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HOLOMORPHIC DISCS WITH DENSE IMAGES

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Abstract. Let Δ be the open unit disc in $\mathbb{C}$, X a connected complex manifold and $\mathcal{D}$ the set of all holomorphic maps $f\colon \Delta \to X$ with $\overline{f(\Delta)} = X$. We prove that $\mathcal{D}$ is dense in $Hol(\Delta, X)$.

1. Introduction

Let $\Delta_r = \{z \in \mathbb{C} \colon |z| < r\}$ and $\Delta = \Delta_1$. In [8] the second author proved that *for any irreducible complex space X there exists a holomorphic map $\Delta \to X$ with dense image*, and he raised the question whether the set of all holomorphic maps $\Delta \to X$ with dense image forms a dense subset of the set $Hol(\Delta, X)$ of all holomorphic maps $\Delta \to X$ with respect to the topology of locally uniform convergence.

In this paper we show that the answer to this question is positive if X is smooth, but negative for some singular space.

Theorem 1. *For any connected complex manifold X the set of holomorphic maps $\Delta \to X$ with dense images forms a dense subset in $Hol(\Delta, X)$. The conclusion fails for some singular complex surface X.*

The situation is quite different for *proper discs*, i.e., proper holomorphic maps $\Delta \to X$. The paper [4] contains an example of a non-pseudoconvex bounded domain $X \subset \mathbb{C}^2$ such that a certain nonempty open subset $U \subset X$ is not intersected by the image of any proper holomorphic disc $\Delta \to X$. On the other hand, proper holomorphic discs exist in great abundance in *Stein manifolds* [6], [1], [2] and, more generally, in q-complete manifolds X for $1 \le q < \dim X$ [3].

2. Preparations

Lemma 1. *Let W_n be a decreasing sequence (i.e., $W_{n+1} \subset W_n$) of open sets with $\Delta \subset W_n \subset \Delta_2$ for every n. Let $K = \cap_n \overline{W}_n$ and assume that the interior of K coincides with Δ. Furthermore assume that there are biholomorphic maps $\phi_n \colon \Delta \to W_n$ with $\phi_n(0) = 0$ for $n = 1, 2, \ldots$.*

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Then there exists an automorphism $\alpha \in \mathrm{Aut}(\Delta)$ *and a subsequence* (ϕ_{n_k}) *of the sequence* (ϕ_n) *such that* $\phi_{n_k} \circ \alpha^{-1}$ *converges locally uniformly to the identity map* id_Δ *on* Δ.

Proof. Montel's theorem shows that, after passing to a suitable subsequence, we have $\lim_{n\to\infty} \phi_n = \alpha\colon \Delta \to K$ and $\lim_{n\to\infty}(\phi_n^{-1}|_\Delta) = \beta\colon \Delta \to \overline{\Delta}$. Since the limit maps are holomorphic and satisfy $\alpha(0) = 0$ and $\beta(0) = 0$, we conclude that $\alpha(\Delta) \subset \mathrm{Int}K = \Delta$ and $\beta(\Delta) \subset \Delta$. Moreover $\alpha \circ \beta = id_\Delta = \beta \circ \alpha$, and hence both α and β are automorphisms of Δ (indeed, rotations $z \to ze^{it}$). □

We also need the following special case of a result of the first author (theorem 3.2 in [5]):

Proposition 1. *Let* X *be a complex manifold,* $0 < r < 1$, E *the real line segment* $[1, 2] \subset \mathbb{C}$, $K = \overline{\Delta} \cup E$, U *an open neighbourhood of* $\overline{\Delta}$ *in* $\mathbb{C}$, S *a finite subset of* K *and* $f\colon U \cup E \to X$ *a continuous map which is holomorphic on* U.

Then there is a sequence of pair of open neighbourhoods $W_n \subset \mathbb{C}$ *of* K *and holomorphic maps* $g_n\colon W_n \to X$ *such that:*

1. $g_n|_K$ *converges uniformly to* $f|_K$ *as* $n \to \infty$, *and*
2. $g_n(a) = f(a)$ *for all* $a \in S$ *and* $n \in \mathbb{N}$.

3. Towards the main result

In this section we prove the following proposition which is the main technical result in the paper. The first statement in theorem 1 (§1) is an immediate corollary.

Proposition 2. *Let* X *be a connected complex manifold endowed with a complete Riemannian metric and induced distance* d, S *a countable subset of* X, $f\colon \Delta \to X$ *a holomorphic map,* $\epsilon > 0$ *and* $0 < r < 1$.

Then there exists a holomorphic map $F\colon \Delta \to X$ *such that*

(a) $S \subset F(\Delta)$, *and*
(b) $d(f(z), F(z)) \le \epsilon$ *for all* $z \in \Delta_r$.

Proof. Let $s_1, s_2, s_3, ..$ be an enumeration of the elements of S. We shall inductively construct a sequence of holomorphic maps $f_n\colon \Delta \to X$, numbers $r_n \in (0, 1)$ and points $a_{1,n}, \ldots, a_{n,n} \in \Delta$ satisfying the following properties for $n = 0, 1, 2, \ldots$:

1. $f_0 = f$ and $r_0 = r$,
2. $(r_n + 1)/2 < r_{n+1} < 1$,
3. $f_n(a_{j,n}) = s_j$ for $n \ge 1$ and $j = 1, 2, \ldots, n$,
4. $d(f_n(z), f_{n+1}(z)) < 2^{-(n+1)}\epsilon$ for all $z \in \Delta_{r_n}$, and
5. $d_\Delta(a_{j,n}, a_{j,n+1}) < 2^{-n}$ for $j = 1, 2, \ldots, n$ where d_Δ denotes the Poincaré distance on the unit disc.

Assume inductively that the data for level n (i.e., f_n, r_n, $a_{j,n}$) have been chosen. (For $n = 0$ we do not have any points $a_{j,0}$.) With n fixed we choose an increasing sequence of real numbers λ_k with $\lambda_k > r_n$ and $\lim_{k\to\infty} \lambda_k = 1$.

For every $k \in \mathbb{N}$ the map $\widetilde{g}_k(z) \stackrel{def}{=} f_n(\lambda_k z) \in X$ is defined and holomorphic on the disc $\Delta_{1/\lambda_k} \supset \overline{\Delta}$. After a slight shrinking of its domain we can extend it continuously to the segment $E = [1, 2] \subset \mathbb{C}$ such that the right end point 2 of E is mapped to the next point $s_{n+1} \in S$ (this is possible since X is connected).

Applying proposition 1 to the extended map $\widetilde{g}_k$ we obtain for every $k \in \mathbb{N}$ an open neighbourhood $V_k \subset \mathbb{C}$ of $K = \overline{\Delta} \cup E$ and a holomorphic map $g_k \colon V_k \to X$ such that

(i) $|g_k(z) - f_n(\lambda_k z)| < 2^{-k}$ for all $z \in \overline{\Delta}$,
(ii) $g_k(2) = s_{n+1}$, and
(iii) $g_k(a_{j,n}/\lambda_k) = f_n(a_{j,n}) = s_j$ for $j = 1, \dots, n$.

Next we choose a decreasing sequence of simply connected open sets $W_k \subset \mathbb{C}$ ($k \in \mathbb{N}$) with $K \subset W_k \subset V_k$ and $K = \cap_k \overline{W}_k$. Notice that $\mathrm{Int} K = \Delta$. By lemma 1 there is a sequence of biholomorphic maps $\phi_k \colon \Delta \to W_k$ with $\lim_{k \to \infty} \phi_k = id_\Delta$.

Consider the holomorphic maps $h_k = g_k \circ \phi_k \colon \Delta \to X$. By our construction we know that $\lim_{k \to \infty} h_k = f_n$ locally uniformly on Δ.

To fulfill the inductive step it thus suffices to choose $f_{n+1} = h_k$ for a sufficiently large k, $a_{j,n+1} = a_{j,n}/\lambda_k$ ($j = 1, \dots, n$), $a_{n+1,n+1} = \phi_k^{-1}(2)$. Finally we choose a number r_{n+1} satisfying

$$\max\{|a_{n+1,n+1}|, \frac{r_n + 1}{2}\} < r_{n+1} < 1.$$

This completes the inductive step.

By properties (2) and (4) the sequence f_n converges locally uniformly in Δ to a holomorphic map $F \colon \Delta \to X$. Aided by property (1) we also control $d(f(z), F(z))$ for $z \in \Delta_r$. Since the Poincaré metric is complete, property (5) insures that for every fixed $j \in \mathbb{N}$ the sequence $a_{j,n} \in \Delta$ ($n = j, j+1, \dots$) has an accumulation point b_j inside of Δ, and (3) implies $F(b_j) = s_j$ for $j = 1, 2, \dots$. Hence $S \subset F(\Delta)$. □

4. Singular spaces

We use an example of Kaliman and Zaidenberg [7] to show that for a complex spaces X with singularities the set of maps $\Delta \to X$ with dense image need not be dense in $Hol(\Delta, X)$. We denote by $Sing(X)$ the singular locus of X.

Proposition 3. *There is a singular compact complex surface S, a non-constant holomorphic map $f \colon \Delta \to S$ and an open neighbourhood Ω of f in $Hol(\Delta, S)$ such that $g(\Delta) \subset Sing(S)$ for every $g \in \Omega$.*

Proof. In [7] Kaliman and Zaidenberg constructed an example of a singular surface S with normalization $\pi \colon Z \to S$ such that S contains a rational curve $C \simeq \mathbb{P}^1$ while Z is smooth and hyperbolic. Denote by d_Z the Kobayashi distance function on Z. We choose two distinct points $p, q \in C$ and open relatively compact neighbourhoods V of p and W of q in S such that $\overline{V} \cap \overline{W} = \emptyset$. The preimages $\pi^{-1}(\overline{V})$ and $\pi^{-1}(\overline{W})$ in Z are also compact, and since Z is hyperbolic we have

$$r = \min\{d_Z(x, y) \colon x \in \pi^{-1}(\overline{V}), y \in \pi^{-1}(\overline{W})\} > 0.$$

Fix a point $a \in \Delta$ with $0 < d_\Delta(0, a) < r$ and let Ω consist of all holomorphic maps $g \colon \Delta \to S$ satisfying $g(0) \in V$ and $g(a) \in W$. Since both p and q are lying on the rational curve C, there is a holomorphic map $g \colon \Delta \to C$ with $g(0) = p \in V$ and $g(a) = q \in W$; hence the set Ω is not empty. Clearly Ω is open in $Hol(\Delta, S)$.

To conclude the proof it remains to show that $g(\Delta) \subset Sing(S)$ for all $g \in \Omega$. Indeed, a holomorphic map $g \colon \Delta \to S$ with $g(\Delta) \not\subset Sing(S)$ admits a holomorphic lifting $\widetilde{g} \colon \Delta \to Z$ with $\pi \circ \widetilde{g} = g$. If $g \in \Omega$ then by construction

$$d_Z(\widetilde{g}(0), \widetilde{g}(a)) \geq r > d_\Delta(0, a)$$

which violates the distance decreasing property for the Kobayashi pseudometric. This contradiction establishes the claim. □

In particular, we see that in this example the set of all holomorphic maps $f \colon \Delta \to S$ with dense image *does not* constitute a dense subset of $Hol(\Omega, S)$.

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