \quad 2x_1^2y_2^2\theta'' < (1 - \theta')(x_1^2\theta'^2 + \theta).$$

Observe that $0 \leq y_2^2 \leq |s|^2 = \theta(x_1^2)$, and y_2^2 assumes both extreme values 0 and $\theta(x_1^2)$ when (y_1, y_2) traces the circle $y_1^2 + y_2^2 = \theta(x_1^2)$. Thus the second inequality above holds at all points of this circle precisely when it holds at the point $y_1 = 0, y_2 = \sqrt{\theta(x_1^2)}$. This gives the conditions

$$\theta' < 1, \quad 2x_1^2\theta\theta'' < (1 - \theta')(x_1^2\theta'^2 + \theta)$$

characterizing strong pseudoconvexity of D_- along the mentioned circle in Σ . Since $x_1 = |r|$, the above is equivalent to the pair of inequalities (2) at $p = r + is$. Similarly we see that negativity of $\mathcal{L}_\rho(p; v)$ for all choices of λ and v'' (which characterizes strong pseudoconvexity of D_+) is equivalent to the reverse inequalities in (2).

Assume now that θ satisfies (4). As before we reduce to the case $p = x + iy = (x_1 + iy_1, iy_2, 0, \dots, 0)$. From (5) we obtain

$$\begin{aligned} 2\mathcal{L}_\rho(p; v) &= |\lambda|^2(-2x_1^2y_2^2\theta'' + 2x_1^2\theta\theta'') + (1 - \theta')|v''|^2 \\ &= 2|\lambda|^2x_1^2y_1^2\theta'' + (1 - \theta')|v''|^2. \end{aligned}$$

(We used $\theta(x_1^2) - y_1^2 = y_2^2$.) From (4) we see that θ'' is of the same sign as $1 - \theta'$. Thus $\theta' < 1$ implies $\mathcal{L}_\rho(p; v) \geq 0$, with equality precisely when $v'' = 0$ and $0 = x_1y_1 = x \cdot y$. In this case $D_- = \{\rho < 0\}$ is weakly pseudoconvex along $\Sigma = \{\rho = 0\}$, strongly pseudoconvex on $\{x + iy \in \Sigma : x \cdot y \neq 0\}$, and has one zero eigenvalue of the Levi form at each point of $\{x + iy \in \Sigma : x \cdot y = 0\}$. When $\theta' > 1$ the analogous conclusions hold for $D_+ = \{\rho > 0\}$. If $\theta' = 1$ holds identically then $\rho = |y|^2 - |x|^2 + c = -\Re(\sum_{j=1}^n z_j^2) + c$ is pluriharmonic. \square

The second inequality in (2) simplifies further in the variables $|x|, |y|$:

COROLLARY 2.2. *Let $U \subset \mathbb{R}^n \setminus \{0\}$ be an $O(n)$ -invariant open set and $f: I \rightarrow (0, +\infty)$ a \mathcal{C}^2 function on $I = \{|x| : x \in U\}$. The domain $D_- = \{x + iy \in \mathbb{C}^n : x \in U, |y| < f(|x|)\}$ is strongly pseudoconvex along the hypersurface $\Sigma = \{|y| = f(|x|)\}$ if and only if*

$$(6) \quad \frac{ff'}{|x|} < 1, \quad f \cdot \left(f'' + \frac{f'^3}{|x|} \right) < 1$$

for all $x \in U$, where f and its derivatives are evaluated at $|x|$. The domain $D_+ = \{x + iy : x \in U, |y| > f(|x|)\}$ is strongly pseudoconvex along Σ when the reverse inequalities hold in (6). If f satisfies the differential equation

$$(7) \quad f \cdot \left(f'' + \frac{f'^3}{|x|} \right) = 1$$

then $ff'/|x| < 1$ implies that D_- is weakly pseudoconvex along Σ while $ff'/|x| > 1$ implies that D_+ is weakly pseudoconvex along Σ .

Proof. Set $t = |x| > 0$ for $x \in U$. The functions f and θ are related by $f(t)^2 = \theta(t^2)$. Differentiation gives $f(t)f'(t) = t\theta'(t^2)$ whence $\theta' < 1$ is equivalent to $ff'/t < 1$. Another differentiation of $f(t)f'(t) = t\theta'(t^2)$ gives

$$ff'' + f'^2 = 2t^2\theta'' + \theta' = 2|x|^2\theta'' + ff'/t.$$

Hence $2|x|^2\theta'' = ff'' + f'^2 - ff'/t$. Multiplying by $\theta = f^2$ we obtain the first line in the following display. In the second line we used $|x|^2\theta'^2 =$

$$(t\theta')^2 = f^2 f'^2.$$

$$\begin{aligned} 2|x|^2\theta\theta'' &= f^2\left(ff'' + f'^2 - \frac{ff'}{t}\right), \\ (1 - \theta')(|x|^2\theta'^2 + \theta) &= \left(1 - \frac{ff'}{t}\right)(f^2 f'^2 + f^2) \\ &= f^2\left(1 - \frac{ff'^3}{t} + f'^2 - \frac{ff'}{t}\right). \end{aligned}$$

Comparing the two sides, dividing by $f^2 > 0$ and cancelling the common terms $f'^2 - ff'/t$ we see that the second inequality in (2) is equivalent to $f(f'' + f'^3/t) < 1$. Similarly one treats the other cases. \square

REMARKS 2.3. (A) The differential inequalities (2) and (6) are invariant up to the sign with respect to taking the inverses. More precisely, assume $\theta'(|x_0|^2) \neq 0$ for $x_0 \in U$ and denote by τ the local inverse of θ . At points where $\theta' > 0$ the inequalities (2) transform into the reverse inequalities for τ :

$$\tau' > 1, \quad 2|y|^2\tau\tau'' > (1 - \tau')(|y|^2\tau'^2 + \tau).$$

On the other hand, near points where $\theta' < 0$ the inequalities (2) transform into the same inequalities for $\tau = \theta^{-1}$. This can be explained geometrically as follows. If $\theta'(|x_0|^2) > 0$ then for x near x_0 we have $|y|^2 < \theta(|x|^2)$ if and only if $|x|^2 > \tau(|y|^2)$, and strong pseudoconvexity of the latter region is equivalent to the above inequality for τ according to Proposition 2.1. If $\theta'(|x_0|^2) < 0$ then for x near x_0 we have $|y|^2 < \theta(|x|^2)$ if and only if $|x|^2 < \tau(|y|^2)$, and pseudoconvexity is now characterized by (2). Similarly the equations (4) and (7) are invariant with respect to taking the inverses.

(B) If $f(t)$ ($t \in \mathbb{R}$) is a function of class \mathcal{C}^1 and piecewise \mathcal{C}^2 , we adopt the convention that f satisfies the second inequality in (6) at a point of discontinuity t_0 of the second derivative f'' when both the left and the right limit of f'' at t_0 satisfies it. (At endpoints we consider only the one sided limit.) A similar convention is adopted for (2).

EXAMPLE 2.4. We illustrate the above by looking at model domains defined by the quadratic function

$$\rho_\lambda(z) = \rho_\lambda(x + iy) = \lambda|x|^2 - |y|^2 \quad (\lambda \in \mathbb{R}, z \in \mathbb{C}^n)$$

which will be used in the following section. Setting $g_{\lambda,a}(t) = +\sqrt{\lambda t^2 + a}$ we have $\{\rho_\lambda < -a\} = \{x + iy \in \mathbb{C}^n : |y| > g_{\lambda,a}(|x|)\}$. From $\left(\frac{\partial^2 \rho_\lambda}{\partial z_j \partial \bar{z}_k}\right) = \frac{(\lambda-1)}{2}I$ we see that ρ_λ is strongly plurisubharmonic when $\lambda > 1$, strongly

plurisuperharmonic when $\lambda < 1$, and $\rho_1(x+iy) = |x|^2 - |y|^2 = \Re\left(\sum_{j=1}^n z_j^2\right)$ is pluriharmonic. It is easily verified directly that $g_{\lambda,a}$ satisfies (6) on $\{t \in \mathbb{R}_+ : \lambda t^2 + a \geq 0\}$ if $\lambda < 1$, and it satisfies the reverse inequalities in (6) if $\lambda > 1$. If $\lambda \neq 0$ then $g = g_{\lambda,a}$ satisfies the differential equation $g\left(g'' + \frac{g'^3}{\lambda t}\right) = \lambda$.

3. Strongly pseudoconvex handlebodies

In this section we find functions $f: I \rightarrow (0, +\infty)$ on intervals $I \subset \mathbb{R}_+ = [0, +\infty)$ which satisfy one of the following pairs of differential inequalities:

$$(8) \quad f \cdot \left(f'' + \frac{f'^3}{t}\right) < 1 \quad \text{and} \quad \frac{ff'}{t} < 1,$$

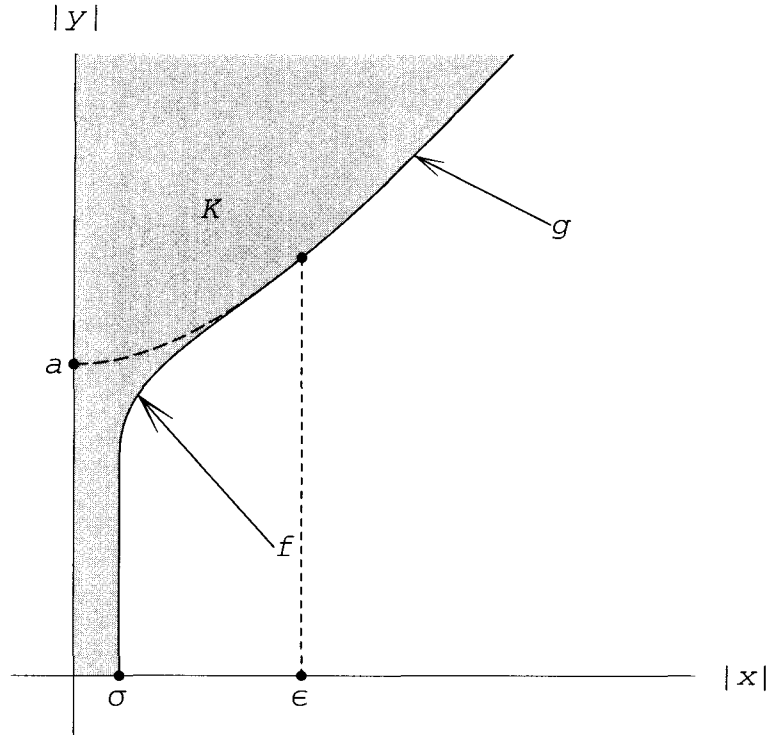
$$(9) \quad f \cdot \left(f'' + \frac{f'^3}{t}\right) > 1 \quad \text{and} \quad \frac{ff'}{t} > 1.$$

If f is of class \mathcal{C}^1 and piecewise \mathcal{C}^2 then at a point of discontinuity of f'' it should be understood that f satisfies the first inequality in (8) resp. in (9) if the one-sided limits of f'' at that point satisfy it. By Corollary 2.2 the condition (8) characterizes strong pseudoconvexity of the domain $\{x + iy \in \mathbb{C}^n : |y| < f(|x|)\}$ along $\Sigma = \{|y| = f(|x|)\}$ while (9) does the same for $\{x + iy \in \mathbb{C}^n : |y| > f(|x|)\}$.

PROPOSITION 3.1. *Let $\lambda > 1, a > 0$ and $g(t) = +\sqrt{\lambda t^2 + a}$. For every sufficiently small $\epsilon > 0$ there exists a number $\sigma = \sigma(\epsilon) \in (0, \epsilon)$ and a continuous, positive, strictly increasing function $f = f_\epsilon: [\sigma, +\infty) \rightarrow [f(\sigma), +\infty)$ which is \mathcal{C}^∞ on $(\sigma, +\infty)$, satisfies (9) and also the following:*

- (i) $f(t) = g(t)$ for $t \geq \epsilon$,
- (ii) $f(t) < g(t)$ for $\sigma \leq t < \epsilon$,
- (iii) $f'(\sigma+) = \lim_{t \downarrow \sigma} f'(t) = +\infty$, and
- (iv) the inverse function $f^{-1}: \mathbb{R}_+ \rightarrow [\sigma, +\infty)$ is of class \mathcal{C}^∞ and satisfies (8) provided that we set $f^{-1}(u) = \sigma$ for $0 \leq u \leq f(\sigma)$.

COROLLARY 3.2. *Let $\lambda > 1, a > 0$, $D = \{x + iy \in \mathbb{C}^n : |y|^2 \geq \lambda|x|^2 + a\}$ and $M = \{iy : y \in \mathbb{R}^n, |y| \leq a\}$. If f satisfies Proposition 3.1 then $K = \{x + iy \in \mathbb{C}^n : |x| \leq f^{-1}(|y|)\}$ (Figure 2) is a smooth strongly pseudoconvex handlebody with center $E = D \cup M$, satisfying $D \cup \{|x| \leq \sigma\} \subset K \subset D \cup \{|x| < \epsilon\}$.*

FIGURE 2. The handlebody K

REMARK. We have already said in the Introduction that Proposition 3.1 (and Corollary 3.2) is the same as Lemma 3.4.3 in [4]. The handlebodies on figures 2 and 4 are shown in the coordinate system $(|x|, |y|) \in \mathbb{R}_+^2$; the actual handlebody is the preimage under the map $x + iy \rightarrow (|x|, |y|)$.

Proof of Proposition 3.1. Without loss of generality we may take $a = 1$ and $g(t) = \sqrt{\lambda t^2 + 1}$ (the general case follows by rescaling). A calculation gives for $t > 0$

$$g'(t) = \frac{\lambda t}{g(t)} > 0, \quad g''(t) = \frac{\lambda}{g(t)^3} > 0, \quad g'''(t) = -\frac{3\lambda^2 t}{g(t)^5} < 0$$

which shows that g is increasing, convex, and g' is concave. We also obtain $g'(t) - tg''(t) = \lambda^2 t^3 / g(t)^3 > 0$. Fix a small $\epsilon > 0$ and let $c := g'(\epsilon) - \epsilon g''(\epsilon) > 0$. Choose a number $0 < \eta < \min(\epsilon, c^3/3)$ and let $c_1 := c + \eta g''(\epsilon)$. Let $\sigma > 0$ be a number satisfying $2\sigma < \eta < \epsilon$ (its

precise value will be determined later). We shall first obtain a solution f of class C^1 and piecewise C^2 on $(\sigma, +\infty)$; the final solution will be obtained by smoothing. Let

$$f(t) = g(\epsilon) + \int_{\epsilon}^t f'(\tau) d\tau \quad (\sigma \leq t < +\infty)$$

where f' is a continuous and piecewise C^1 function defined as follows:

$$f'(t) = \begin{cases} g'(t), & \text{if } \epsilon \leq t; \\ g'(\epsilon) + g''(\epsilon)(t - \epsilon), & \text{if } \eta \leq t < \epsilon; \\ c_1 + \eta \log(\eta/t), & \text{if } 2\sigma \leq t < \eta; \\ 2\sqrt{\sigma}/\sqrt{t - \sigma}, & \text{if } \sigma < t < 2\sigma. \end{cases}$$

The graph of f' is shown on Figure 3. (However, due to technical difficulties we show the case for large ϵ . For small $\epsilon > 0$ the derivative of the linear part of the graph should be close to $\lambda > 1$. The same remark applies to Figure 5.)

Note that f' is continuous at $t = \eta$, with $f'(\eta) = c_1$. To insure the continuity of f' at $t = 2\sigma$ we choose σ to be the solution of $c_1 + \eta \log(\eta/2\sigma) = 2$. Clearly $f(t) = g(t)$ for $t \geq \epsilon$. It is also clear that $f'(t) > g'(t)$ for $\sigma < t < \epsilon$: on $t \in [\eta, \epsilon]$ the graph of f' is the tangent line to the graph of g' at $(\epsilon, g'(\epsilon))$ which stays above g' due to the concavity of g' ; on $(\sigma, \eta]$ this is clear since g' is increasing while f' is decreasing. Hence f is strictly increasing and satisfies $f(t) < g(t)$ for $\sigma \leq t < \epsilon$. Also $f'(\sigma+) = +\infty$. It remains to show that $f(\sigma+) > 0$ and that f satisfies (9) on (σ, ϵ) .

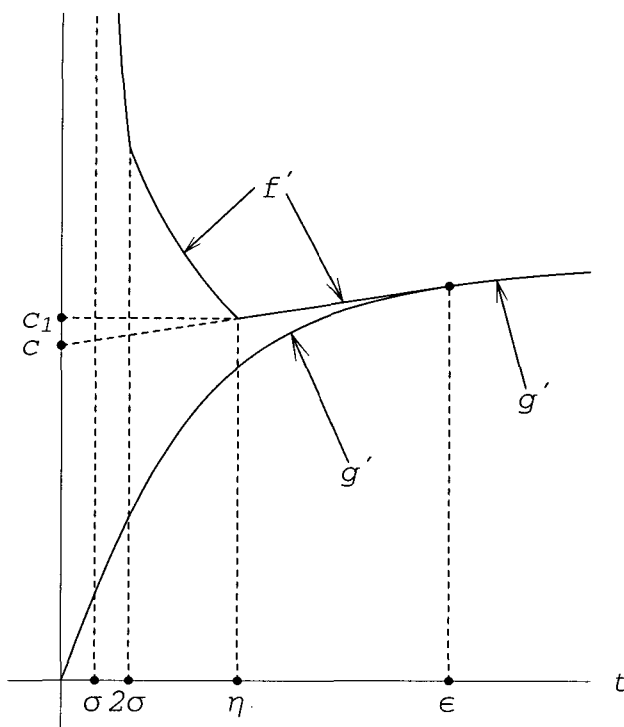
Case 1: $\eta \leq t < \epsilon$. On this interval

$$f'(t) = g'(\epsilon) + g''(\epsilon)(t - \epsilon) = c + tg''(\epsilon) > tg''(\epsilon).$$

The graph of f' is the tangent line to the graph of g' at the point $(\epsilon, g'(\epsilon))$. Since g' is strongly concave, we conclude $f'(t) > g'(t)$ for all $t \in [\eta, \epsilon)$. We have

$$f(t) > g(\epsilon) - \int_0^{\epsilon} (g'(\epsilon) + g''(\epsilon)(\tau - \epsilon)) d\tau > g(\epsilon) - \epsilon g'(\epsilon) = 1/g(\epsilon).$$

Since $f'(t) > 0$ and $f''(t) = g''(\epsilon)$, we get $f(f'' + f'^3/t) > ff'' > g''(\epsilon)/g(\epsilon) = \lambda/g(\epsilon)^4$ which is > 1 if ϵ is small (since $\lambda > 1$ and $g(\epsilon) \approx g(0) = 1$). From $f(t) > 1/g(\epsilon)$ and $f'(t) > tg''(\epsilon)$ we also get $f(t)f'(t)/t > g''(\epsilon)/g(\epsilon) > 1$.

FIGURE 3. The graph of f'

Case 2: $2\sigma \leq t < \eta$. Using $f(\eta) > 1/g(\epsilon)$, $f'(\eta) < g'(\epsilon)$ (Case 1) we get

$$f(t) > f(\eta) - \int_0^\eta (f'(\eta) - \eta \log(\tau/\eta)) d\tau > \frac{1}{g(\epsilon)} - \eta g'(\epsilon) - \eta^2 =: M.$$

Clearly $M > 1/2$ when $\epsilon > 0$ is small. From $f''(t) = -\eta/t$, $f'(t) = f'(\eta) + \eta \log(\eta/t) > f'(\eta) > c > 0$ and $0 < 3\eta < c^3$ we obtain

$$\begin{aligned} f\left(f'' + \frac{f'^3}{t}\right) - 1 &> M\left(\frac{-\eta}{t} + \frac{c^3}{t}\right) - 1 \\ &> \frac{M}{t}(c^3 - \eta - 2t) > \frac{M}{\eta}(c^3 - 3\eta) > 0. \end{aligned}$$

Also, $ff'/t > Mf'(\eta)/\eta > Mc/\eta > 3M/c^2 > 1$ (since $c > 0$ is small) which verifies the second inequality in (9).

Case 3: $\sigma < t < 2\sigma$. As before we easily obtain a lower bound $f(t) > 1/2$ provided that $\epsilon > 0$ is sufficiently small. We have $f'(t) = 2\sqrt{\sigma}/\sqrt{t-\sigma}$, $f''(t) = -\sqrt{\sigma}/\sqrt{t-\sigma}^3$, and hence

$$\begin{aligned}
 f(f'' + f'^3/t) &> \frac{1}{2} \left(\frac{-\sqrt{\sigma}}{\sqrt{t-\sigma}^3} + \frac{8\sigma\sqrt{\sigma}}{t\sqrt{t-\sigma}^3} \right) \\
 &> \frac{\sqrt{\sigma}}{2\sqrt{t-\sigma}^3}(-1 + 4) \geq \frac{3}{2\sigma} > 1.
 \end{aligned}$$

The second inequality in (2) is trivial as in Case 2.

The function f constructed above is invertible and its inverse function $f^{-1}: [f(\sigma), +\infty) \rightarrow [\sigma, +\infty)$ is of class \mathcal{C}^1 , piecewise \mathcal{C}^2 (actually piecewise real-analytic), and satisfies (8). We extend f^{-1} to $[0, +\infty)$ by taking $f^{-1}(u) = \sigma$ for $u \in [0, f(\sigma)]$; this extension satisfies the same properties also near the point $u = f(\sigma)$. The final solution is obtained by smoothing $h := f^{-1}$ in a small neighborhood of any point of discontinuity of its second derivative. (We interpolate smoothly between the left and the right limit of h'' at such a point and integrate twice to obtain the new h . This does not change h and h' very much and hence the inequality (8) is preserved.) This completes the proof. \square

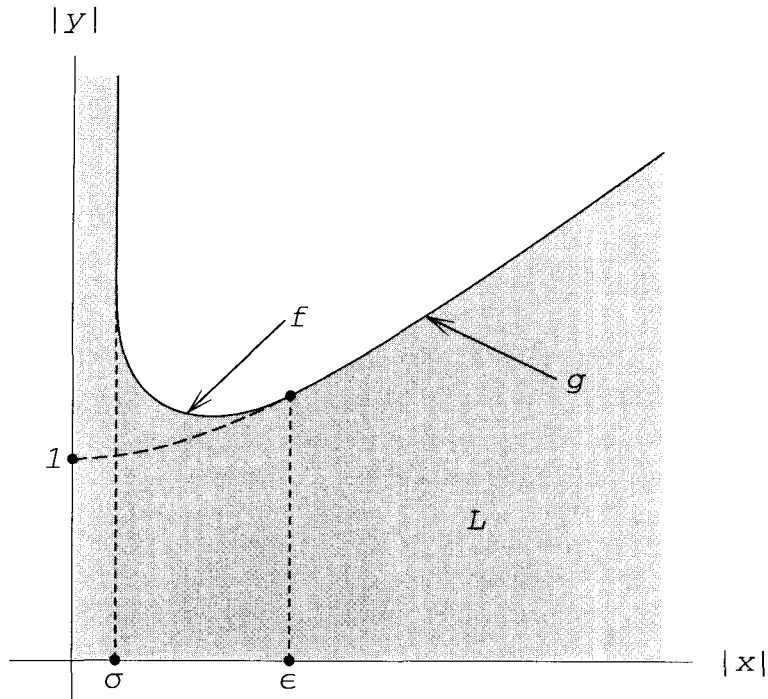
A small modification of the above construction gives strongly pseudoconvex handlebodies $L \subset \mathbb{C}^n$ with center

$$E = \{x + iy \in \mathbb{C}^n : |y|^2 \leq \lambda|x|^2 + 1\} \cup i\mathbb{R}^n \quad (\lambda < 1).$$

A typical L is shown on Figure 4. Observe that $D = \{|y|^2 \leq \lambda|x|^2 + 1\}$ is strongly pseudoconvex precisely when $\lambda < 1$. It is an unbounded hyperboloid when $0 < \lambda < 1$, a tube when $\lambda = 0$ and a bounded ellipsoid when $\lambda < 0$. The Lagrangian plane $i\mathbb{R}^n$ is an (unbounded) handle attached to D along the sphere $\{iy : y \in \mathbb{R}^n, |y| = 1\}$. Boonstra [3] found explicit handlebodies for $\lambda < 0$ and gave an indirect ‘bumping and patching’ construction for $0 \leq \lambda < 1$. We give an explicit construction for all $\lambda < 1$. (Our example is easily modified to obtain handlebodies with center $D \cup M$ where $M \subset i\mathbb{R}^n$ is a compact domain such that $D \cap i\mathbb{R}^n$ is contained in the relative interior of M .) Set

$$L = \{x + iy : |x| > \sigma, |y| \leq f(|x|)\} \cup \{x + iy : |x| \leq \sigma\}$$

where f is given by the following proposition.

FIGURE 4. The handlebody L

PROPOSITION 3.3. Let $\lambda < 1$ and $g(t) = +\sqrt{\lambda t^2 + 1}$. For every sufficiently small $\epsilon > 0$ there exists a number $\sigma = \sigma(\epsilon) \in (0, \epsilon)$ and a continuous function $f: [\sigma, +\infty) \rightarrow (0, +\infty)$, smooth on $(\sigma, +\infty)$, which satisfies the inequalities (8) and the following:

- (i) $f(t) = g(t)$ for $t \geq \epsilon$,
- (ii) $f(t) > g(t)$ for $\sigma \leq t < \epsilon$,
- (iii) $f'(\sigma+) = \lim_{t \downarrow \sigma} f'(t) = -\infty$,
- (iv) there exists a smooth inverse function f^{-1} near the point $u = f(\sigma)$, with $f^{-1}(u) = \sigma$ for $u \geq f(\sigma)$, satisfying the inequalities (8) on its domain.

Proof. Choose numbers $0 < \sigma < 2\sigma < \eta < \epsilon$; additional conditions will be imposed later. We have $g'(\epsilon)/\epsilon = \lambda/g(\epsilon) < 1$. Choose a number k satisfying $g'(\epsilon)/\epsilon < k < 1$ and let $c := g'(\epsilon) - k\epsilon < 0$. Clearly $c > -1$ if ϵ is small. Choosing $\eta > 0$ sufficiently small we have $c_1 := g'(\epsilon) + k(\eta - \epsilon) = c + k\eta < 0$ and $\eta + c_1^3 < 0$. Let $\sigma \in (0, \eta/2)$ solve $c_1 - \eta \log(\eta/2\sigma) = -2$.

With these choices we define f on $(\sigma, +\infty)$ by $f(t) = g(\epsilon) + \int_{\epsilon}^t f'(\tau) d\tau$ where

$$f'(t) = \begin{cases} g'(t), & \text{if } \epsilon \leq t; \\ g'(\epsilon) + k(t - \epsilon), & \text{if } \eta \leq t < \epsilon; \\ c_1 - \eta \log(\eta/t), & \text{if } 2\sigma \leq t < \eta; \\ -2\sqrt{\sigma}/\sqrt{t - \sigma}, & \text{if } \sigma < t < 2\sigma. \end{cases}$$

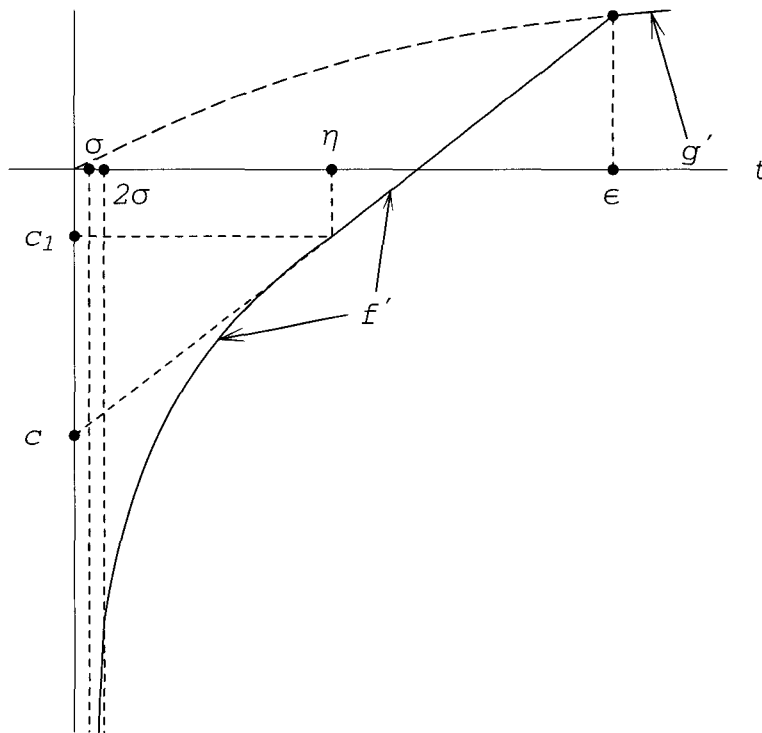


FIGURE 5. The graph of f'

The graph of f' is shown on Figure 5. We verify that f satisfies (8). For $t \geq \epsilon$ this is clear since $f(t) = g(t)$. For $\eta \leq t < \epsilon$ we have

$$g(t) < f(t) \leq g(\epsilon) + \left| \int_t^\epsilon (g'(\epsilon) + k(\tau - \epsilon)) d\tau \right| < g(\epsilon) + \epsilon(1 + |g'(\epsilon)|).$$

By our choice of k the graph of f lies below the secant line through $(0, 0)$ and $(\epsilon, g'(\epsilon))$, and the secant is below g' due to concavity of g' . This

gives $f'(t) < g'(\epsilon)t/\epsilon = \lambda t/g(\epsilon)$. Also, $f''(t) = k$. At points $t \in [\eta, \epsilon]$ where $f''(t) + f'(t)^3/t > 0$ we thus have

$$f(t)(f''(t) + f'(t)^3/t) \leq (g(\epsilon) + \epsilon(1 + |g'(\epsilon)|)) (k + \lambda^3 \epsilon^2/g(\epsilon)^3) < 1$$

provided that $\epsilon > 0$ is sufficiently small (since $k < 1$, $g(\epsilon) \approx 1$ and the other quantities are $O(\epsilon)$). At points where $f''(t) + f'(t)^3/t \leq 0$ the same estimate holds since $f(t) > 0$. Also, $f(t)f'(t)/t < (g(\epsilon) + O(\epsilon))\lambda/g(\epsilon) = \lambda + O(\epsilon) < 1$ if $\epsilon > 0$ is small.

For $t \in [2\sigma, \eta]$ the estimates (8) are almost trivial: from $f'(t) \leq f'(\eta) = c_1 < 0$ and $f''(t) = \eta/t$ we get $f''(t) + f'(t)^3/t \leq (\eta + c_1^3)/t < 0$ which implies the first estimate in (8) (since $f(t) > g(t) > 0$). Also $f(t)f'(t)/t < 0$ and hence (8) holds. Similarly we verify (8) on $(\sigma, 2\sigma]$. We complete the proof as in Proposition 3.1 by smoothing f^{-1} . \square

4. Handlebodies on general quadratic domains

In this section we consider handlebodies modeled on general quadratic strongly plurisubharmonic functions $\rho: \mathbb{C}^n \rightarrow \mathbb{R}$. Choose a $k \in \{0, 1, \dots, n\}$ and write the coordinates on \mathbb{C}^n in the form $\zeta = (z, w)$, with $z = x + iy \in \mathbb{C}^k$ and $w = u + iv \in \mathbb{C}^{n-k}$. Let A, B be positive definite real symmetric matrices of dimension k resp. $n - k$. Denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on any \mathbb{R}^m . Given these choices let

$$(10) \quad \rho(z, w) = Q(y, w) - |x|^2, \quad Q(y, w) = \langle Ay, y \rangle + \langle Bv, v \rangle + |u|^2.$$

It is easily seen that ρ is strongly plurisubharmonic if and only if all eigenvalues of A are larger than 1. (Equivalently, the matrix $A - I$ must be positive definite which we denote by $A > I$.) Clearly ρ has a Morse critical point of index k at the origin and no other critical points. It is proved in [12] that every Morse critical point of a strongly plurisubharmonic function is of this form in some local holomorphic coordinates, modulo terms of order > 2 .

Assume now that $k \geq 1$. Let $\Lambda^k = \{(x + i0, 0) \in \mathbb{C}^n : x \in \mathbb{R}^k\}$. We identify $x \in \mathbb{R}^k$ with $(x + i0, 0) \in \Lambda^k \subset \mathbb{C}^n$ when appropriate.

PROPOSITION 4.1. (Notation as above.) *Let ρ be given by (10) where $A > I$, $B > 0$. Given $r > 0$, $\epsilon > 0$ there exist constants $0 < r < c_0 < R$, $\delta > 0$ and a smooth, increasing, weakly convex function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\tau(z, w) := Q(y, w) - h(|x|^2)$ is a strongly plurisubharmonic function on \mathbb{C}^n , with a Morse critical point of index k at $0 \in \mathbb{C}^n$, satisfying*

- (i) for $|x|^2 \leq r$ we have $\tau(z, w) = Q(y, w) - \delta|x|^2$,

- (ii) for $|x|^2 \geq R$ we have $\tau(z, w) = \rho(z, w) + c_0$, and
- (iii) $\{\rho \leq -c_0\} \cup \Lambda^k \subset \{\tau \leq 0\} \subset \{\rho < -r\} \cup \{Q < \epsilon\}$.

COROLLARY 4.2. For every sufficiently small $c > 0$ the set $K_c = \{\tau \leq c\}$ is a strongly pseudoconvex handlebody with center

$$E_{c-c_0} = \{\zeta \in \mathbb{C}^n : \rho(\zeta) \leq c - c_0\} \cup \Lambda^k,$$

satisfying $E_{c-c_0} \subset K_c \subset \{\rho < -r\} \cup \{Q < \epsilon\}$ (Figure 6).

Proof of Proposition 4.1. We modify slightly the construction in Lemma 6.7 of [7] (the function constructed there was not Morse). Let $t_0 = r + \epsilon$. Choose $0 < \delta < 1, \mu > 1$ such that $\delta t_0 < \epsilon$ and $1 < \mu + \delta < \lambda_1$ where $\lambda_1 > 1$ denotes the smallest eigenvalue of A . Set $R = \mu^2 t_0 / (\mu + \delta - 1)^2$ and

$$h(t) = \begin{cases} \delta t, & \text{if } 0 \leq t \leq t_0; \\ \delta t + \mu(\sqrt{t} - \sqrt{t_0})^2, & \text{if } t_0 < t \leq R; \\ t - R + h(R), & \text{if } R < t. \end{cases}$$

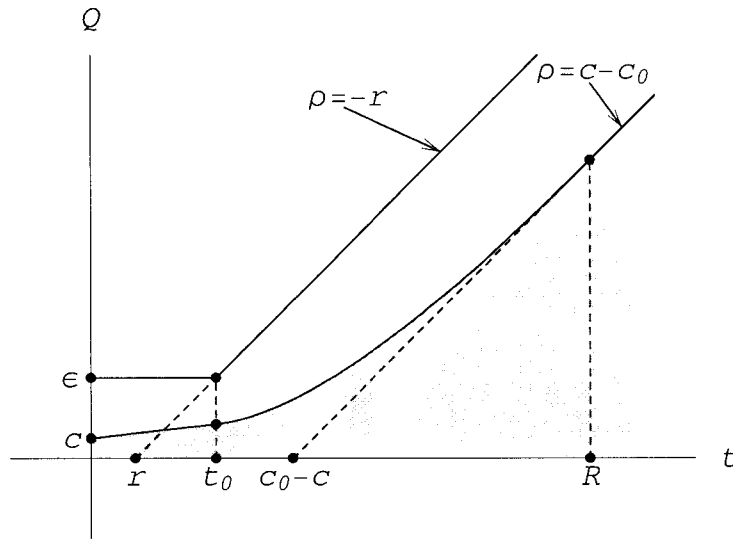


FIGURE 6. The handlebody $K_c = \{\tau \leq c\}$

It is easily verified that h is an increasing convex function of class \mathcal{C}^1 and piecewise \mathcal{C}^2 on \mathbb{R} which satisfies

$$2t\ddot{h} + \dot{h} = \mu + \delta < \lambda_1 \quad (t_0 \leq t \leq R)$$

and $\delta = \dot{h}(t_0) \leq \dot{h}(t) \leq 1 = \dot{h}(R)$ for all $t \in \mathbb{R}$. By smoothing h we obtain an increasing convex C^∞ function, still denoted h , which equals δt for $0 \leq t \leq t_0$, it equals $t - R + h(R)$ for $t \geq R$, and satisfies

$$\dot{h}(t) < \lambda_1, \quad 2t\ddot{h}(t) + \dot{h}(t) < \lambda_1 \quad (t \in \mathbb{R}).$$

A simple calculation shows that, as a consequence of these inequalities, the associated function τ is strongly plurisubharmonic on \mathbb{C}^n and satisfies Proposition 4.1 with $c_0 = R - h(R)$. (See the proof of Lemmas 6.7 and 6.8 in [7] for the details of this calculation.) \square

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Institute of Mathematics, Physics and Mechanics
University of Ljubljana
Jadranska 19, 1000 Ljubljana, Slovenia
E-mail: franc.forstneric@fmf.uni-lj.si
jernej.kozak@fmf.uni-lj.si