Let $\Delta$ be the open unit disc in $\mathbb{C}$, $X$ a connected complex manifold and $D$ the set of all holomorphic maps $f : \Delta \rightarrow X$ with $f(\Delta) = X$. We prove that $D$ is dense in $\text{Hol}(\Delta, X)$.

1. Introduction

Let $\Delta_r = \{ z \in \mathbb{C} : |z| < r \}$ and $\Delta = \Delta_1$. In [8] the second author proved that for any irreducible complex space $X$ there exists a holomorphic map $\Delta \rightarrow X$ with dense image, and he raised the question whether the set of all holomorphic maps $\Delta \rightarrow X$ with dense image forms a dense subset of the set $\text{Hol}(\Delta, X)$ of all holomorphic maps $\Delta \rightarrow X$ with respect to the topology of locally uniform convergence.

In this paper we show that the answer to this question is positive if $X$ is smooth, but negative for some singular space.

Theorem 1. For any connected complex manifold $X$ the set of holomorphic maps $\Delta \rightarrow X$ with dense images forms a dense subset in $\text{Hol}(\Delta, X)$. The conclusion fails for some singular complex surface $X$.

The situation is quite different for proper discs, i.e., proper holomorphic maps $\Delta \rightarrow X$. The paper [4] contains an example of a non-pseudoconvex bounded domain $X \subset \mathbb{C}^2$ such that a certain nonempty open subset $U \subset X$ is not intersected by the image of any proper holomorphic disc $\Delta \rightarrow X$. On the other hand, proper holomorphic discs exist in great abundance in Stein manifolds [6], [1], [2] and, more generally, in $q$-complete manifolds $X$ for $1 \leq q < \text{dim} \ X$ [3].

2. Preparations

Lemma 1. Let $W_n$ be a decreasing sequence (i.e., $W_{n+1} \subset W_n$) of open sets with $\Delta \subset W_n \subset \Delta_2$ for every $n$. Let $K = \cap_n W_n$ and assume that the interior of $K$ coincides with $\Delta$. Furthermore assume that there are biholomorphic maps $\phi_n : \Delta \rightarrow W_n$ with $\phi_n(0) = 0$ for $n = 1, 2, \ldots$. 

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Then there exists an automorphism \( \alpha \in \text{Aut}(\Delta) \) and a subsequence \((\phi_{n_k})\) of the sequence \((\phi_n)\) such that \( \phi_{n_k} \circ \alpha^{-1} \) converges locally uniformly to the identity map \( \text{id}_{\Delta} \) on \( \Delta \).

**Proof.** Montel’s theorem shows that, after passing to a suitable subsequence, we have \( \lim_{n \to \infty} \phi_n = \alpha : \Delta \to K \) and \( \lim_{n \to \infty} (\phi_n^{-1}|_{\Delta}) = \beta : \Delta \to \overline{\Delta} \). Since the limit maps are holomorphic and satisfy \( \alpha(0) = 0 \) and \( \beta(0) = 0 \), we conclude that \( \alpha(\Delta) \subset \text{Int}K = \Delta \) and \( \beta(\Delta) \subset \Delta \). Moreover \( \alpha \circ \beta = \text{id}_{\Delta} = \beta \circ \alpha \), and hence both \( \alpha \) and \( \beta \) are automorphisms of \( \Delta \) (indeed, rotations \( z \to ze^{it} \)). \( \square \)

We also need the following special case of a result of the first author (theorem 3.2 in [5]):

**Proposition 1.** Let \( X \) be a complex manifold, \( 0 < r < 1 \), \( E \) the real line segment \([1, 2] \subset \mathbb{C} \), \( K = \overline{\Delta} \cup E \), \( U \) an open neighbourhood of \( \overline{\Delta} \) in \( \mathbb{C} \), \( S \) a finite subset of \( K \) and \( f : U \cup E \to X \) a continuous map which is holomorphic on \( U \).

Then there is a sequence of pairs of open neighbourhoods \( W_n \subset \mathbb{C} \) of \( K \) and holomorphic maps \( g_n : W_n \to X \) such that:

1. \( g_n|_K \) converges uniformly to \( f|_K \) as \( n \to \infty \), and
2. \( g_n(a) = f(a) \) for all \( a \in S \) and \( n \in \mathbb{N} \).

### 3. Towards the main result

In this section we prove the following proposition which is the main technical result in the paper. The first statement in theorem 1 (§1) is an immediate corollary.

**Proposition 2.** Let \( X \) be a connected complex manifold endowed with a complete Riemannian metric and induced distance \( d \), \( S \) a countable subset of \( X \), \( f : \Delta \to X \) a holomorphic map, \( \epsilon > 0 \) and \( 0 < r < 1 \).

Then there exists a holomorphic map \( F : \Delta \to X \) such that

- \( S \subset F(\Delta) \), and
- \( d(F(z), F(z)) \leq \epsilon \) for all \( z \in \Delta_r \).

**Proof.** Let \( s_1, s_2, s_3, \ldots \) be an enumeration of the elements of \( S \). We shall inductively construct a sequence of holomorphic maps \( f_n : \Delta \to X \), numbers \( r_n \in (0, 1) \) and points \( a_{1,n}, \ldots, a_{n,n} \in \Delta \) satisfying the following properties for \( n = 0, 1, 2, \ldots \):

1. \( f_0 = f \) and \( r_0 = r \),
2. \( (r_{n} + 1)/2 < r_{n+1} < 1 \),
3. \( f_n(a_{j,n}) = s_j \) for \( n \geq 1 \) and \( j = 1, 2, \ldots, n \),
4. \( d(f_n(z), f_{n+1}(z)) < 2^{-(n+1)} \epsilon \) for all \( z \in \Delta_{r_n} \), and
5. \( d_{\Delta}(a_{j,n}, a_{j,n+1}) < 2^{-n} \) for \( j = 1, 2, \ldots, n \) where \( d_{\Delta} \) denotes the Poincaré distance on the unit disc.

Assume inductively that the data for level \( n \) (i.e., \( f_n, r_n, a_{j,n} \)) have been chosen. (For \( n = 0 \) we do not have any points \( a_{j,0} \).) With \( n \) fixed we choose an increasing sequence of real numbers \( \lambda_k \) with \( \lambda_k > r_n \) and \( \lim_{k \to \infty} \lambda_k = 1 \).
For every \( k \in \mathbb{N} \) the map \( \tilde{g}_k(z) \overset{def}{=} f_n(\lambda_k z) \in X \) is defined and holomorphic on the disc \( \Delta_1/\lambda_k \supset \Delta \). After a slight shrinking of its domain we can extend it continuously to the segment \( E = [1, 2] \subset \mathbb{C} \) such that the right end point 2 of \( E \) is mapped to the next point \( s_{n+1} \in S \) (this is possible since \( X \) is connected).

Applying proposition 1 to the extended map \( \tilde{g}_k \) we obtain for every \( k \in \mathbb{N} \) an open neighbourhood \( V_k \subset \mathbb{C} \) of \( K = \Delta \cup E \) and a holomorphic map \( g_k : V_k \to X \) such that

(i) \( |g_k(z) - f_n(\lambda_k z)| < 2^{-k} \) for all \( z \in \Delta \),
(ii) \( g_k(2) = s_{n+1} \), and
(iii) \( g_k(a_{j,n}/\lambda_k) = f_n(a_{j,n}) = s_j \) for \( j = 1, \ldots, n \).

Next we choose a decreasing sequence of simply connected open sets \( W_k \subset \mathbb{C} \) (\( k \in \mathbb{N} \)) with \( K \subset W_k \subset V_k \) and \( K = \cap_k \overline{W_k} \). Notice that \( \text{Int}K = \Delta \). By lemma 1 there is a sequence of biholomorphic maps \( \phi_k : \Delta \to W_k \) with \( \lim_{k \to \infty} \phi_k = \text{id}_\Delta \).

Consider the holomorphic maps \( h_k = g_k \circ \phi_k : \Delta \to X \). By our construction we know that \( \lim_{k \to \infty} h_k = f_n \) locally uniformly on \( \Delta \).

To fulfill the inductive step it thus suffices to choose \( f_{n+1} = h_k \) for a sufficiently large \( k \), \( a_{j,n+1} = a_{j,n}/\lambda_k \) \( (j = 1, \ldots, n) \), \( a_{n+1,n+1} = \phi_k^{-1}(2) \). Finally we choose a number \( r_{n+1} \) satisfying

\[
\max\{|a_{n+1,n+1}|, \frac{r_n + 1}{2}\} < r_{n+1} < 1.
\]

This completes the inductive step.

By properties (2) and (4) the sequence \( f_n \) converges locally uniformly in \( \Delta \) to a holomorphic map \( F : \Delta \to X \). Aided by property (1) we also control \( d(F(z), F(z')) \) for \( z \in \Delta \). Since the Poincaré metric is complete, property (5) insures that for every fixed \( j \in \mathbb{N} \) the sequence \( a_{j,n} \in \Delta (n = j, j+1, \ldots) \) has an accumulation point \( b_j \) inside of \( \Delta \), and (3) implies \( F(b_j) = s_j \) for \( j = 1, 2, \ldots \).

Hence \( S \subset F(\Delta) \).

\section*{4. Singular spaces}

We use an example of Kaliman and Zaidenberg \cite{7} to show that for a complex spaces \( X \) with singularities the set of maps \( \Delta \to X \) with dense image need not be dense in \( \text{Hol}(\Delta, X) \). We denote by \( \text{Sing}(X) \) the singular locus of \( X \).

\textbf{Proposition 3.} There is a singular compact complex surface \( S \), a non-constant holomorphic map \( f : \Delta \to S \) and an open neighbourhood \( \Omega \) of \( f \) in \( \text{Hol}(\Delta, S) \) such that \( g(\Delta) \subset \text{Sing}(S) \) for every \( g \in \Omega \).

\textbf{Proof.} In \cite{7} Kaliman and Zaidenberg constructed an example of a singular surface \( S \) with normalization \( \pi : Z \to S \) such that \( S \) contains a rational curve \( C \simeq \mathbb{P}^1 \) while \( Z \) is smooth and hyperbolic. Denote by \( d_Z \) the Kobayashi distance function on \( Z \). We choose two distinct points \( p, q \in C \) and open relatively compact neighbourhoods \( V \) of \( p \) and \( W \) of \( q \) in \( S \) such that \( V \cap W = \emptyset \). The
preimages $\pi^{-1}(V)$ and $\pi^{-1}(W)$ in $Z$ are also compact, and since $Z$ is hyperbolic we have

$$r = \min\{d_Z(x,y) : x \in \pi^{-1}(V), y \in \pi^{-1}(W)\} > 0.$$  

Fix a point $a \in \Delta$ with $0 < d_\Delta(0,a) < r$ and let $\Omega$ consist of all holomorphic maps $g : \Delta \to S$ satisfying $g(0) \in V$ and $g(a) \in W$. Since both $p$ and $q$ are lying on the rational curve $C$, there is a holomorphic map $g : \Delta \to C$ with $g(0) = p \in V$ and $g(a) = q \in W$; hence the set $\Omega$ is not empty. Clearly $\Omega$ is open in $\text{Hol}(\Delta, S)$.

To conclude the proof it remains to show that $g(\Delta) \subset \text{Sing}(S)$ for all $g \in \Omega$. Indeed, a holomorphic map $g : \Delta \to S$ with $g(\Delta) \not\subset \text{Sing}(S)$ admits a holomorphic lifting $\tilde{g} : \Delta \to Z$ with $\pi \circ \tilde{g} = g$. If $g \in \Omega$ then by construction

$$d_Z(\tilde{g}(0), \tilde{g}(a)) \geq r > d_\Delta(0,a)$$

which violates the distance decreasing property for the Kobayashi pseudometric. This contradiction establishes the claim.

In particular, we see that in this example the set of all holomorphic maps $f : \Delta \to S$ with dense image does not constitute a dense subset of $\text{Hol}(\Omega, S)$.

References