STRONGLY PSEUDOCONVEX DOMAINS AS SUBVARIETIES OF COMPLEX MANIFOLDS

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Dedicated to Edgar Lee Stout.

Abstract. In this paper we obtain existence and approximation results for closed complex subvarieties that are normalized by strongly pseudoconvex Stein domains. Our sufficient condition for the existence of such subvarieties in a complex manifold X is expressed in terms of the Morse indices and the number of positive Levi eigenvalues of an exhaustion function on X. Examples show that our conditions cannot be weakened in general. We obtain optimal results for subvarieties of this type in complements of compact complex submanifolds with Griffiths positive normal bundle; in the projective case these generalize classical theorems of Remmert, Bishop and Narasimhan concerning proper holomorphic maps and embeddings to $\mathbb{C}^n = \mathbb{P}^n \backslash \mathbb{P}^{n-1}$.

1. Introduction. An interesting and difficult problem in analytic geometry is to describe the closed complex subvarieties of a given complex (or algebraic) manifold X. The set of all compact subvarieties — the *Douady space* $\mathcal{D}(X)$ and its close relative, the *cycle space* $\mathcal{C}(X)$ (the *Chow variety* in the quasi-projective setting) — is itself a finite dimensional complex analytic space (see [2], [5], [13]). Noncompact subvarieties are in many aspects harder to deal with, and consequently not as well understood.

In the present paper we continue the investigation, begun in [14], of the existence and plenitude of subvarieties that arise as proper holomorphic images of strongly pseudoconvex Stein domains. In [14] we analyzed the one dimensional case — complex curves normalized by bordered Riemann surfaces. Here we study higher dimensional subvarieties of this type and obtain optimal results in terms of the Levi geometry and the Morse indices of an exhaustion function on the ambient manifold.

Let X be a complex manifold with the complex structure operator $J \in \mathrm{End}_{\mathbb{R}} TX$, $J^2 = -I$. The *Levi form* of a $\mathcal{C}^2$-function $\rho\colon X \to \mathbb{R}$ is

$$\mathcal{L}_\rho(x; v) = \langle dd^c \rho, v \wedge Jv \rangle, \quad x \in X, \ v \in T_x X,$$

where $d^c = -J^* \circ d = \mathrm{i}(\bar\partial - \partial)$ is the conjugate differential defined by $\langle d^c\rho, v\rangle = -\langle d\rho, Jv\rangle$. We have $dd^c = 2\,\mathrm{i}\partial\bar\partial$. Choosing local holomorphic coordinates $z =$

Manuscript received October 14, 2007; revised July 9, 2009.
Research supported by grants P1-0291 and J1-2043-0101, Republic of Slovenia.
American Journal of Mathematics 132 (2010), 0–00.

$(z_1,\ldots,z_n)$ near a point $x\in X$ and writing $v=\eta+\bar\eta$, where $\eta=\sum_{j=1}^n \eta_j \frac{\partial}{\partial z_j}|_x \in T_x^{1,0}X$, we have

$$\mathcal{L}_\rho(x;v)=\langle \partial\bar\partial\rho(x),\eta\wedge\bar\eta\rangle=\sum_{j,k=1}^n \frac{\partial^2\rho(x)}{\partial z_j\partial\bar z_k}\,\eta_j\,\bar\eta_k.$$

Our main result is the following.

THEOREM 1.1. *Assume that X is an n-dimensional complex manifold, Ω is an open subset of X, $\rho\colon\ \Omega\to(0,+\infty)$ is a smooth Morse function whose Levi form has at least r positive eigenvalues at every point of Ω for some $r\le n$, and for any pair of real numbers $0<c_1<c_2$ the set*

$$\Omega_{c_1,c_2}=\{x\in\Omega\colon\ c_1\le\rho(x)\le c_2\}$$

is compact. Let D be a smoothly bounded, relatively compact, strongly pseudoconvex domain in a Stein manifold S, and let $f_0\colon\ \bar D\to X$ be a continuous map that is holomorphic in D and satisfies $f_0(bD)\subset\Omega$. If

(a) *$r\ge 2d$, where $d=\dim_{\mathbb{C}} S$,*

or if

(b) *$r\ge d+1$ and ρ has no critical points of index $>2(n-d)$ in Ω,*

then f_0 can be approximated, uniformly on compacts in D, by holomorphic maps $f\colon\ D\to X$ such that $f(z)\in\Omega$ for every $z\in D$ sufficiently close to bD, and

$$\lim_{z\to bD}\rho(f(z))=+\infty.$$

Moreover, given an integer $k\in\mathbb{Z}_+$, f can be chosen to agree with f_0 to order k at each point in a given finite set $\sigma\subset D$.

The most interesting case is when X is noncompact, Ω is a union of connected components of $X\backslash K$ for some compact subset K of X, and $\rho\to+\infty$ along the noncompact ends of Ω. Theorem 1.1 then furnishes *proper holomorphic maps* $f\colon\ D\to X$ that approximate a given map f_0 uniformly on compacts in D. A typical situation is $\Omega=\{\rho>0\}$ where $\rho\colon\ X\to\mathbb{R}$ is an exhaustion function satisfying the stated properties on Ω.

A $\mathcal{C}^2$-function ρ on an n-dimensional complex manifold X whose Levi form has at least r positive eigenvalues at every point in an open set $\Omega\subset X$ is said to be *$(n-r+1)$-convex on* Ω (see [26]). All Morse indices of such function are $\le r+2(n-r)=2n-r$. (See Lemma 2.1 below for a quadratic normal form of such a function at a critical point. The Morse condition can always be achieved by a small perturbation of ρ in the fine $\mathcal{C}^2$-topology, keeping the above Levi convexity property of ρ.) Hence condition (a) in Theorem 1.1 implies that all

Morse indices of ρ in Ω are $\leq 2(n-d)$, and therefore condition (b) holds as well. When $d = 1$ (i.e., D is a bordered Riemann surface), conditions (a) and (b) are both equivalent to $r \geq 2$, and in this case Theorem 1.1 is essentially the same as [14, Theorem 1.1]. When $d > 1$, condition (b) is weaker than (a).

An n-dimensional complex manifold X is said to be *q-convex* if it admits an exhaustion function ρ: $X \to \mathbb{R}$ that is q-convex on $\{\rho > c\}$ for some $c \in \mathbb{R}$; X is *q-complete* if ρ can be chosen q-convex on all of X (see [1], [26]). By approximation we can assume that ρ is $\mathcal{C}^\infty$-smooth. We have the following corollary of Theorem 1.1(a).

COROLLARY 1.2. *Let X be an n-dimensional complex manifold, and let $D \Subset S$ be a d-dimensional strongly pseudoconvex domain as in Theorem 1.1. Assume that $2d \leq n$ and $q \in \{1, \dots, n-2d+1\}$. Then the following hold:*

(a) *If X is q-convex then there exists a proper holomorphic map $D \to X$.*

(b) *If X is q-complete then every continuous map $\bar{D} \to X$ that is holomorphic in D can be approximated, uniformly on compacts in D, by proper holomorphic maps $D \to X$.*

Theorem 1.1 and Corollary 1.2 apply to any Stein manifold D with compact closure $\bar{D}$ and smooth strongly pseudoconvex boundary bD. Indeed, such D is equivalent to a smoothly bounded strongly pseudoconvex domain in a Stein manifold, and even in an affine algebraic manifold, by a biholomorphism extending smoothly to the boundary (see [6], [36], [52]). For the general theory of Stein manifolds we refer to [31], [37].

The image $V = f(D)$ of a proper holomorphic map f: $D \to X$ is a closed complex subvariety of X (see Remmert [47]). If the generic fiber of f is a single point of D (which is easily ensured by a suitable choice of the initial map f_0), then f: $D \to V$ is a normalization map of the subvariety V.

Our proof of Theorem 1.1 (see §5) involves three main analytic techniques. When $d = \dim D = 1$, D is a bordered Riemann surface, and in this case Theorem 1.1 essentially coincides with [14, Theorem 1.1]. The higher dimensional case requires a considerably more delicate technique for lifting the boundary of D (considered as a subset of X via a map $\bar{D} \to X$) to higher levels of ρ. The main local lifting lemma (see Lemma 5.3) employs special holomorphic peak functions that reach their maximum along certain Legendrian (complex tangential) submanifolds of maximal real dimension $d - 1$ in bD. Its proof mainly relies on the work of Dor [12] (see also Hakim [32] and Stensønes [51]). The idea of using such peak functions goes back to the construction of inner functions by Hakim and Sibony (see [33]) and Løw (see [40]); these undoubtedly belong among the most intricate and beautiful results in complex analysis.

Each local modification is patched with the previous global map $\bar{D} \to X$ by the method of *gluing holomorphic sprays* developed in [14] (see §4 below). This technique effectively replaces the $\bar{\partial}$-equation which cannot be used directly in a nonlinear setting. However, the lemma on gluing of sprays from [14] depends on the existence of a bounded linear solution operator for the $\bar{\partial}$-equation at the level of $(0,1)$-forms.

To avoid the critical points of ρ (where the estimates in the lifting process cannot be controlled) we adapt a method that was developed (for strongly plurisubharmonic functions) in [18]. We first ensure that the boundary of D (considered as a subset of X) avoids the stable manifold of any critical point of ρ; this is possible by general position, provided that all Morse indices of ρ are $\leq 2(n-d)$. In order to lift bD over the critical level at a critical point $p \in \Omega$ we construct a new noncritical function τ, with the same Levi convexity properties as ρ, such that $\{\tau \leq 0\}$ contains $\{\rho \leq c\}$ for some $c < \rho(p)$, and it also contains the local stable manifold of p for the gradient flow of ρ (see Lemma 3.1). Using the lifting procedure with τ we can push bD into $\{\rho > \rho(p)\}$, and the construction may proceed.

In the remainder of this introduction we discuss further corollaries and examples related to Theorem 1.1.

Example 1.3. Condition (b) in Theorem 1.1 cannot be weakened for any pair of dimensions $1 \leq d < n$.

Given integers $1 \leq d < n$, set $m = n - d + 1 \in \{2, \ldots, n\}$. Let $\mathbb{T}^m = \mathbb{C}^m/\Gamma$ be a complex torus of dimension m that is not projective-algebraic, and that furthermore does not contain any closed complex curves. (Most tori of dimension > 1 are such; for a specific example with $m = 2$ see [53, p. 222].) Set

$$X = \mathbb{T}^m\backslash\{p\} \times \mathbb{C}^{n-m} = \mathbb{T}^m\backslash\{p\} \times \mathbb{C}^{d-1}. \tag{1.1}$$

Choose an exhaustion function $\tau\colon\ \mathbb{T}^m\backslash\{p\} \to \mathbb{R}$ that equals $|y - y(p)|^{-2}$ in some local holomorphic coordinates y on $\mathbb{T}^m$ near p. The exhaustion function $\rho(y,w) = \tau(y) + |w|^2$ on X has no critical points in a deleted neighborhood of p, and its Levi form has $1 + n - m = d$ positive eigenvalues near $\{p\} \times \mathbb{C}^{d-1}$. Thus X satisfies condition (b) in Theorem 1.1 for domains D of dimension $< d$, but not for domains of dimension $\geq d$.

We claim that no d-dimensional Stein manifold D admits a proper holomorphic map to the manifold X (1.1). Indeed, suppose that $f\colon\ D \to X$ is such a map. Let $\pi\colon\ X \to \mathbb{C}^{d-1}$ denote the projection $\pi(y,w) = w$ onto the second factor. Consider the holomorphic map $\pi \circ f\colon\ D \to \mathbb{C}^{d-1}$. By dimension reasons there exists a point $w \in \mathbb{C}^{d-1}$ for which the fiber $\Sigma = \{z \in D\colon\ \pi(f(z)) = w\}$ is a subvariety of positive dimension in D. Since D is Stein, Σ contains a one dimensional subvariety C, and $f(C)$ is then a closed complex curve in $\mathbb{T}^m\backslash\{p\} \times \{w\}$. Since a point is a removable singularity for positive dimensional analytic subvarieties [48], it follows that $\overline{f(C)}$ is a nontrivial closed complex curve in $\mathbb{T}^m \times \{w\}$, a contradiction.

Interestingly enough, the manifold X (1.1) admits plenty of *nonproper* holomorphic maps $S \to X$ from any Stein manifold S. Indeed, X enjoys the following *Oka property* (see [19, Corollary 1.5 (ii)]):

Any continuous map $f_0\colon S \to X$ from a Stein manifold S is homotopic to a holomorphic map $f\colon S \to X$; if in addition f_0 is holomorphic in a neighborhood of a compact $\mathcal{O}(S)$-convex subset $K \subset S$, then f can be chosen to approximate f_0 as close as desired uniformly on K.

This shows that properness of a holomorphic map $f\colon D \to X$ is a very restrictive condition irrespectively of the codimension $\dim X - \dim D$.

Theorem 1.1 gives interesting information on the existence of proper holomorphic maps of strongly pseudoconvex domains into complements of certain complex submanifolds. For example, if A is a compact complex submanifold of complex codimension q in a projective space $X = \mathbb{P}^n$, then $\Omega = \mathbb{P}^n \backslash A$ admits a q-convex exhaustion function without critical points close to A (Barth [3]; the manifold $\mathbb{P}^n \backslash \mathbb{P}^{n-q}$ is even q-complete.) Thus condition (b) in Theorem 1.1 holds when $r = n - q + 1 > \dim D$ or, equivalently, $\dim D \le \dim A$. This gives the first part of the following corollary; for the second part we apply a result of Schneider (see [49, Corollary 2]).

Corollary 1.4. *If A is a compact complex submanifold of $\mathbb{P}^n$ then every smoothly bounded, relatively compact, strongly pseudoconvex Stein domain D of dimension $\dim D \le \dim A$ admits a proper holomorphic map $D \to \mathbb{P}^n \backslash A$. In particular, if $\dim D < n$ then D admits a proper holomorphic map into the complement $\mathbb{P}^n \backslash A$ of any nonsingular complex hypersurface A in $\mathbb{P}^n$. The analogous conclusion holds for maps $D \to X \backslash A$, where A is a compact complex submanifold of a complex manifold X with Griffiths positive normal bundle $N_{A|X}$.*

Corollary 1.4 generalizes Bishop's theorem (see [4]) on the existence of proper holomorphic maps $D \to \mathbb{C}^n = \mathbb{P}^n \backslash \mathbb{P}^{n-1}$ for $n > \dim D$. While Bishop's theorem holds for any Stein manifold D of dimension $< n$, in the general situation considered here one must restrict to Kobayashi hyperbolic domains since the complement $\mathbb{P}^n \backslash A$ of a generic hypersurface $A \subset \mathbb{P}^n$ of sufficiently high degree is hyperbolic.

The conclusion of Corollary 1.4 fails when $\dim D > \dim A$. Indeed, the closure of $V = f(D)$ in $\mathbb{P}^n$ would be a closed complex subvariety of $\mathbb{P}^n$ by the Remmert-Stein theorem (see [48, 27, p. 354]), hence $f(D) = \overline{V} \backslash A$ would be quasi-projective algebraic (the difference of two closed projective varieties). This is clearly impossible. It is easily seen that a proper holomorphic map $f\colon D \to X \backslash A$ cannot extend continuously (as a map to X) to any boundary point of D.

Our next corollary generalizes classical results of Remmert [46], Bishop [4], and Narasimhan [42] on immersions and embeddings of strongly pseudoconvex domains into Euclidean spaces, as well as results of Dor [11], [12] where the target manifold X is a domain of holomorphy in $\mathbb{C}^n$.

Corollary 1.5. *Assume that D is a smoothly bounded, relatively compact, strongly pseudoconvex domain in a Stein manifold S, X is a Stein manifold, and $f_0\colon \bar{D} \to X$ is a continuous map that is holomorphic in D.*

(i) *If* $\dim X \geq 2\dim D$, *then* f_0 *can be approximated uniformly on compacts in* D *by proper holomorphic immersions* $D \to X$.

(ii) *If* $\dim X \geq 2\dim D+1$, *then* f_0 *can be approximated uniformly on compacts in* D *by proper holomorphic embeddings* $D \hookrightarrow X$.

Corollary 1.5 is a consequence of Theorem 1.1 (condition (a) holds since a Stein manifold X is 1-complete), except for the claim that f can be chosen an immersion (resp. an embedding). The latter conditions are easily built into the construction by applying a general position argument at every step of the inductive process.

Further results on holomorphic immersions and embeddings in $X = \mathbb{C}^n$ can be found in [16], [23], [22], [45], [50], [54], [55]; for embeddings into special domains such as balls and polydiscs see also [10], [17], [24], [32], [41], [51].

Assume now that X is a quasi-projective algebraic manifold. We shall see that in this case every subvariety $V = f(D) \subset X$, obtained by the proof of Theorem 1.1, is a limit of domains contained in algebraic varieties in X and normalized by D.

By a theorem of Stout [52] (see also [9], [39]) we can assume that D in Theorem 1.1 is a smoothly bounded, strongly pseudoconvex, Runge domain in an affine algebraic manifold $S \subset \mathbb{C}^N$ of pure dimension d. A holomorphic map f from an open set $U \subset S$ to a quasi-projective algebraic variety X is said to be *Nash algebraic* (see Nash [43]) if its graph

$$G_f = \{(z,f(z)) \in S \times X\colon\ z \in U\}$$

is contained in a pure d-dimensional algebraic subvariety of $S\times X$. We then have the following result (c.f. [14, Corollary 1.2] for $d = 1$).

COROLLARY 1.6. *Assume that* X *is a quasi-projective algebraic manifold,* $\rho\colon X \to \mathbb{R}$ *is a smooth exhaustion function that satisfies one of the conditions in Theorem 1.1 on the set* $\Omega = \{x \in X\colon\ \rho(x) > 0\}$, *and* $D \Subset S$ *is a smoothly bounded, strongly pseudoconvex Runge domain in an affine algebraic manifold* S. *Given a map* $f_0\colon \bar{D} \to X$ *as in Theorem 1.1, with* $f_0(bD) \subset \Omega$, *there is a sequence of Nash algebraic maps* $f_j\colon U_j \to X$, *defined in open sets* $\bar{D} \subset U_j \subset S$, *such that* $f_j(bD) \subset \Omega$,

$$\lim_{j\to\infty} \big(\inf\{\rho\circ f_j(z)\colon\ z \in bD\}\big) \to +\infty,$$

and the sequence $f_j|_D$ *converges to a proper holomorphic map* $f\colon D \to X$ *as* $j \to \infty$. *Furthermore,* f *can be chosen to approximate* f_0 *as close as desired uniformly on a given compact subset of* D.

The image $f_j(U_j)$ of the Nash algebraic map f_j in Corollary 1.6 is contained in a pure d-dimensional algebraic subvariety Γ_j of X (the projection to X of an algebraic subvariety in $S\times X$ containing the graph of f_j). As $j \to \infty$, the domains

$f_j(D) \subset \Gamma_j$ converge to the subvariety $f(D) \subset X$, while their boundaries $f_j(bD)$ tend to infinity in X.

Corollary 1.6 is seen exactly as [14, Corollary 1.2] by combining the proof of Theorem 1.1 with the approximation theorems of Demailly, Lempert and Shiffman (see [9, Theorem 1.1]) and Lempert (see [38, Theorem 1.1]).

Here is another natural question:

Question 1.7. *When is a continuous map $D \to X$ from a strongly pseudoconvex Stein domain D to a complex manifold X homotopic to a proper holomorphic map?*

The following result in this direction generalizes Corollaries 1.5 and 1.6 in [14] which concern the one dimensional case; the proofs given there also apply in our case by using Theorem 1.1. The Oka property was defined in Example 1.3; for more details see [19].

COROLLARY 1.8. *Let $D \Subset S$ be a smoothly bounded, strongly pseudoconvex domain in a d-dimensional Stein manifold S, and let X be a complex manifold of dimension $n \geq 2d$ that is $(n-2d+1)$-complete. Let J_S (resp. J_X) denote the complex structure operator on S (resp. on X).*

(i) *If $d \neq 2$ then for every continuous map $f_0\colon \bar{D} \to X$ there exists a Stein structure $\widetilde{J}_S$ on S that is homotopic to J_S and such that D is strongly $\widetilde{J}_S$-pseudoconvex, and there exists a proper $(\widetilde{J}_S, J_X)$-holomorphic map $f\colon D \to X$ homotopic to $f_0|_D$.*

(ii) *If $d = 2$ then the conclusion (i) holds after changing the $\mathcal{C}^\infty$ structure on S (i.e., the new Stein structure $\widetilde{J}_S$ may be exotic).*

(iii) *If X enjoys the Oka property then every continuous map $\bar{D} \to X$ is homotopic to a proper (J_S, J_X)-holomorphic map $D \to X$.*

For further results see Theorem 7.2, Theorem 7.3 and Corollary 7.4.

Organization of the paper. In §2 and §3 we analyze the behavior of a q-convex function near a Morse critical point. In §4 we recall the relevant results from [14] on the theory of holomorphic sprays. Theorem 1.1 is proved in §5. In §6 we recall the notions of Griffiths positivity and signature of a Hermitian holomorphic vector bundle, as well as their connection with the Levi convexity properties. This information is used in §7 where we study the existence of subvarieties as in Theorem 1.1 in complements of certain compact complex submanifolds.

Acknowledgments. We wish to thank the referee for very helpful remarks and suggestions which helped us to improve the presentation.

2. Quadratic normal form for critical points of q-convex functions. In this section we describe a quadratic normal form of a q-convex function ρ at a nondegenerate critical point p. In the following section this normal form will be used in the construction of a q-convex function τ that allows us to pass the critical level $\{\rho = \rho(p)\}$ by applying the noncritical case of our lifting construction with τ (instead of ρ).

Since our considerations are completely local, we assume that ρ is a real valued $\mathcal{C}^2$-function in an open neighborhood of the origin in $\mathbb{C}^n$, with a nondegenerate (Morse) critical point at 0, and $\rho(0) = 0$. Suppose that ρ is q-convex at 0 for some $q \in \{1, 2, \ldots, n+1\}$; this means that its Levi form $\mathcal{L}_\rho(0)$ has at least $r = n - q + 1$ positive eigenvalues (the remaining $s = q - 1$ eigenvalues can be of any sign). By a complex linear change of coordinates on $\mathbb{C}^n$ we can achieve that the subspace $\mathbb{C}^r \times \{0\}^s$ is spanned by (some of the) eigenvectors corresponding to the positive eigenvalues of $\mathcal{L}_\rho(0)$ and that 0 is a Morse critical point of $\rho(\cdot\,, 0)$. We denote the coordinates on $\mathbb{C}^n = \mathbb{C}^r \times \mathbb{C}^s = \mathbb{C}^r \times \mathbb{R}^{2s}$ by $z = (\zeta, u)$, where $\zeta = x + \mathrm{i}y \in \mathbb{C}^r$ $(x, y \in \mathbb{R}^r)$ and $u \in \mathbb{R}^{2s}$. By shrinking the domain of ρ to a sufficiently small polydisc $P = P^r \times P^s \subset \mathbb{C}^n$ around 0 we can assume that the function $\zeta \to \rho(\zeta, u)$ is strongly plurisubharmonic on P^r for each fixed $u \in P^s$.

Lemma 3 from [34], applied to the strongly plurisubharmonic function $\zeta \to \rho(\zeta, 0)$, gives a complex linear change of coordinates on $\mathbb{C}^r$ and a number $k \in \{0, 1, \ldots, r\}$ such that, in the new coordinates, we have

$$\rho(\zeta, 0) = \sum_{j=1}^{r} (\delta_j x_j^2 + \lambda_j y_j^2) + o(|\zeta|^2),$$

where $\lambda_j > 1$, $\delta_j = -1$ for $j = 1, \ldots, k$, and $\lambda_j \geq 1$, $\delta_j = +1$ for $j = k+1, \ldots, r$. Note that k is the Morse index of $\rho(\cdot\,, 0)$ at $\zeta = 0$.

Writing $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_r)$ we obtain

$$\rho(\zeta, 0) = -|x'|^2 + |x''|^2 + \sum_{j=1}^{r} \lambda_j y_j^2 + o(|\zeta|^2).$$

Consider the full second order Taylor expansion of ρ at $0 \in \mathbb{C}^n$:

$$\rho(z) = \rho(\zeta, u) = \rho(\zeta, 0) + \sum_{j=1}^{2s} u_j a_j(x, y) + \sum_{i,j=1}^{2s} c_{ij}\, u_i u_j + o(|z|^2).$$

Here $a_j(x, y) = \sum_{l=1}^{r} \alpha_{jl} x_l + \beta_{jl} y_l$ are real-valued linear functions on $\mathbb{C}^r = \mathbb{R}^{2r}$ and $c_{ij} = c_{ji}$ are real constants.

Our next aim is to remove the mixed terms $u_j a_j(x, y)$ by using a shear of the form $(\zeta, u) \mapsto (\zeta + h(u), u)$ for a suitable $\mathbb{R}$-linear map h: $\mathbb{R}^{2s} \to \mathbb{C}^r$; such transformation is holomorphic (indeed, affine linear) in the ζ-coordinates, and hence it preserves (strong) ζ-plurisubharmonicity. To find such h, we consider the critical point equation $\partial_\zeta \rho^{(2)}(\zeta, u) = 0$, where $\rho^{(2)}$ denotes the 2nd order homogeneous polynomial of ρ:

$$\frac{\partial \rho^{(2)}}{\partial x_i}(\zeta, u) = 2\delta_i x_i + \sum_{j=1}^{2s} u_j \alpha_{ji} = 0; \quad \frac{\partial \rho^{(2)}}{\partial y_i}(\zeta, u) = 2\lambda_i y_i + \sum_{j=1}^{2s} u_j \beta_{ji} = 0.$$

This system has a unique (linear) solution $\zeta = x + iy = h(u)$, and the quadratic map $\zeta \to \rho^{(2)}(\zeta + h(u), u)$ has a unique critical point at $\zeta = 0$ for every u. Writing $\rho(\zeta, u) = \widetilde{\rho}(\zeta + h(u), u)$, the function $\widetilde{\rho}$ is of the same form as ρ, but with $a_j(x, y) = 0$ for all $j = 1, \ldots, 2s$. We drop the tilde and denote the new function again by ρ.

The classical theorem of Sylvester furnishes an $\mathbb{R}$-linear transformation of the u-coordinates which puts $\sum_{i,j=1}^{2s} c_{ij}\, u_i u_j$ into a normal form $-|u'|^2 + |u''|^2$, where $u' = (u_1, \ldots, u_m)$ and $u'' = (u_{m+1}, \ldots, u_{2s})$ for some $m \in \{0, 1, \ldots, 2s\}$. This gives $\rho(\zeta, u) = \widetilde{\rho}(\zeta, u) + o(|\zeta|^2 + |u|^2)$ where

$$\widetilde{\rho}(\zeta, u) = -|x'|^2 - |u'|^2 + |x''|^2 + |u''|^2 + \sum_{j=1}^{r} \lambda_j y_j^2, \tag{2.1}$$

$\lambda_j > 1$ for $j = 1, \ldots, k$, and $\lambda_j \geq 1$ for $j = k+1, \ldots, r$. We shall say that (2.1) is a *quadratic normal form* for critical points of q-convex functions. Note that $k + m$ is the Morse index of ρ (or $\widetilde{\rho}$) at 0.

We summarize the above discussion in the following lemma; for the strongly pseudoconvex case see [35, Lemma 2.5].

LEMMA 2.1. (Quadratic normal form of a q-convex critical point) *Assume that X is an n-dimensional complex manifold and that $\rho\colon X \to \mathbb{R}$ is a $\mathcal{C}^2$-function with a nondegenerate critical point at $p_0 \in X$. If ρ is q-convex at p_0 for some $q \in \{1, \ldots, n+1\}$ then there exist:*

(i) *a local holomorphic coordinate map $z = (\zeta, w)\colon U \to \mathbb{C}^r \times \mathbb{C}^s$ on an open neighborhood $U \subset X$ of p_0, with $z(p_0) = 0$, $r = n - q + 1$ and $s = q - 1$,*

(ii) *a change of coordinates $\psi(z) = \psi(\zeta, w) = (\zeta + h(w), g(w))$ on $\mathbb{C}^n$ that is $\mathbb{R}$-linear in $w \in \mathbb{C}^s = \mathbb{R}^{2s}$, and*

(iii) *a normal form $\widetilde{\rho}(\zeta, u)$ of type (2.1),*

such that, setting $\phi(p) = \psi(z(p)) \in \mathbb{C}^n$ for $p \in U$, we have

$$\rho(p) = \rho(p_0) + \widetilde{\rho}(\phi(p)) + o(|\phi(p)|^2), \quad p \in U.$$

Furthermore, we can approximate ρ as close as desired in the $\mathcal{C}^2$-topology by a q-convex function ρ' that agrees with ρ outside of U and has a nice critical point *at p_0, in the sense that $\rho'(p) = \rho(p_0) + \widetilde{\rho}(\phi(p))$ near p_0. If ρ is $\mathcal{C}^r$-smooth for some $r \in \{2, 3, \ldots, \infty\}$ then ρ' can also be chosen $\mathcal{C}^r$-smooth.*

Proof. Everything except the claim in the penultimate sentence has been proved above. The latter is seen by taking

$$\rho'(p) = \rho(p_0) + \widetilde{\rho}(\phi(p)) + \chi\left(\epsilon^{-1}\phi(p)\right)\, o(|\phi(p)|^2),$$

where $\chi\colon \mathbb{C}^n \to [0, 1]$ is a smooth function that equals zero in the unit ball $\mathbb{B} \subset \mathbb{C}^n$, and equals one outside of $2\mathbb{B}$. When $\epsilon > 0$ decreases to zero, the $\mathcal{C}^2$-norm of the last summand tends to zero uniformly on U. □

3. Crossing a critical level of a q-convex function. Let $p_0 \in X$ be a *nice critical point* of a $\mathcal{C}^2$-function $\rho\colon X \to \mathbb{R}$ that is q-convex near p_0 and satisfies $\rho(p_0) = 0$ (see Lemma 2.1). Choose a neighborhood $U \subset X$ of p_0 and a coordinate map $\phi\colon U \to P$ onto a polydisc $P \Subset \mathbb{C}^n$ as in Lemma 2.1 such that the function $\widetilde{\rho} = \rho \circ \phi^{-1}\colon P \to \mathbb{R}$ is a q-convex normal form (2.1). Set $Q(y, x'', u'') = \sum_{j=1}^r \lambda_j y_j^2 + |x''|^2 + |u''|^2$; hence

$$\widetilde{\rho}(x + \mathrm{i}y, u) = -|x'|^2 - |u'|^2 + Q(y, x'', u''). \tag{3.1}$$

Let $c_0 \in (0, 1)$ be chosen sufficiently small such that

$$\{(x + \mathrm{i}y, u) \in \mathbb{C}^r \times \mathbb{R}^{2s}\colon\ |x'|^2 + |u'|^2 \le c_0,\ Q(y, x'', u'') \le 4c_0\} \subset P.$$

Set

$$\widetilde{E} = \{(x + \mathrm{i}y, u) \in \mathbb{C}^r \times \mathbb{R}^{2s}\colon\ |x'|^2 + |u'|^2 \le c_0,\ y = 0,\ x'' = 0,\ u'' = 0\}.$$

Its preimage $E = \phi^{-1}(\widetilde{E}) \subset U$ is an embedded disc of dimension $k+m$ (the Morse index of ρ at p_0) that is attached from the outside to the sublevel set $\{\rho \le -c_0\}$ along the sphere $bE \subset \{\rho = -c_0\}$. In the metric on U, inherited by ϕ from the standard metric in $\mathbb{C}^n$, E is the (local) stable manifold of p_0 for the gradient flow of ρ.

The following lemma generalizes [18, Lemma 6.7] to q-convex functions. (The cited lemma applies to $q = 1$, that is, to a strongly plurisubharmonic function ρ.)

Lemma 3.1. *(Notation and assumptions as above.) Assume that ρ is q-convex in the set $K_{c_0} = \{p \in X\colon -c_0 \le \rho(p) \le 3c_0\} \Subset X$ and that p_0 is the only critical point of ρ in K_{c_0}. Assume that a normal form of ρ at p_0 is given by (2.1), where $k \in \{1, \ldots, r\}$ and $\lambda = \min\{\lambda_1, \ldots, \lambda_k\} > 1$. Given a number t_0 with $0 < t_0 < (1 - \frac{1}{\lambda})^2 c_0$, there is a $\mathcal{C}^2$-function $\tau\colon \{\rho \le 3c_0\} \to \mathbb{R}$ enjoying the following properties (see Figure 1):*

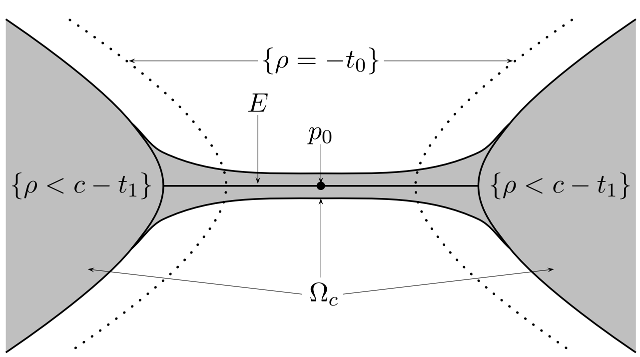

Figure 1. The set $\Omega_c = \{\tau < c\}$.

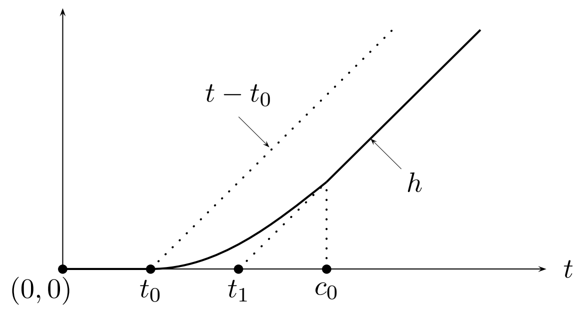

Figure 2. The function h.

(i) $\{\rho \le -c_0\} \cup E \subset \{\tau \le 0\} \subset \{\rho \le -t_0\} \cup E$,
(ii) $\{\rho \le c_0\} \subset \{\tau \le 2c_0\} \subset \{\rho < 3c_0\}$,
(iii) *τ is q-convex at every point of K_{c_0}, and*
(iv) *τ has no critical values in* $(0, 3c_0) \subset \mathbb{R}$.

If ρ is $\mathcal{C}^r$-smooth for some $r \in \{2, 3, \dots, \infty\}$ then τ can also be chosen smooth of class $\mathcal{C}^r$.

Each sublevel set $\Omega_c = \{\tau < c\}$ for $c \in (0, 2c_0)$ is a domain with $\mathcal{C}^2$ strongly q-convex boundary that contains $\{\rho \le -c_0\} \cup E$, and the latter set is a strong deformation retract of Ω_c (see Figure 1). As c decreases to 0, the sets $\Omega_c \cap \{\rho \ge -t_0\}$ decrease to the disc $E' = E \cap \{\rho \ge -t_0\}$. Finally, the domain Ω_{2c_0} contains the set $\{\rho < c_0\}$.

Proof. In [18, proof of Lemma 6.7, p. 178] the second named author constructed a smooth convex increasing function $h\colon \mathbb{R} \to [0, +\infty)$ enjoying the following properties (see Figure 2):
(i) $h(t) = 0$ for $t \le t_0$,
(ii) for $t \ge c_0$ we have $h(t) = t - t_1$ with $t_1 = c_0 - h(c_0) \in (t_0, c_0)$,
(iii) for $t_0 \le t \le c_0$ we have $t - t_1 \le h(t) \le t - t_0$, and
(iv) for all $t \in \mathbb{R}$ we have $0 \le \dot h(t) \le 1$ and $2t\ddot h(t) + \dot h(t) < \lambda$.

Given such h we define a function $\widetilde\tau\colon \mathbb{C}^n = \mathbb{C}^r \times \mathbb{R}^{2s} \to \mathbb{R}$ by

$$\widetilde\tau(\zeta, u) = -h\left(|x'|^2 + |u'|^2\right) + Q(y, x'', u''). \tag{3.2}$$

Its critical locus is $\{|x'|^2 + |u'|^2 \le t_0,\ x'' = 0,\ y = 0,\ u'' = 0\} \subset \widetilde E$ and the corresponding critical value is zero. From the property (iv) of h and [18, Lemma 6.8] we see that $\widetilde\tau(\cdot\,, u)$ is strongly plurisubharmonic on $\mathbb{C}^r$ for every fixed value of $u \in \mathbb{R}^{2s}$. The cited lemma applies directly when $u' = 0$; in general we consider the translated function $h_c(t) = h(t + c)$ with $c = |u'|^2 > 0$; we have $\dot h_c(t) = \dot h(t + c) \le 1 < \lambda$ and $2t\ddot h_c(t) + \dot h_c(t) \le 2(t + c)\ddot h(t + c) + \dot h(t + c) < \lambda$ (we used $\ddot h \ge 0$ and the property (iv) of h). Lemma 6.8 in [18] now gives the desired conclusion concerning the function $\zeta \to \widetilde\tau(\zeta, u) = -h_c(|x'|^2) + Q(y, x'', u'')$, where $c = |u'|^2$.

Comparing the definitions of $\widetilde\rho$ (3.1) and $\widetilde\tau$ (3.2), and taking into account the properties of h, we see that the following conditions hold:

(a) $\widetilde\rho \le \widetilde\tau \le \widetilde\rho + t_1$ (since $t - t_1 \le h(t) \le t$ for all $t \ge 0$),

(b) $\widetilde\rho + t_0 \le \widetilde\tau$ on the set $\{|x'|^2 + |u'|^2 \ge t_0\}$ (from (ii) and (iii)), and

(c) $\widetilde\tau = \widetilde\rho + t_1$ on the set $\{|x'|^2 + |u'|^2 \ge c_0\}$ (from (ii)).

Let $V = \{p \in X\colon\ \rho(p) \le 3c_0\}$. We define a function $\tau\colon\ V \to \mathbb{R}$ by

$$\tau = \widetilde\tau \circ \phi \ \text{ on } U \cap V, \quad \tau = \rho + t_1 \ \text{ on } V \backslash U.$$

Property (c) implies that both definitions of τ agree on the set

$$\{p \in U \cap V\colon\ |x'(p)|^2 + |u'(p)|^2 \ge c_0\}.$$

(Here $x'(p)$ and $u'(p)$ denote the corresponding components of $\phi(p) \in \mathbb{C}^n$.) Since $\{p \in U \cap V\colon\ |x'(p)|^2 + |u'(p)|^2 \le c_0\} \subset \{p \in U\colon\ |x'(p)|^2 + |u'(p)|^2 \le c_0,\ Q(y(p), x''(p), u''(p)) \le 4c_0\}$ and the latter set is compactly contained in U, we see that τ is well defined on V. The stated properties now follow immediately. In particular, since $\widetilde\tau$ is strongly plurisubharmonic on $\mathbb{C}^r \times \{u\} \subset \mathbb{C}^n$ ($u \in \mathbb{R}^{2s}$) and the coordinate map $\phi\colon\ U \to P$ is holomorphic on $\Sigma_u = \phi^{-1}(\mathbb{C}^r \times \{u\}) \subset U$, the restriction of τ to each $r = (n - q + 1)$-dimensional complex submanifold Σ_u of U is strongly plurisubharmonic. Since these submanifolds form a smooth nonsingular foliation of U with holomorphic leaves, τ is q-convex in $U \cap V$, while on $V \backslash U$ it is just a translate of ρ by a constant. □

Remark 3.2. *In Lemma 3.1 we exclude the case $k = 0$ when ρ has a local minimum at the critical point p_0 in the Levi-positive directions. This case need not be considered in the proof of Theorem 1.1 since the boundary of D in X cannot approach such a point from below during the lifting process.*

4. Holomorphic sprays. In the proof of Theorem 1.1 we use sprays of maps to globalize local corrections made near a small part of the boundary. For this purpose we recall from [14], [15], [20] the relevant results concerning holomorphic sprays, adjusting them to the applications in this paper.

Definition 4.1. Let $\ell \ge 2$, $r \in \{0, \ldots, \ell\}$ and $k \in \mathbb{Z}_+$ be integers. Assume that X is a complex manifold, D is a relatively compact strongly pseudoconvex domain with $\mathcal{C}^\ell$ boundary in a Stein manifold S, and σ is a finite set of points in D. A *spray of maps of class $\mathcal{A}^r(D)$ with the exceptional set σ of order k* (and with values in X) is a map $f\colon\ \bar D \times P \to X$, where P (the *parameter set* of the spray) is an open subset of a Euclidean space $\mathbb{C}^m$ containing the origin, such that the following hold:

(i) f is holomorphic on $D \times P$ and of class $\mathcal{C}^r$ on $\bar D \times P$,

(ii) the maps $f(\cdot\,, 0)$ and $f(\cdot\,, t)$ agree on σ up to order k for $t \in P$, and

(iii) for every $z \in \bar{D}\backslash\sigma$ and $t \in P$ the map

$$\partial_t f(z,t)\colon\ T_t\mathbb{C}^n = \mathbb{C}^n \to T_{f(z,t)}X$$

is surjective (the *domination property*).
We shall call $f_0 = f(\cdot\,,0)$ the *core* (or *central*) map of the spray f.

The following lemma is essentially [14, Lemma 4.2] for the case of sprays of maps. As it is remarked in the first line of its proof in [14], the assumption $r \geq 2$ is needed only for the existence of a Stein neighborhood. Using [20, Corollary 1.3] instead of [14, Theorem 2.6] we obtain the same result for all $r \in \mathbb{Z}_+$. In §5 below we shall use these results with $r = 0$.

Lemma 4.2. (Existence of sprays) *Assume that ℓ, r, k, D, σ and X are as in Definition 4.1. Given a map $f_0\colon \bar{D} \to X$ of class $\mathcal{A}^r(D)$ to a complex manifold X, there exists a spray $f\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$, with the exceptional set σ of order k, such that $f(\cdot\,,0) = f_0$.*

Definition 4.3. Let $\ell \geq 2$ be an integer. A pair of open subsets $D_0, D_1 \Subset S$ in a Stein manifold S is said to be a *Cartan pair* of class $\mathcal{C}^\ell$ if

(i) D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are strongly pseudoconvex with $\mathcal{C}^\ell$ boundaries, and

(ii) $\overline{D_0\backslash D_1} \cap \overline{D_1\backslash D_0} = \emptyset$ (the separation property).

The following is the main result on gluing sprays (see [14, Proposition 4.3]). The key ingredient in the proof is a Cartan-type splitting lemma; for a simple proof see [20, Lemma 3.2].

Proposition 4.4. (Gluing sprays) *Let (D_0, D_1) be a Cartan pair of class $\mathcal{C}^\ell$ ($\ell \geq 2$) in a Stein manifold S (Def. 4.3). Set $D = D_0 \cup D_1$, $D_{0,1} = D_0 \cap D_1$. Let X be a complex manifold. Given integers $r \in \{0, 1, \ldots, \ell\}$, $k \in \mathbb{Z}_+$, and a spray $f\colon \bar{D}_0 \times P_0 \to X$ of class $\mathcal{A}^r(D_0)$ with the exceptional set σ of order k such that $\sigma \cap \bar{D}_{0,1} = \emptyset$, there is an open set $P \Subset P_0$ containing $0 \in \mathbb{C}^n$ satisfying the following.*

For every spray $f'\colon \bar{D}_1 \times P_0 \to X$ of class $\mathcal{A}^r(D_1)$, with the exceptional set σ' of order k, such that f' is sufficiently $\mathcal{C}^r$-close to f on $\bar{D}_{0,1} \times P_0$ and $\sigma' \cap \bar{D}_{0,1} = \emptyset$, there exists a spray $g\colon \bar{D} \times P \to X$ of class $\mathcal{A}^r(D)$, with the exceptional set $\sigma \cup \sigma'$ of order k, enjoying the following properties:

(i) *the restriction $g\colon \bar{D}_0 \times P \to X$ is close to $f\colon \bar{D}_0 \times P \to X$ in the $\mathcal{C}^r$-topology (depending on the $\mathcal{C}^r$-distance of f and f' on $\bar{D}_{0,1} \times P_0$),*

(ii) *the core map $g_0 = g(\cdot\,,0)$ is homotopic to $f_0 = f(\cdot\,,0)$ on $\bar{D}_0$, and g_0 is homotopic to $f'_0 = f'(\cdot\,,0)$ on $\bar{D}_1$, and*

(iii) *g_0 agrees with f_0 up to order k on σ, and it agrees with f'_0 up to order k on σ'.*

Remark 4.5. *It follows from the proof in [14] that, in addition to the above, we have $g(z,t) \in \{f'(z,s)\colon s \in P_0\}$ for each $z \in D_1$ and $t \in P$.*

5. Proof of Theorem 1.1. The scheme of proof is exactly as in [14, proof of Theorem 1.1]. A holomorphic map $f\colon D \to X$ satisfying the conclusion of Theorem 1.1 is obtained as a locally uniform limit $f = \lim_{j\to\infty} f_j$ in D of a sequence of continuous maps $f_j\colon \bar{D} \to X$ that are holomorphic in D. At every step of the inductive construction we obtain the next map f_{j+1} from f_j by first lifting a part of the boundary $f_j(bD)$, lying in a local chart of X, to higher levels of ρ, while at the same time taking care not to drop the boundary substantially lower with respect to ρ. The local modification, provided by Lemma 5.3, uses special holomorphic peak functions; its proof relies on the work of A. Dor [12]. To pass a critical level of ρ we use methods developed in §3 above.

For technical reasons we work with sprays of maps (see §4). This allows us to patch any local modification, furnished by Lemma 5.3, with the given global map $\bar{D} \to X$ by appealing to Proposition 4.4 above. When talking of sprays, we adopt the following convention:

All sprays in this section are assumed to be of class $\mathcal{A}^0(D)$, and (unless otherwise specified) their exceptional set σ of order k equals the finite set σ from Theorem 1.1. We shall accordingly omit the phrases "of class $\mathcal{A}^0(D)$" and "with exceptional set σ of order k" when there is no ambiguity.

In the lifting process we have to consider two cases: The first is to lift the boundary of D across noncritical levels of ρ, and the second is crossing a critical level set of ρ. We reduce the second case to the first one by using Lemma 3.1 (see Lemma 5.5 below).

For maps from strongly pseudoconvex domains to a Euclidean space, relevant lifting techniques using holomorphic peak functions have been developed by several authors. The following result was proved by A. Dor (see [12, Lemma 1]; here we use Dor's original notation). The term "normalized-3" indicates that the complex Hessian is globally bounded from below with factor 3, that is, $\mathcal{L}_\rho(x; v) \geq 3|v|^2$. Since we do not wish to normalize our exhaustion function, we shall need one additional constant (denoted μ_0 in Lemma 5.2 below) in the corresponding estimates.

LEMMA 5.1. (A. Dor, [12, Lemma 1]) *Assume that $N \geq 2$ and $M \geq N+1$ are integers, Ω is a domain in $\mathbb{C}^M$, $\rho\colon \Omega \to \mathbb{R}$ is a smooth normalized-3 plurisubharmonic function ($\mathcal{L}_\rho(x; v) \geq 3|v|^2$), $D \Subset \mathbb{C}^N$ is a strongly pseudoconvex domain with smooth boundary, and $z_0 \in bD$. Let $K_1 \subset \Omega$ be a compact subset such that $d\rho \neq 0$ on K_1. Then there exist*

- *a constant $\epsilon_0 \in (0, 1)$ that depends only on N and on the domain D,*
- *constants $\gamma_0 \in (0, 1)$ and $C > 1$ that depend only on K_1, ρ and Ω, and*
- *a neighborhood $U \subset bD$ of z_0 that depends only on z_0 and D,*

such that the following hold. Given a smooth map $f\colon \bar{D} \to \Omega$ that is holomorphic on D, a compact subset $K \subset D$, a number $\epsilon > 0$, and a continuous map $\gamma\colon bD \to (0, \gamma_0]$, there is a smooth map $g\colon \bar{D} \to \mathbb{C}^M$ that is holomorphic on D and enjoys the following properties:

(i) *$f(z) + g(z) \in \Omega$ for $z \in \bar{D}$,*

(ii) $C|g(z)|^2 + \epsilon > \rho((f+g)(z)) - \rho(f(z)) > |g(z)|^2 - \epsilon$ *for* $z \in \bar{D}$,
(iii) $|g(z)| > \epsilon_0 \gamma(z)$ *for* $z \in U \cap f^{-1}(K_1)$,
(iv) $|g(z)| < \epsilon_0^{-1} \gamma(z)$ *for* $z \in bD$, *and*
(v) $|g(z)| < \epsilon$ *for* $z \in K$.

Although Dor stated this result only for strongly plurisubharmonic functions ρ, in the final pages of his paper he also proved it for q-convex functions with $q \le M - N$; he called such functions "locally $(N+1)$-dimensional plurisubharmonic". The proof in [12] is split into three parts. In the first part the author chooses a good system of peak functions near z_0 and obtains a constant $\epsilon_0 > 0$ that depends only on the geometry of the boundary bD near the chosen point $z_0 \in bD$, but is independent of the target domain X and of the function ρ. In the second step he constructs a local correction map that is defined in a neighborhood U of z_0 and enjoys the stated properties on U. In the last step this local map is patched with the original global map by solving a $\bar{\partial}$-equation on $\bar{D}$.

Since in our case X is a manifold (and not a Euclidean space as in [12]), we perform Dor's corrections on small pieces of $\bar{D}$ near the boundary of bD that are mapped to local charts of X, and then glue these corrections with the initial spray by the methods explained in §4. So we only need the following local version of Lemma 5.1 (before globalization). We adjust the notation to the one used in the remainder of this section, writing p instead of z_0, ϵ_p instead of ϵ_0, and g instead of $f+g$. We emphasize that Lemma 5.2 is what Dor actually proved in [12], and hence it does not require a proof.

LEMMA 5.2. *Let* $d \ge 2$, $n \ge d+1$, *and* $q \le n-d$ *be integers. Assume that* ω *is a domain in* $\mathbb{C}^n$, $\rho\colon \omega \to \mathbb{R}$ *is a smooth* q*-convex function,* K_ω *is a compact subset of* ω *such that* $d\rho \ne 0$ *on* K_ω, $D \Subset \mathbb{C}^d$ *is a domain with smooth boundary, and* $p \in bD$ *is a strongly pseudoconvex boundary point of* D. *Then there exist*
(a) *small balls* $V_p \Subset U_p$ *in* $\mathbb{C}^d$ *such that* $p \in V_p$,
(b) *a constant* $\epsilon_p \in (0,1)$ *depending only on* $U_p \cap bD$,
(c) *a constant* $\mu_0 > 0$ *depending only on* ρ *and* K_ω, *and*
(d) *a constant* $\gamma_0 \in (0,1)$ *depending on* $U_p \cap bD$, ρ *and* K_ω,
such that the following hold. Given an open subset $D_1 \subset U_p \cap D$, *an open subset* C *of* bD *contained in* V_p *such that* $\mathrm{dist}_{\mathbb{C}^d}(\overline{C}, bD_1 \backslash bD) > 0$, *a smooth map* $f\colon \bar{D}_1 \to \omega$ *that is holomorphic on* D_1, *a number* $\epsilon > 0$, *and a continuous map* $\gamma\colon bD \cap bD_1 \to (0, \gamma_0]$, *there is a smooth map* $g\colon \bar{D}_1 \to \omega$ *that is holomorphic on* D_1 *and enjoys the following properties:*
(i) $\rho(g(z)) - \rho(f(z)) > \mu_0 |g(z) - f(z)|^2 - \epsilon$ *for* $z \in \bar{D}_1$,
(ii) $|g(z) - f(z)| > \epsilon_p \gamma(z)$ *for* $z \in C \cap f^{-1}(K_\omega)$,
(iii) $|g(z) - f(z)| < \epsilon_p^{-1} \gamma(z)$ *for* $z \in bD \cap bD_1$, *and*
(iv) $|g(z) - f(z)| < \epsilon$ *for points* $z \in \bar{D}_1$ *such that* $\mathrm{dist}_{\mathbb{C}^d}(z, C) > \epsilon$.

The main sets in Lemma 5.2 are illustrated on Figure 3, with C shown as the dashed arc on $bD \cap bD_1$.

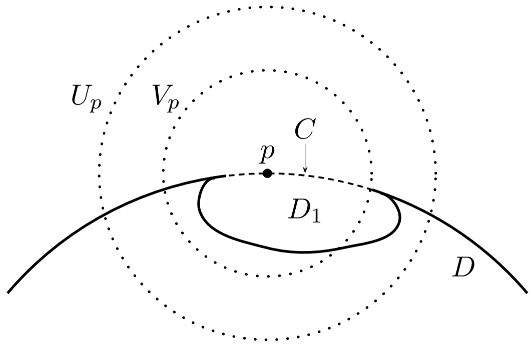

Figure 3. The sets in Lemma 5.2.

Using Lemma 5.2 we now prove our main modification lemma for the non-critical case. Note that f_0 always denotes the core map of a spray f.

LEMMA 5.3. *Let $d \geq 2$, $n \geq d+1$ and $k \geq 0$ be integers. Assume that X is an n-dimensional complex manifold endowed with a complete metric* dist, *Ω is an open subset of X, and ρ: $\Omega \to \mathbb{R}$ is a smooth function such that for a pair of real numbers $c_1 < c_2$ the set*

$$\Omega_{c_1,c_2} = \{x \in \Omega:\ c_1 \leq \rho(x) \leq c_2\}$$

is compact, $d\rho \neq 0$ on Ω_{c_1,c_2}, and the Levi form $\mathcal{L}_\rho$ of ρ has at least $d+1$ positive eigenvalues at every point of Ω_{c_1,c_2}.

Let $D \Subset S$ be a smoothly bounded, strongly pseudoconvex domain in a Stein manifold S and let σ be a finite set of points in D. Choose real numbers c_1', c_2' such that $c_1 < c_1' < c_2' < c_2$. Then there is a number $\delta > 0$ with the following property. Given a number $c \in [c_1', c_2']$, a compact set $K_D \subset D$, and a spray of maps f: $\bar{D} \times P \to X$ with the exceptional set σ of order k such that

$$f_0(z) \in \Omega\ (\forall z \in \overline{D\backslash K_D}), \quad \rho(f_0(z)) > c - \delta\ (\forall z \in bD),$$

there exist for each $\epsilon > 0$ an open set $P_0 \Subset P$ containing the origin and a spray g: $\bar{D} \times P_0 \to X$ with the exceptional set σ of order k such that

(i) *$g_0(z) \in \Omega$ and $\rho(g_0(z)) > c + \delta$ for $z \in bD$,*

(ii) *$g_0(z) \in \Omega$ and $\rho(g_0(z)) > \rho(f_0(z)) - \epsilon$ for $z \in \overline{D\backslash K_D}$,*

(iii) dist$(f_0(z), g_0(z)) < \epsilon$ *for $z \in K_D$, and*

(iv) *the maps f_0 and g_0 have the same k-jets at every point in σ, and g_0 is homotopic to f_0 relative to σ.*

Proof. We first explain the main idea. Lemma 5.2 provides a local step on both sides — locally with respect to the boundary bD, and locally on the level set of ρ in X. In every step of the inductive construction we lift the part of the image of the boundary bD that lies in a small coordinate neighborhood to higher levels of the exhaustion function ρ (see Sublemma 5.4). In finitely many such steps we push the image of the boundary outside a certain bigger sublevel set of ρ. Each step in the construction consists of finitely many substeps, and at each substep we only make corrections on the part of the boundary lying in a suitable coordinate neighborhood in S. Sprays are used at every substep to patch the local correction with the previous global map.

Now to the details. Let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{C}^n$, and let $s\mathbb{B}^n$ denote the ball of radius $s > 0$. Fix a number $c \in [c'_1, c'_2]$. We shall find a number $\delta > 0$ satisfying the conclusion of Lemma 5.3 for this value of c. It will be clear from the construction that δ can be chosen uniformly for all c' sufficiently close to c, and hence (by compactness) for all $c \in [c'_1, c'_2]$.

Since the level set $\{\rho = c\}$ is compact, there are finitely many holomorphic coordinate maps $h_i\colon \frac{5}{4}\mathbb{B}^n \to X$ ($i = 1, \ldots, N$) such that

$$\{\rho = c\} \subset \bigcup_{i=1}^{N} h_i(\tfrac{1}{4}\mathbb{B}^n) \subset \bigcup_{i=1}^{N} h_i(\tfrac{5}{4}\mathbb{B}^n) \subset \Omega_{c_1,c_2}.$$

For each point $p \in bD$ one can choose local holomorphic coordinates in S, and in this coordinate patch we obtain the sets U_p, V_p and a constant ϵ_p as in Lemma 5.2 (parts (a) and (b)). Choose open coverings $\{V_j\}_{j=1}^M$ and $\{U_j\}_{j=1}^M$ of bD such that each pair $V_j \Subset U_j$ corresponds to $V_{p_j} \Subset U_{p_j}$ for some point $p_j \in bD$, and $U_j \cap \sigma = \emptyset$ for each j. We also obtain the corresponding numbers $\epsilon_j = \epsilon_{p_j} > 0$.

Let $\epsilon_0 = \min\{\epsilon_1, \ldots, \epsilon_M\} > 0$. Using Lemma 5.2 (parts (c) and (d)) for the data $\bar{D} \cap U_j$ (in the local coordinates), $\omega = \frac{5}{4}\mathbb{B}^n$, $K_\omega = \overline{\mathbb{B}}^n$, and with ρ replaced by the function $\rho \circ h_i$, we also obtain constants γ_i^j and μ_i^j for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ (these correspond to γ_0, resp. to μ_0, in Lemma 5.2). Choose constants $\alpha > 0$ and $\beta > 0$ such that the following hold for $i = 1, \ldots, N$:

$$w, w' \in \overline{\mathbb{B}}^n \Longrightarrow \operatorname{dist}(h_i(w), h_i(w')) \le \alpha |w - w'|, \tag{5.1}$$

$$\Big(x \in h_i(\tfrac{7}{8}\mathbb{B}^n),\ x' \in X,\ \operatorname{dist}(x, x') < M\beta\Big) \Longrightarrow \Big(x' \in h_i(\mathbb{B}^n),\ |h_i^{-1}(x) - h_i^{-1}(x')| < \tfrac{1}{8N}\Big). \tag{5.2}$$

Set

$$\gamma = \min\{\gamma_i^j, \tfrac{\epsilon_0 \beta}{3\alpha}\} > 0, \qquad \mu = \min\{\mu_i^j\} > 0, \tag{5.3}$$

the minima being taken over all indices $i = 1, \ldots, N$ and $j = 1, \ldots, M$. (This choice of γ insures that our correction is small compared to the size of ω.) Finally, we choose a number δ such that

$$0 < \delta < \tfrac{1}{3}\mu\epsilon_0^2\gamma^2, \qquad \Omega_{c-\delta, c+2\delta} \subset \bigcup_{i=1}^{N} h_i\left(\tfrac{1}{4}\mathbb{B}^n\right). \tag{5.4}$$

We shall prove that this δ satisfies Lemma 5.3. We need the following:

SUBLEMMA 5.4. *Fix an index $i \in \{1, 2, \ldots, N\}$. Given a spray of maps $f'\colon \bar{D} \times P \to X$, with the exceptional set σ of order k and such that $f'_0(z) \in \Omega$ for $z \in \overline{D\backslash K_D}$, there exist for each $\epsilon' > 0$ an open set $P' \Subset P$ containing the origin and a spray of maps $g'\colon \bar{D} \times P' \to X$ with the exceptional set σ of order k such that $g'_0(z) \in \Omega$ for $z \in \overline{D\backslash K_D}$ and the following hold:*

(i′) *$\rho(g'_0(z)) > \rho(f'_0(z)) + 3\delta - \epsilon'$ for $z \in bD$ such that $f'_0(z) \in h_i(\frac{1}{2}\mathbb{B}^n)$,*

(ii′) *$\rho(g'_0(z)) > \rho(f'_0(z)) - \epsilon'$ for $z \in \overline{D\backslash K_D}$,*

(iii′) *$\mathrm{dist}(f'_0(z), g'_0(z)) < M\beta$ for $z \in \bar{D}$,*

(iv′) *$\mathrm{dist}(f'_0(z), g'_0(z)) < \epsilon'$ for $z \in K_D$, and*

(v′) *the maps f'_0 and g'_0 have the same k-jets at every point in σ.*

Proof. Let $f^0 = f'$ and $P^0 = P$. Recall that $bD \subset \bigcup_{j=1}^{M} V_j$. We inductively construct a finite decreasing sequence of parameter sets $P^0 \supset P^1 \supset \cdots \supset P^M$, with $0 \in P^{j+1} \Subset P^j$ for $j = 0, \ldots, M-1$, and a sequence of sprays $f^j\colon \bar{D} \times P^j \to X$ with the exceptional set σ of order k such that the following hold for $j = 0, 1, \ldots, M-1$:

(i†) $\rho(f_0^{j+1}(z)) > \rho(f_0^j(z)) + 3\delta - \frac{\epsilon'}{M}$ for every point $z \in bD \cap V_{j+1}$ such that $f_0^j(z) \in h_i(\frac{3}{4}\mathbb{B}^n)$,

(ii†) $\rho(f_0^{j+1}(z)) > \rho(f_0^j(z)) - \frac{\epsilon'}{M}$ for $z \in \overline{D\backslash K_D}$,

(iii†) $\mathrm{dist}\left(f_0^{j+1}(z), f_0^j(z)\right) < \beta$ for $z \in \bar{D}$,

(iv†) $\mathrm{dist}\left(f_0^{j+1}(z), f_0^j(z)\right) < \frac{\epsilon'}{M}$ for $z \in K_D$, and

(v†) the maps f_0^{j+1} and f_0^j have the same k-jets at every point in σ.

Assume for a moment that we have already constructed the sequences P^j and f^j. Let $P' = P^M$ and $g' = f^M$. Using (ii†)–(v†) repeatedly M times we see that properties (ii′)–(v′) in Sublemma 5.4 hold. To see that (i′) holds, fix a point $z \in bD$ such that $f_0^0(z) \in h_i(\frac{1}{2}\mathbb{B}^n)$. Choose an index j such that $z \in V_j$. By (iii†) and (5.2) it follows that $f_0^{j-1}(z) \in h_i(\frac{3}{4}\mathbb{B}^n)$, and thus (i†) gives $\rho(f_0^j(z)) > \rho(f_0^{j-1}(z)) + 3\delta - \frac{\epsilon'}{M}$. Using (ii†) repeatedly this implies $\rho(g'_0(z)) > \rho(f'_0(z)) + 3\delta - \epsilon'$. Therefore g' enjoys all required properties.

It remains to construct the sequences P^j and f^j. Assume inductively that we have already constructed $P^0, \ldots, P^j$ and $f^0, \ldots, f^j$ for some $j \in \{0, 1, \ldots, M-1\}$; we now explain how to find P^{j+1} and f^{j+1}. Set

$$C = bD \cap V_{j+1} \cap (f_0^j)^{-1}(h_i(\tfrac{13}{16}\mathbb{B}^n)).$$

Observe that the open set $D \cap V_{j+1} \cap \left(f_0^j\right)^{-1}\left(h_i\left(\frac{13}{16}\mathbb{B}^n\right)\right)$ is pseudoconvex, contained in $U_{j+1} \cap \left(f_0^j\right)^{-1}\left(h_i\left(\frac{7}{8}\mathbb{B}^n\right)\right) \subset U_{j+1} \cap \bar{D}$, and has positive distance to $\bar{D}\backslash\left(U_{j+1} \cap \left(f_0^j\right)^{-1}\left(h_i\left(\frac{7}{8}\mathbb{B}^n\right)\right)\right)$. Hence there is a smoothly bounded, strongly pseudoconvex domain D_1 contained in D such that

$$V_{j+1} \cap \left(f_0^j\right)^{-1}\left(h_i\left(\tfrac{13}{16}\mathbb{B}^n\right)\right) \subset \bar{D}_1 \subset U_{j+1} \cap \left(f_0^j\right)^{-1}\left(h_i\left(\tfrac{7}{8}\mathbb{B}^n\right)\right)$$

and

$$\mathrm{dist}_{\mathbb{C}^d}(\overline{C}, bD_1 \backslash bD) > 0.$$

The situation is as shown in Figure 3, with U_p replaced by U_{j+1} and V_p replaced by V_{j+1}.

Choose a smoothly bounded, strongly pseudoconvex domain $D_0 \subset D$, obtained by denting D slightly inward in a neighborhood of C, such that $D \backslash D_0 \subset U_{j+1}$, $bD_0 \cap \overline{C} = \emptyset$, and (D_0, D_1) is a Cartan pair such that $D_0 \cup D_1 = D$ (see Definition 4.3). Set

$$A_{i,j}(z,t) = h_i^{-1} \circ f^j(z,t) - h_i^{-1} \circ f^j(z,0).$$

There exists a smaller parameter set $0 \in P_0^j \subset P^j$ such that for $w \in \frac{15}{16}\mathbb{B}^n$, $z \in \bar{D}_1$ and $t \in P_0^j$ the following hold:

$$|A_{i,j}(z,t)| < \tfrac{1}{16}, \tag{5.5}$$

$$|\rho(h_i(w)) - \rho\left(h_i(w + A_{i,j}(z,t))\right)| < \tfrac{\epsilon'}{3M}, \tag{5.6}$$

$$\mathrm{dist}\left(h_i(w), h_i(w + A_{i,j}(z,t))\right) < \tfrac{\beta}{3}. \tag{5.7}$$

Applying Lemma 5.2 to the map $h_i^{-1} \circ f_0^j$ on $\bar{D}_1$, the constant function $\gamma(z) = \gamma$ (with γ as in (5.3)), and a number $\epsilon > 0$ (to be specified later), we obtain a map $g\colon \bar{D}_1 \to \mathbb{C}^n$ enjoying the following properties:

$$\rho(h_i(g(z))) > \rho(f_0^j(z)) + \mu|h_i^{-1} \circ f_0^j(z) - g(z)|^2 - \epsilon \quad \text{for } z \in \bar{D}_1, \tag{5.8}$$

$$|h_i^{-1} \circ f_0^j(z) - g(z)| > \epsilon_0\gamma \quad \text{for } z \in C, \tag{5.9}$$

$$|h_i^{-1} \circ f_0^j(z) - g(z)| < \epsilon_0^{-1}\gamma \quad \text{for } z \in bD_1 \cap bD, \text{ and} \tag{5.10}$$

$$|h_i^{-1} \circ f_0^j(z) - g(z)| < \epsilon \quad \text{for } z \in \bar{D}_1 \text{ such that } \mathrm{dist}_{\mathbb{C}^d}(z, C) > \epsilon. \tag{5.11}$$

If $\epsilon < \min\{\frac{\gamma}{\epsilon_0}, \mathrm{dist}_{\mathbb{C}^d}(C, bD_1 \backslash bD)\}$ then (5.10), (5.11) and the maximum principle imply that $|h_i^{-1} \circ f_0^j(z) - g(z)| < \frac{\gamma}{\epsilon_0}$ for $z \in \bar{D}_1$. By (5.1) we get

$$\mathrm{dist}\left(h_i(g(z)), f_0^j(z)\right) < \tfrac{\beta}{3} \text{ for } z \in \bar{D}_1.$$

By (5.2) and (5.7) this allows us to define a spray $f'\colon \bar{D}_1 \times P_0^j \to X$ (with empty exceptional set) by setting

$$f'(z,t) = h_i\left(g(z) + A_{i,j}(z,t)\right) \quad \text{for } z \in \bar{D}_1,\ t \in P^j_0.$$

If $\epsilon < \frac{\epsilon'}{3M}$ then it follows from (5.6), (5.8), (5.9) and the definition of δ that

$$\rho(f'(z,t)) > \rho(f^j_0(z)) + 3\delta - \tfrac{2\epsilon'}{3M} \quad \text{for } z \in C, \tag{5.12}$$

$$\rho(f'(z,t)) > \rho(f^j_0(z)) - \tfrac{2\epsilon'}{3M} \ \text{for } z \in \bar{D}_1. \tag{5.13}$$

If $\epsilon > 0$ is small enough then for every $z \in \overline{D_0 \cap D_1}$ we have $\mathrm{dist}_{\mathbb{C}^d}(z, C) > \epsilon$, and the properties (5.11) and (5.1) imply that

$$\mathrm{dist}\left(f'(z,t), f^j(z,t)\right) \le \alpha\epsilon \quad \text{for } (z,t) \in \overline{D_0 \cap D_1} \times P^j_0.$$

Finally, if $\epsilon > 0$ is small enough then we can glue the sprays f^j and f' by Proposition 4.4. This gives a smaller parameter set $0 \in P^{j+1} \subset P^j_0$ and a new spray $f^{j+1}\colon \bar{D} \times P^{j+1} \to X$ whose restriction $f^{j+1}\colon \bar{D}_0 \times P^{j+1} \to X$ is as close as desired to the spray $f^j\colon \bar{D}_0 \times P^{j+1} \to X$ in the $\mathcal{C}^0$-topology, and the range of f^{j+1} over $\overline{D}_1$ is contained in the range of the spray f'. The good approximation of f^j by f^{j+1} over D_0 and properties (5.12) and (5.13) ensure that properties (i†)–(v†) hold. □

We now conclude the proof of Lemma 5.3. Let $\epsilon' = \min\{\frac{\epsilon}{N}, \frac{\delta}{N}\}$, $f^0 = f$ and $P^0 = P$. We construct a decreasing sequence of open parameter sets $P^0 \supset P^1 \supset \cdots \supset P^N$, with $0 \in P^j \Subset P^{j-1}$ for $j = 1, \ldots, N$, and a sequence of sprays $f^j\colon \bar{D} \times P^j \to X$ with the exceptional set σ of order k such that the following hold for $j = 1, \ldots, N$:

(i$''$) $\rho(f^j_0(z)) > \rho(f^{j-1}_0(z)) + 3\delta - \epsilon'$ when $z \in bD$ and $f^{j-1}_0(z) \in h_j(\frac{1}{2}\mathbb{B}^n)$,

(ii$''$) $\rho(f^j_0(z)) > \rho(f^{j-1}_0(z)) - \epsilon'$ for $z \in \overline{D\backslash K_D}$,

(iii$''$) if $z \in bD$ and $f_0(z) \in h_i\left(\frac{1}{4}\mathbb{B}^n\right)$ for some $i \in \{1, 2, \ldots, N\}$ then $f^j_0(z) \in h_i\left((\frac{1}{4} + \frac{j}{4N})\mathbb{B}^n\right)$,

(iv$''$) $\mathrm{dist}\left(f^j_0(z), f^{j-1}_0(z)\right) < \frac{\epsilon}{N}$ for $z \in K_D$, and

(v$''$) the maps f^j_0 and f^{j-1}_0 have the same k-jets at every point in σ.

Assume inductively that we have already constructed $f^0, \ldots, f^{j-1}$ and $P^0, \ldots, P^{j-1}$ for some $j \in \{1, \ldots, N\}$. We use Sublemma 5.4 for $f' = f^{j-1}$ to obtain the next spray $f^j = g'\colon \bar{D} \times P^j \to X$. Properties (i$'$), (ii$'$), (iv$'$) and (v$'$) in the Sublemma imply the corresponding properties (i$''$), (ii$''$), (iv$''$) and (v$''$) above. Property (iii$''$) follows from (iii$'$) and (5.2). This completes the induction step and hence gives the desired sequences.

We now show that Lemma 5.3 holds for the parameter set $P_0 = P^N$ and the spray $g = f^N\colon \bar{D} \times P^N \to X$. The properties (ii)–(iv) follow easily from the inductive construction above. To prove (i), choose a point $z \in bD$. By (5.4) we either have $\rho(f_0(z)) > c + 2\delta$ (and in this case the property (ii$''$) implies that $\rho(g_0(z)) > c + \delta$), or else $f_0(z) \in h_i(\frac{1}{4}\mathbb{B}^n)$ for some $i \in \{1, \ldots, N\}$. In the latter case we get by (iii$''$) that $f_0^{i-1}(z) \in h_i(\frac{1}{2}\mathbb{B}^n)$, and therefore property (i$''$) implies $\rho(f_0^i(z)) > \rho(f_0^{i-1}(z)) + 3\delta - \epsilon'$. Using this together with (ii$''$) and $\rho(f_0(z)) > c - \delta$ we obtain

$$\rho(g_0(z)) \geq \rho(f_0(z)) + 3\delta - N\epsilon' \geq \rho(f_0(z)) + 2\delta > c + \delta.$$

This proves Lemma 5.3. □

Using Lemma 5.3 we now prove the following result that provides the lifting construction in the proof of Theorem 1.1.

LEMMA 5.5. *Let X, K, Ω, ρ, r, and $D \Subset S$ be as in Theorem 1.1. Choose a complete metric* dist *on X inducing the manifold topology. Let P be an open set in $\mathbb{C}^m$ containing the origin, and let* $0 < M_1 < M_2$. *Assume that $f\colon \bar{D} \times P \to X$ is a spray of maps with the exceptional set σ of order k and $U \Subset D$ is an open subset such that $f_0(\bar{D}\backslash U) \subset \{x \in \Omega\colon \rho(x) > M_1\}$. Given $\epsilon > 0$, there exist a domain $P' \subset P$ containing* $0 \in \mathbb{C}^m$ *and a spray of maps $g\colon \bar{D} \times P' \to X$ with the exceptional set σ of order k enjoying the following:*

(i) $g_0(z) \in \{x \in \Omega\colon \rho(x) > M_2\}$ *for $z \in bD$,*

(ii) $g_0(z) \in \{x \in \Omega\colon \rho(x) > M_1\}$ *for $z \in \bar{D}\backslash U$,*

(iii) $\mathrm{dist}(g_0(z), f_0(z)) < \epsilon$ *for $z \in \overline{U}$,*

(iv) *f_0 and g_0 have the same k-jets at each of the points in σ, and*

(v) *g_0 is homotopic to f_0 relative to σ.*

Proof. After a small change of M_1 and M_2 we may assume that these are regular values of ρ. By a finite subdivision of $[M_1, M_2]$ it suffices to consider the following two cases:

Case 1. ρ has no critical values on $[M_1, M_2]$. In this *noncritical case* we obtain g by applying Lemma 5.3 finitely many times.

Case 2. ρ has exactly one critical point p in $\{x \in \Omega\colon M_1 \leq \rho(x) \leq M_2\}$ (the *critical case*).

In Case 2 we follow [14, proof of Theorem 1.1, §6]. We have $M_1 < \rho(p) < M_2$. Choose $c_0 > 0$ so small that $M_1 + 3c_0 < \rho(p) < M_2 - 3c_0$ and $f_0(z) \in \{x \in \Omega\colon \rho(x) > M_1 + 3c_0\}$ for all $z \in \bar{D}\backslash U$. Set

$$K_{c_0} = \{x \in \Omega\colon \rho(p) - c_0 \leq \rho(x) \leq \rho(p) + 3c_0\}.$$

Lemma 3.1 furnishes a constant $t_0 \in (0, c_0)$, a smooth function

$$\tau\colon \{x \in \Omega\colon \rho(x) \le \rho(p) + 3c_0\} \to \mathbb{R},$$

and an embedded disc $E \subset \Omega$ of dimension equal to the Morse index of ρ at p, enjoying the following:

(a) $\{\rho \le \rho(p) - c_0\} \cup E \subset \{\tau \le 0\} \subset \{\rho \le \rho(p) - t_0\} \cup E$,
(b) $\{\rho \le \rho(p) + c_0\} \subset \{\tau \le 2c_0\} \subset \{\rho < \rho(p) + 3c_0\}$,
(c) τ is q-convex at every point of K_{c_0}, and
(d) τ has no critical values in $(0, 3c_0) \subset \mathbb{R}$.

Applying Lemma 5.3 finitely many times we get a spray $\widetilde{f}\colon \bar{D} \times \widetilde{P} \to X$ with exceptional set σ of order k having the following properties:

(i′) $\widetilde{f}_0(z) \in \{x \in \Omega\colon \rho(x) > \rho(p) - t_0\}$ for $z \in bD$,
(ii′) $\widetilde{f}_0(z) \in \{x \in \Omega\colon \rho(x) > M_1 + 2c_0\}$ for $z \in \bar{D}\backslash U$,
(iii′) $\mathrm{dist}(\widetilde{f}_0(z), f_0(z)) < \frac{\epsilon}{3}$ for $z \in \overline{U}$, and
(iv′) f_0 and $\widetilde{f}_0$ have the same k-jets at each of the points in σ.

For the parameter values $t \in \widetilde{P}$ sufficiently close to $t = 0$ we also have $\widetilde{f}_t(bD) \subset \{x \in \Omega\colon \rho(x) > \rho(p) - t_0\}$ by (i′). Since $\dim_{\mathbb{R}} E \le 2n - 2d$ and $\dim_{\mathbb{R}} bD = 2d - 1$, Sard's lemma gives a $t' \in \widetilde{P}$ arbitrarily close to the origin such that $\widetilde{f}_{t'}(bD) \cap E = \emptyset$. By a translation in the t-variable we can choose $\widetilde{f}_{t'}$ as the new central map; the new spray (still denoted $\widetilde{f}$) then enjoys the following properties for all t sufficiently near 0:

(i″) $\widetilde{f}_t(z) \in \{x \in \Omega\colon \rho(x) > \rho(p) - t_0\}\backslash E$ for $z \in bD$,
(ii″) $\widetilde{f}_t(z) \in \{x \in X\colon \rho(x) > M_1 + c_0\}$ for $z \in \bar{D}\backslash U$,
(iii″) $\mathrm{dist}(\widetilde{f}_t(z), f_0(z)) < \frac{2\epsilon}{3}$ for $z \in \overline{U}$, and
(iv″) f_0 and $\widetilde{f}_t$ have the same k-jets at each of the points in σ.

Since $\{\tau \le 0\} \subset \{\rho \le \rho(p) - t_0\} \cup E$ by property (a), (i″) ensures that $\tau > 0$ on $\widetilde{f}_t(bD)$. Since τ has no critical values on $(0, 3c_0)$ by property (d), we can use the noncritical case (Case 1 above), with τ instead of ρ, to push the boundary of the central map into the set $\{\tau > 2c_0\}$. As $\{\rho \le \rho(p) + c_0\} \subset \{\tau \le 2c_0\}$ by property (b), the image of bD now lies in $\{\rho > \rho(p) + c_0\}$. We have thus crossed the critical level $\{\rho = \rho(p)\}$ and may continue with the noncritical case procedure, applied again with the function ρ. In a finite number of steps we obtain a spray g with the required properties. □

Proof of Theorem 1.1. We follow [14, proof of Theorem 1.1], using Lemma 5.5 instead of [14, Proposition 6.3]. We begin by embedding the initial map $f_0\colon \bar{D} \to X$ into a spray of maps $f = f^0\colon \bar{D} \times P \to X$ (Definition 4.1 and Lemma 4.2) such that $f(\cdot, 0) = f_0$ and $f(bD \times P) \subset \Omega$. By inductively applying Lemma 5.5 we obtain a sequence of sprays $f^j\colon \bar{D} \times P_j \to X$ $(j = 1, 2, \ldots)$ with decreasing parameter sets $\mathbb{C}^m \supset P = P_0 \supset P_j \supset P_2 \supset \cdots$ containing the origin $0 \in \mathbb{C}^m$ such that the maps $f_0^j = f^j(\cdot, 0)\colon \bar{D} \to X$ converge uniformly on compacts in D to a holomorphic map $f\colon D \to X$ satisfying the conclusion of Theorem 1.1. □

6. Positivity and convexity of holomorphic vector bundles. In this section we recall the notions of *positivity* and *signature* of a Hermitian holomorphic vector bundle (Griffiths [29], [30]) and its connection with the Levi convexity properties of the squared norm function.

Let $\pi\colon E \to M$ be a holomorphic Hermitian vector bundle with fiber $\mathbb{C}^r$ over a complex manifold M of dimension m. We identify M with the zero section of E. The metric on E is given in a local frame $(e_1, \cdots, e_r)$ by a Hermitian matrix function $h = (h_{\rho\sigma})$ with

$$h_{\rho\sigma}(x) = \langle e_\sigma(x), e_\rho(x)\rangle, \quad \rho, \sigma = 1, \ldots, r.$$

The Chern connection matrix θ and the Chern curvature form Θ are given in any local holomorphic frame by

$$\theta = h^{-1}\partial h, \qquad \Theta = \bar\partial\,\theta = -h^{-1}\partial\bar\partial h + h^{-1}\partial h \wedge h^{-1}\bar\partial h.$$

(See [8, Chapter 5] or [53, Chapter III].) For a line bundle ($r = 1$) with the metric $h = e^{-\psi}$ the above equal

$$\theta = h^{-1}\partial h = \partial \log h = -\partial\,\psi, \quad \Theta = -\partial\bar\partial \log h = -\bar\partial\partial\,\psi = \partial\bar\partial\,\psi.$$

In local holomorphic coordinates $z = (z^1, \ldots, z^m)$ on M we have

$$\Theta = \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij} e^*_\sigma \otimes e_\rho \cdot dz^i \wedge d\bar z^j.$$

For any point $x_0 \in M$ there exists a local holomorphic frame $(e_1, \ldots, e_r)$ which is *special at* x_0, in the sense that the associated matrix h satisfies $h(x_0) = I$ and $dh(x_0) = 0$ (see [30, p. 195]). In this case we get

$$\text{(6.1)}\quad \theta(x_0) = 0, \quad \Theta(x_0) = -\partial\bar\partial h(x_0), \quad \overline{\Theta^{\sigma}_{\rho ij}(x_0)} = \Theta^{\rho}_{\sigma ji}(x_0) = -\frac{\partial^2 h_{\rho\sigma}}{\partial z^i \partial \bar z^j}(x_0).$$

To each vector $e = \sum_{\rho=1}^r \xi^\rho e_\rho(x_0) \in E_{x_0}$ we associate the $(1,1)$-covector

$$\Theta\{e\} = \frac{\mathrm{i}}{2}\,\langle \Theta\, e, e\rangle = \frac{\mathrm{i}}{2} \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij}(x_0)\xi^\sigma \bar\xi^\rho\, dz^i \wedge d\bar z^j.$$

Its coefficients $A_{ij}(x_0, \xi) = \sum_{\rho,\sigma=1,\ldots,r} \Theta^{\rho}_{\sigma ij}(x_0)\xi^\sigma\bar\xi^\rho$ form a Hermitian matrix. Denote by $s(e)$ (resp. $t(e)$) the number of positive (resp. negative) eigenvalues of

$\Theta\{e\}$; that is, $(s(e), t(e))$ is the signature of the Hermitian quadratic form

$$\mathbb{C}^m \ni \eta \to \sum_{i,j} A_{ij}(x_0, \xi)\eta^i \bar{\eta}^j = \sum_{\substack{\rho,\sigma=1,\ldots,r \\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij}(x_0)\xi^{\sigma}\bar{\xi}^{\rho}\eta^i\bar{\eta}^j. \tag{6.2}$$

The numbers $s(e), t(e)$ only depend on the Hermitian metric on E, and not on the particular choices of coordinates and frames. The following notions are due to Griffiths [28], [29], [30]; see also [1] and [8, Chapter 7].

Definition 6.1. The pair of numbers $(s(e), t(e))$ defined above is the *signature* of the Hermitian holomorphic vector bundle $E \to M$ at the point $e \in E\backslash M$. The signature of E is (s, t) where

$$s = \min\{s(e)\colon\ e \in E\backslash M\}, \quad t = \min\{t(e)\colon\ e \in E\backslash M\}.$$

E is of *pure signature* (s, t) is $s = s(e)$ and $t = t(e)$ for all $e \in E\backslash M$; it is *positive* (resp. *negative*) if it has pure signature $(m, 0)$ (resp. $(0, m)$).

Let $\phi\colon\ E \to \mathbb{R}_+$ denote the function $\phi(e) = ||e||^2$. For $c \in (0, \infty)$ set

$$W_c = \{e \in E\colon\ \phi(e) < c\}, \quad \Sigma_c = bW_c = \{e \in E\colon\ \phi(e) = c\}.$$

The following explains the connection between the curvature properties of a Hermitian metric and the Levi convexity properties of ϕ. (See Andreotti and Grauert [1, §23] and Griffiths [29, p. 426].)

Proposition 6.2. *Let $\pi\colon\ E \to M$ be a Hermitian holomorphic vector bundle with fiber $\mathbb{C}^r$ over an m-dimensional complex manifold M. Set $n = m + r = \dim E$. Then the following hold:*

(i) *If E has signature (s, t) at a point $e \in E\backslash M$ then the Levi form of the hypersurface $\Sigma_{\phi(e)}$ has signature $(t + r - 1, s)$ at e from the side $\{\phi < \phi(e)\}$.*

(ii) *If E has signature (s, t) then the Levi form of ϕ has signature $(t + r, s)$ (and hence ϕ is $(m - t + 1)$-convex) on $E\backslash M$, and the Levi form of $\frac{1}{\phi}$ has signature $(s + 1, t + r - 1)$ (and hence $\frac{1}{\phi}$ is $(n - s)$-convex) on $E\backslash M$.*

(iii) *In particular, if E is positive then $\frac{1}{\phi}$ is r-convex on $E\backslash M$, and if E is negative then ϕ is strongly plurisubharmonic on $E\backslash M$.*

Proof. Fix $e_0 \in E\backslash M$ and let $\Sigma = \Sigma_{\phi(e_0)}$. Choose local holomorphic coordinates $z = (z^1, \ldots, z^m)$ at $x_0 = \pi(e_0)$ and a local holomorphic frame (e_ρ) that is special at x_0. Then $e = \sum_{\sigma=1}^r \xi^\sigma e_\sigma$, $e_0 = \sum_{\rho=1}^r \xi_0^\rho e_\rho(x_0)$, and $\phi(e) = \sum_{\rho,\sigma=1}^r h_{\rho\sigma}\xi^\sigma\bar{\xi}^\rho$. Using

(6.1), a simple calculation [29, p. 426] gives

$$\begin{aligned}
\partial\bar\partial\phi(e_0) &= \partial_\xi\bar\partial_\xi\big|_{\xi=\xi_0}\sum_{\rho=1}^{r}\xi^\rho\bar\xi^\rho + \sum_{\rho,\sigma=1,\ldots,r}\partial_z\bar\partial_z h_{\rho\sigma}(x_0)\xi^\sigma\bar\xi^\rho \\
&= \sum_{\rho=1}^{r} d\xi^\rho\wedge d\bar\xi^\rho - \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}}\Theta^\rho_{\sigma ij}(x_0)\xi^\sigma\bar\xi^\rho dz^i\wedge d\bar z^j \\
&= \sum_{\rho=1}^{r} d\xi^\rho\wedge d\bar\xi^\rho - \sum_{i,j=1,\ldots,m} A_{ij}(x_0,\xi)\, dz^i\wedge d\bar z^j.
\end{aligned}$$

The maximal complex tangent space to Σ at e_0 consists of the vectors $\gamma = (\zeta^1,\ldots,\zeta^r;\eta^1,\ldots,\eta^m)$ with $\sum_{\rho=1}^{r}\xi_0^\rho\bar\zeta^\rho = 0$. In the ζ-direction (tangential to E_{x_0}) we thus get $r-1$ positive Levi eigenvalues for Σ; in the η-direction (the horizontal direction in $T_{e_0}E$ with respect to the Chern connection) we get $s(e_0)$ negative and $t(e_0)$ positive eigenvalues. Hence the Levi signature of Σ at e_0 is $(t(e_0)+r-1, s(e_0))$. The remaining Levi eigenvalue of ϕ in the radial direction is positive. All claims follow immediately. □

7. Subvarieties in complements of submanifolds. In this section we combine our analytic techniques with the differential geometric information from §6 to study the existence of subvarieties as in Theorem 1.1 in total spaces of Hermitian holomorphic vector bundles, and in complements of certain compact complex submanifolds.

THEOREM 7.1. *Let $E\to M$ be a holomorphic vector bundle with fiber $\mathbb{C}^r$ over a compact m-dimensional complex manifold M. Set $n = m+r = \dim E$ and identify M with the zero section of E. Assume that $D \Subset S$ is a smoothly bounded, strongly pseudoconvex domain in a d-dimensional Stein manifold S and $f_0\colon \bar D\to E$ is a continuous map that is holomorphic in D and such that $f_0(bD)\subset E\backslash M$. Then the following hold:*

(i) *If E admits a Hermitian metric with signature (s,t) and $d < t+r$, then f_0 can be approximated uniformly on compacts in D by proper holomorphic maps $f\colon D\to E$ such that $f_0^{-1}(M) = f^{-1}(M)$. In particular, if E is positive then this holds when $d < r = \operatorname{rank} E$, and if E is negative then it holds when $d < n = \dim E$.*

(ii) *If E admits a Hermitian metric with signature (s,t) with $d\le s$, then f_0 can be approximated uniformly on compacts in D by holomorphic maps $f\colon D\to E$ such that $f_0^{-1}(M) = f^{-1}(M)$ and the cluster set of f at bD belongs to the zero section M. In particular, if E is positive then the above holds when $d\le m = \dim M$.*

Recall that the *cluster set* of a map $f\colon D\to E$ at bD is the set of all sequential limits $\lim_{j\to\infty} f(p_j)\in E$ along sequences $\{p_j\}\subset D$ without accumulation points in D (therefore tending to bD).

Proof. Part (i) follows from Theorem 1.1 and Proposition 6.2 (i), applied to $X = E$, $\rho = \phi$ and $\Omega = E\backslash M$. (Note that ϕ is noncritical on $E\backslash M$.) To ensure that $f_0^{-1}(M) = f^{-1}(M)$, it suffices to construct f such that it agrees with f_0 to sufficiently high order at the finite set of points in $f_0^{-1}(M) \subset D$. (If $d > r$ then $f_0(bD) \cap M = \emptyset$ implies $f_0(D) \cap M = \emptyset$ since the analytic set $f_0^{-1}(M) \subset D$, if nonempty, would have positive dimension.)

To prove Part (ii), choose $c > 0$ such that $f_0(\bar{D}) \subset W_c = \{\phi < c\}$ and apply Theorem 1.1 with $\Omega = W_c\backslash M$ and $\rho = \frac{1}{\phi} - \frac{1}{c}$. □

The situation in Theorem 7.1 is a special case of the following one.

THEOREM 7.2. *Let A be a compact complex submanifold in a complex manifold X whose normal bundle $N_{A|X}$ has signature (s,t) with respect to some Hermitian metric. There is an open tubular neighborhood $V \subset X$ of A with the following property. If D is a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold with* $\dim D \leq s$, *z_0 is a point in D and $f_0\colon \bar{D} \to X$ is a continuous map that is holomorphic in D and such that $f_0(bD) \subset V\backslash A$, then f_0 can be approximated uniformly on D by holomorphic maps $f\colon D \to X$ such that $f^{-1}(A) = f_0^{-1}(A)$, $f(z_0) = f_0(z_0)$, and the cluster set of f at bD belong to A. If $N_{A|X}$ is positive then the above holds when* $\dim D \leq \dim A$.

Proof. Schneider proved in [49] that there exist a neighborhood $V \subset X$ of A and a smooth function $\rho\colon V\backslash A \to \mathbb{R}$ that tends to $+\infty$ at A, and whose Levi form $\mathcal{L}_\rho$ has at least $s+1$ positive eigenvalues at every point of $V\backslash A$. We recall Schneider's construction to see that his function is also noncritical on a deleted tubular neighborhood of A. Once this is clear, it remains to apply Theorem 1.1.

Assume first that A is a smooth complex hypersurface in X. Let $E \to X$ denote the hyperplane section bundle of the divisor determined by A. Then $E|_A \simeq N_{A|X}$, and there is a holomorphic section $\sigma\colon X \to E$ such that $A = \{x \in X\colon \sigma(x) = 0\}$. Such σ is given by a collection (g_i) of holomorphic functions $g_i\colon U_i \to \mathbb{C}$ on an open covering $\{U_i\}$ of X such that $\{g_i = 0\} = A \cap U_i$ and $dg_i \neq 0$ on $A \cap U_i$. The associated 1-cocycle $g_{ij} = \frac{g_i}{g_j}$ defines the line bundle $E \to X$.

The Hermitian metric of signature (s,t) on the normal bundle $E|_A = N_{A|X}$ extends to a Hermitian metric h on E. On $E|_{U_i} \simeq U_i \times \mathbb{C}$ the metric is given by a positive function $h_i\colon U_i \to (0,\infty)$. Let $||\sigma||_h^2\colon X \to [0,\infty)$ be the squared length of the section $\sigma\colon X \to E$. (On U_i this equals $h_i|g_i|^2$.) Schneider showed that for a sufficiently large constant $C > 0$ the metric ϕ on E, defined over U_i by $\phi_i = \frac{h_i}{1+Ch_i|g_i|^2}$, has signature $(s+1,t)$ over a neighborhood of A (see [49, p. 225]). Set $g = ||\sigma||_\phi^2\colon X \to [0,\infty)$, so

$$g|_{U_i} = \phi_i|g_i|^2 = \frac{h_i|g_i|^2}{1 + Ch_i|g_i|^2} = \frac{||\sigma||_h^2}{1 + C||\sigma||_h^2}.$$

It follows that

$$-i\partial\bar\partial \log g|_{U_i} = -i\partial\bar\partial \log \phi_i$$

and hence the Levi form of $-\log g = -\log ||\sigma||^2_\phi$ has at least $s+1$ positive eigenvalues in a deleted neighborhood of A in M. Clearly the same holds for $e^{-\log g} = \frac{1}{g} = \frac{1+C||\sigma||^2_h}{||\sigma||^2_h}$ and hence for $\rho = \frac{1}{||\sigma||^2_h}$. The latter function is noncritical near A and it blows up along A.

The general case reduces to the hypersurface case by blowing up X along A [49, §3]. Assume that A has complex dimension m and codimension r in X. Let $\hat A = \mathbb{P}(N)$ denote the total space of the fiber bundle over A whose fiber over a point $x \in A$ is $\mathbb{P}(N_x) \simeq \mathbb{P}^{r-1}$, the projective space of complex lines in $N_x \simeq \mathbb{C}^r$. Replacing A by $\hat A$ changes X to a new manifold $\hat X$ such that $\hat X\backslash \hat A$ is biholomorphic to $X\backslash A$, and $\hat A$ is a smooth complex hypersurface in $\hat X$. The restriction of the normal bundle $N_{\hat A|\hat X}$ to the submanifold $\mathbb{P}(N_x) \subset \hat A$ is the universal bundle over $\mathbb{P}(N_x) \simeq \mathbb{P}^{r-1}$ (the inverse of the hyperplane section bundle). This bundle is negative with respect to the Fubini-Study metric on $\mathbb{P}^{r-1}$, and a simple calculation shows that $N_{\hat A|\hat X}$ has signature $(s, t+r-1)$ if $N_{A|X}$ has signature (s,t). It remains to apply the previous argument (in the hypersurface case) to a deleted neighborhood of $\hat A$ in $\hat X$ (that is the same as a deleted neighborhood of A in X). □

Assume now that X is a complex manifold and A is a compact complex submanifold of X with positive normal bundle $N_{A|X}$. Let D be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold, with $\dim D \le \dim A$. Given a pair of points $z_0 \in D$ and $x_0 \in X\backslash A$, it is a natural question whether there exists a proper holomorphic map $f\colon D \to X\backslash A$ such that $f(z_0) = x_0$. Theorem 7.2 gives an affirmative answer if x_0 lies sufficiently close to A (take $f_0\colon \bar D \to X$ to be the constant map $f(z) = x_0$.)

The answer to this question is negative in general. For example, if X is obtained by blowing up $\mathbb{P}^3$ at one point, with Σ being the exceptional divisor, and if $A \subset X$ is a complex hyperplane disjoint from Σ, then for any 2-dimensional domain $z_0 \in D \subset \mathbb{C}^2$ and holomorphic map $f\colon D \to X\backslash A$ with $f(z_0) \in \Sigma$ the set $f^{-1}(\Sigma)$ is a positive dimensional subvariety of D which accumulates on bD; hence f cannot be proper into $X\backslash A$. (We wish to thank the referee for pointing out this example.)

The situation is different when a connected topological group acts transitively on X by holomorphic automorphisms:

THEOREM 7.3. *Assume that X is a complex manifold and A is a compact complex submanifold of X with positive normal bundle $N_{A|X}$. Let D be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold, with* $\dim D \le \dim A$ *and* $\dim A + \dim D < \dim X$. *Assume that a connected topological group G acts transitively on X by holomorphic automorphisms of X. Given* *points* $z_0 \in D$ *and* $x_0 \in X\backslash A$*, there exists a proper holomorphic map* $f\colon D \to X\backslash A$ *such that* $f(z_0) = x_0$.

Proof. The idea is taken from [21, proof of Theorem 1]. Let $s = \dim A$. By the proof of Theorem 7.2 there exist a neighborhood $V \subset X$ of A and a noncritical smooth function $\rho\colon V\backslash A \to \mathbb{R}$ that tends to $+\infty$ at A and whose Levi form has at least $s + 1$ positive eigenvalues on $V\backslash A$.

Choose an open subset $V_0 \subset X$ such that $A \subset V_0 \Subset V$. There is an open neighborhood U of the identity in G such that $g(V_0) \subset V$ for each $g \in U$.

Choose a continuous map $f_0\colon \bar{D} \to X\backslash A$ that is holomorphic in D and such that $f_0(bD) \subset V_0$. (f_0 may be a constant map.) Choose $g \in G$ such that $gf_0(z_0) = x_0$. As G is connected, there are elements $g_1, \ldots, g_k \in U \subset G$ such that $g = g_k g_{k-1} \cdots g_1$ (product in G).

The map $g_1 f_0$ is continuous on $\bar{D}$, holomorphic in D, and $g_1 f_0(bD) \subset V$. By general position we may assume that its range does not intersect A. Lemma 5.5 furnishes a continuous map $f_1\colon \bar{D} \to X\backslash A$ that is holomorphic in D such that $f_1(bD) \subset V_0$ and $f_1(z_0) = g_1 f_0(z_0)$. (Lemma 5.5 also applies to individual maps in view of Lemma 4.2.)

The map $g_2 f_1$ is continuous on $\bar{D}$, holomorphic in D, and $g_2 f_1(bD) \subset V$. By general position we may assume that its range does not intersect A. Lemma 5.5 furnishes a map $f_2\colon \bar{D} \to X\backslash A$ that is holomorphic in D such that $f_2(bD) \subset V_0$ and $f_2(z_0) = g_2 f_1(z_0) = g_2 g_1 f_0(z_0)$.

After k such steps we obtain a continuous map $f_k\colon \bar{D} \to X\backslash A$ that is holomorphic in D and satisfies

$$f_k(bD) \subset V_0, \quad f_k(z_0) = g_k \cdots g_1 f_0(z_0) = x_0.$$

We now apply Theorem 1.1 to f_k, with $\Omega = V_0\backslash A$ and ρ as above, to obtain a holomorphic map $f\colon D \to X\backslash A$ such that $f(z_0) = x_0$ and the cluster set of f at bD belongs to A. Hence f is a proper map of D to $X\backslash A$. □

Applying Theorem 7.3 with $X = \mathbb{P}^n$ and taking into account that every closed complex submanifold of $\mathbb{P}^n$ has positive normal bundle (see Barth [3]) gives the following corollary. (Compare with Corollary 1.4.)

COROLLARY 7.4. *Let D be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold, and let A be a closed complex submanifold of a projective space* $\mathbb{P}^n$ *such that* $\dim D \leq \dim A$ *and* $\dim A + \dim D < n$*. For every pair of points* $z_0 \in D$ *and* $x_0 \in \mathbb{P}^n\backslash A$ *there exists a proper holomorphic map* $f\colon D \to \mathbb{P}^n\backslash A$ *with* $f(z_0) = x_0$.

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