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Invariance of the Parametric Oka Property

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Dedicated to Linda P. Rothschild

Abstract. Assume that E and B are complex manifolds and that $\pi\colon E \to B$ is a holomorphic Serre fibration such that E admits a finite dominating family of holomorphic fiber-sprays over a small neighborhood of any point in B. We show that the parametric Oka property (POP) of B implies POP of E; conversely, POP of E implies POP of B for contractible parameter spaces. This follows from a parametric Oka principle for holomorphic liftings which we establish in the paper.

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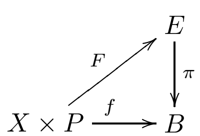
1. Oka properties

The main result of this paper is that a subelliptic holomorphic submersion $\pi\colon E \to B$ between (reduced, paracompact) complex spaces satisfies the *parametric Oka property*. *Subellipticity* means that E admits a finite dominating family of holomorphic fiber-sprays over a neighborhood of any point in B (Def. 2.3). The conclusion means that for any Stein source space X, any compact Hausdorff space P (the parameter space), and any continuous map $f\colon X \times P \to B$ which is X-holomorphic (i.e., such that $f_p = f(\cdot\,, p)\colon X \to B$ is holomorphic for every $p \in P$), a continuous lifting $F\colon X \times P \to E$ of f (satisfying $\pi \circ F = f$) can be homotopically deformed through liftings of f to an X-holomorphic lifting. (See Theorem 4.2 for a precise statement.)



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The following result is an easy consequence. Suppose that E and B are complex manifolds and that $\pi\colon E \to B$ is a subelliptic submersion which is also a Serre fibration (such map is called a *subelliptic Serre fibration*), or is a holomorphic fiber bundle whose fiber satisfies the parametric Oka property. Then the parametric Oka property passes up from the base B to the total space E; it also passes down from E to B if the parameter space P is contractible, or if π is a weak homotopy equivalence (Theorem 1.2).

We begin by recalling the relevant notions. Among the most interesting phenomena in complex geometry are, on the one hand, *holomorphic rigidity*, commonly expressed by Kobayashi-Eisenman hyperbolicity; and, on the other hand, *holomorphic flexibility*, a term introduced in [7]. While Kobayashi hyperbolicity of a complex manifold Y implies in particular that there exist no nonconstant holomorphic maps $\mathbb{C} \to Y$, flexibility of Y means that it admits many nontrivial holomorphic maps $X \to Y$ from any Stein manifold X; in particular, from any Euclidean space $\mathbb{C}^n$.

The most natural flexibility properties are the *Oka properties* which originate in the seminal works of Oka [27] and Grauert [14, 15]. The essence of the classical *Oka-Grauert principle* is that a complex Lie group, or a complex homogeneous manifold, Y, enjoys the following:

Basic Oka Property (BOP) of Y: *Every continuous map $f\colon X \to Y$ from a Stein space X is homotopic to a holomorphic map. If in addition f is holomorphic on (a neighborhood of) a compact $\mathcal{O}(X)$-convex subset K of X, and if $f|_{X'}$ is holomorphic on a closed complex subvariety X' of X, then there is a homotopy $f^t\colon X \to Y$ $(t \in [0,1])$ from $f^0 = f$ to a holomorphic map f^1 such that for every $t \in [0,1]$, f^t is holomorphic and uniformly close to f^0 on K, and $f^t|_{X'} = f|_{X'}$.*

All complex spaces in this paper are assumed to be reduced and paracompact. A map is said to be holomorphic on a compact subset K of a complex space X if it is holomorphic in an open neighborhood of K in X; two such maps are identified if they agree in a (smaller) neighborhood of K; for a family of maps, the neighborhood should be independent of the parameter.

When $Y = \mathbb{C}$, BOP combines the Oka-Weil approximation theorem and the Cartan extension theorem. BOP of Y means that, up to a homotopy obstruction, the same approximation-extension result holds for holomorphic maps $X \to Y$ from any Stein space X to Y.

Denote by $\mathcal{C}(X,Y)$ (resp. by $\mathcal{O}(X,Y)$) the space of all continuous (resp. holomorphic) maps $X \to Y$, endowed with the topology of uniform convergence on compacts. We have a natural inclusion

$$\mathcal{O}(X,Y) \hookrightarrow \mathcal{C}(X,Y). \tag{1.1}$$

BOP of Y implies that every connected component of $\mathcal{C}(X,Y)$ contains a component of $\mathcal{O}(X,Y)$. By [8, Theorem 5.3], BOP also implies the following

One-parametric Oka Property: *A path $f\colon [0,1] \to \mathcal{C}(X,Y)$ such that $f(0)$ and $f(1)$ belong to $\mathcal{O}(X,Y)$ can be deformed, with fixed ends at $t = 0, 1$, to a path in*

$\mathcal{O}(X, Y)$. *Hence* (1.1) *induces a bijection of the path connected components of the two spaces.*

Y enjoys the **Weak Parametric Oka Property** if for each finite polyhedron P and subpolyhedron $P_0 \subset P$, a map $f\colon P \to \mathcal{C}(X, Y)$ such that $f(P_0) \subset \mathcal{O}(X, Y)$ can be deformed to a map $\widetilde{f}\colon P \to \mathcal{O}(X, Y)$ by a homotopy that is fixed on P_0:

$$\begin{array}{ccc} P_0 & \longrightarrow & \mathcal{O}(X,Y) \\ {\scriptstyle incl}\downarrow & \overset{\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle incl} \\ P & \xrightarrow{\;f\;} & \mathcal{C}(X,Y) \end{array}$$

This implies that (1.1) is a weak homotopy equivalence [11, Corollary 1.5].

Definition 1.1. (Parametric Oka Property (POP)) Assume that P is a compact Hausdorff space and that P_0 is a closed subset of P. A complex manifold Y enjoys POP for the pair (P, P_0) if the following holds. Assume that X is a Stein space, K is a compact $\mathcal{O}(X)$-convex subset of X, X' is a closed complex subvariety of X, and $f\colon X \times P \to Y$ is a continuous map such that

(a) the map $f_p = f(\cdot\,, p)\colon X \to Y$ is holomorphic for every $p \in P_0$, and
(b) f_p is holomorphic on $K \cup X'$ for every $p \in P$.

Then there is a homotopy $f^t\colon X \times P \to Y$ ($t \in [0, 1]$) such that f^t satisfies properties (a) and (b) above for all $t \in [0, 1]$, and also

(i) f^1_p is holomorphic on X for all $p \in P$,
(ii) f^t is uniformly close to f on $K \times P$ for all $t \in [0, 1]$, and
(iii) $f^t = f$ on $(X \times P_0) \cup (X' \times P)$ for all $t \in [0, 1]$.

The manifold Y satisfies POP if the above holds for each pair $P_0 \subset P$ of compact Hausdorff spaces. Analogously we define POP for sections of a holomorphic map $Z \to X$. □

Restricting POP to pairs $P_0 \subset P$ consisting of finite polyhedra we get Gromov's Ell_∞ property [16, Def. 3.1.A.]. By Grauert, all complex homogeneous manifolds enjoy POP for finite polyhedral inclusions $P_0 \subset P$ [14, 15]. A weaker sufficient condition, called *ellipticity* (the existence of a dominating spray on Y, Def. 2.1 below), was found by Gromov [16]. A presumably even weaker condition, *subellipticity* (Def. 2.2), was introduced in [4].

If Y enjoys BOP or POP, then the corresponding Oka property also holds for sections of any holomorphic fiber bundle $Z \to X$ with fiber Y over a Stein space X [10]. See also Sect. 2 below and the papers [5, 21, 22, 23].

It is important to know which operations preserve Oka properties. The following result was stated in [8] (remarks following Theorem 5.1), and more explicitly in [9, Corollary 6.2]. (See also [16, Corollary 3.3.C'].)

Theorem 1.2. *Assume that E and B are complex manifolds. If $\pi\colon E \to B$ is a subelliptic Serre fibration* (*Def.* 2.3 *below*)*, or a holomorphic fiber bundle with* POP *fiber, then the following hold:*

(i) *If B enjoys the parametric Oka property* (POP), *then so does E.*
(ii) *If E enjoys* POP *for contractible parameter spaces P (and arbitrary closed subspaces P_0 of P), then so does B.*
(iii) *If in addition $\pi\colon E \to B$ is a weak homotopy equivalence then*

$$\text{POP of } E \Longrightarrow \text{POP of B.}$$

All stated implications hold for a specific pair $P_0 \subset P$ of parameter spaces.

The proof Theorem 1.2, proposed in [9], requires the parametric Oka property for sections of certain continuous families of subelliptic submersions. When Finnur Lárusson asked for explanation and at the same time told me of his applications of this result [24] (personal communication, December 2008), I decided to write a more complete exposition. We prove Theorem 1.2 in Sec. 5 as a consequence of Theorem 4.2. This result should be compared with Lárusson's [24, Theorem 3] where the map $\pi\colon E \to B$ is assumed to be an *intermediate fibration* in the model category that he constructed.

Corollary 1.3. *Let $Y = Y_m \to Y_{m-1} \to \cdots \to Y_0$, where each Y_j is a complex manifold and every map $\pi_j\colon Y_j \to Y_{j-1}$ $(j = 1, 2, \ldots, m)$ is a subelliptic Serre fibration, or a holomorphic fiber bundle with* POP *fiber. Then the following hold:*
(i) *If one of the manifolds Y_j enjoys* BOP, *or* POP *with a contractible parameter space, then all of them do.*
(ii) *If in addition every π_j is acyclic (a weak homotopy equivalence) and if Y is a Stein manifold, then every manifold Y_j in the tower satisfies the implication* BOP $\Longrightarrow$ POP.

Proof. Part (i) is an immediate consequences of Theorem 1.2. For (ii), observe that BOP of Y_j implies BOP of Y by Theorem 1.2 (i), applied with P a singleton. Since Y is Stein, BOP implies that Y is elliptic (see Def. 2.2 below); for the simple proof see [13, Proposition 1.2] or [16, 3.2.A.]. By Theorem 2.4 below it follows that Y also enjoys POP. By part (iii) of Theorem 1.2, POP descends from $Y = Y_m$ to every Y_j. □

Remark 1.4. A main remaining open problem is whether the implication

$$\text{BOP} \Longrightarrow \text{POP} \tag{1.2}$$

holds for all complex manifolds. By using results of this paper and of his earlier works, F. Lárusson proved this implication for a large class of manifolds, including all quasi-projective manifolds [24, Theorem 4]. The main observation is that there exists an affine holomorphic fiber bundle $\pi\colon E \to \mathbb{P}^n$ with fiber $\mathbb{C}^n$ whose total space E is Stein; since the map π is acyclic and the fiber satisfies POP, the implication (1.2) follows from Corollary 1.3 (ii) for any closed complex subvariety $Y \subset \mathbb{P}^n$ (since the total space $E|_Y = \pi^{-1}(Y)$ is Stein). The same applies to complements of hypersurfaces in such Y; the higher codimension case reduces to the hypersurface case by blowing up. □

2. Subelliptic submersions and Serre fibrations

A holomorphic map $h\colon Z \to X$ of complex spaces is a *holomorphic submersion* if for every point $z_0 \in Z$ there exist an open neighborhood $V \subset Z$ of z_0, an open neighborhood $U \subset X$ of $x_0 = h(z_0)$, an open set W in a Euclidean space $\mathbb{C}^p$, and a biholomorphic map $\phi\colon V \to U \times W$ such that $pr_1 \circ \phi = h$, where $pr_1 : U \times W \to U$ is the projection on the first factor.

$$\begin{array}{ccc} Z \supset V & \xrightarrow{\ \phi\ } & U \times W \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle pr_1} \\ X \supset U & \xrightarrow{\ id\ } & U \end{array}$$

Each fiber $Z_x = h^{-1}(x)$ $(x \in X)$ of a holomorphic submersion is a closed complex submanifold of Z. A simple example is the restriction of a holomorphic fiber bundle projection $E \to X$ to an open subset Z of E.

We recall from [16, 4] the notion of a holomorphic spray and domination.

Definition 2.1. Assume that X and Z are complex spaces and $h\colon Z \to X$ is a holomorphic submersion. For $x \in X$ let $Z_x = h^{-1}(x)$.

(i) A *fiber-spray* on Z is a triple (E, π, s) consisting of a holomorphic vector bundle $\pi\colon E \to Z$ together with a holomorphic map $s\colon E \to Z$ such that for each $z \in Z$ we have

$$s(0_z) = z, \quad s(E_z) \subset Z_{h(z)}.$$

(ii) A spray (E, π, s) is *dominating* at a point $z \in Z$ if its differential

$$(\mathrm{d}s)_{0_z} : \mathrm{T}_{0_z}E \to \mathrm{T}_z Z$$

at the origin $0_z \in E_z = \pi^{-1}(z)$ maps the subspace E_z of $\mathrm{T}_{0_z}E$ surjectively onto the *vertical tangent space* $\mathrm{VT}_z Z = \ker \mathrm{d}h_z$. The spray is *dominating* (on Z) if it is dominating at every point $z \in Z$.

(iii) A family of h-sprays (E_j, π_j, s_j) $(j = 1, \ldots, m)$ on Z is dominating at the point $z \in Z$ if

$$(\mathrm{d}s_1)_{0_z}(E_{1,z}) + (\mathrm{d}s_2)_{0_z}(E_{2,z}) \cdots + (\mathrm{d}s_m)_{0_z}(E_{m,z}) = \mathrm{VT}_z Z.$$

If this holds for every $z \in Z$ then the family is *dominating* on Z.

(iv) A spray on a complex manifold Y is a fiber-spray associated to the constant map $Y \to$ *point*.

The simplest example of a spray on a complex manifold Y is the flow $Y \times \mathbb{C} \to Y$ of a $\mathbb{C}$-complete holomorphic vector field on Y. A composition of finitely many such flows, with independent time variables, is a dominating spray at every point where the given collection of vector fields span the tangent space of Y. Another example of a dominating spray is furnished by the exponential map on a complex Lie group G, translated over G by the group multiplication.

The following notion of an *elliptic submersion* is due to Gromov [16, Sect. 1.1.B]; *subelliptic submersions* were introduced in [4]. For examples see [4, 8, 16].

Definition 2.2. A holomorphic submersion $h\colon Z \to X$ is said to be *elliptic* (resp. *subelliptic*) if each point $x_0 \in X$ has an open neighborhood $U \subset X$ such that the restricted submersion $h\colon Z|_U \to U$ admits a dominating fiber-spray (resp. a finite dominating family of fiber-sprays). A complex manifold Y is elliptic (resp. subelliptic) if the trivial submersion $Y \to$ *point* is such.

The following notions appear in Theorem 1.2.

Definition 2.3. (a) A continuous map $\pi\colon E \to B$ is *Serre fibration* if it satisfies the homotopy lifting property for polyhedra (see [32, p. 8]).

(b) A holomorphic map $\pi\colon E \to B$ is an *elliptic Serre fibration* (resp. a *subelliptic Serre fibration*) if it is a surjective elliptic (resp. subelliptic) submersion and also a Serre fibration.

The following result was proved in [10] (see Theorems 1.4 and 8.3) by following the scheme proposed in [13, Sect. 7]. Earlier results include Gromov's Main Theorem [16, Theorem 4.5] (for elliptic submersions onto Stein manifolds, without interpolation), [13, Theorem 1.4] (for elliptic submersions onto Stein manifolds), [4, Theorem 1.1] (for subelliptic submersion), and [8, Theorem 1.2] (for fiber bundles with POP fibers over Stein manifolds).

Theorem 2.4. *Let $h\colon Z \to X$ be a holomorphic submersion of a complex space Z onto a Stein space X. Assume that X is exhausted by Stein Runge domains $D_1 \Subset D_2 \Subset \cdots \subset X = \bigcup_{j=1}^{\infty} D_j$ such that every D_j admits a stratification*

$$D_j = X_0 \supset X_1 \supset \cdots \supset X_{m_j} = \emptyset \tag{2.1}$$

with smooth strata $S_k = X_k \backslash X_{k+1}$ such that the restriction of $Z \to X$ to every connected component of each S_k is a subelliptic submersion, or a holomorphic fiber bundle with POP *fiber. Then sections $X \to Z$ satisfy* POP.

Remark 2.5. In previous papers [11, 12, 13, 4, 8, 9] POP was only considered for pairs of parameter spaces $P_0 \subset P$ such that

(*) P is a nonempty compact Hausdorff space, and P_0 is a closed subset of P that is a strong deformation neighborhood retract (SDNR) in P.

Here we dispense with the SDNR condition by using the Tietze extension theorem for maps into Hilbert spaces (see the proof of Proposition 4.4).

Theorem 2.4 also hold when P is a locally compact and countably compact Hausdorff space, and P_0 is a closed subspace of P. The proof requires only a minor change of the induction scheme (applying the diagonal process).

On the other hand, all stated results remain valid if we restrict to pairs $P_0 \subset P$ consisting of finite polyhedra; this suffices for most applications. $\square$

Theorem 2.4 implies the following result concerning holomorphic liftings.

Theorem 2.6. *Let $\pi\colon E \to B$ be a holomorphic submersion of a complex space E onto a complex space B. Assume that B admits a stratification $B = B_0 \supset B_1 \supset \cdots \supset B_m = \emptyset$ by closed complex subvarieties such that the restriction of π to every connected component of each difference $B_j \backslash B_{j+1}$ is a subelliptic submersion, or a holomorphic fiber bundle with* POP *fiber.*

Given a Stein space X and a holomorphic map $f\colon X \to B$, every continuous lifting $F\colon X \to E$ of f $(\pi \circ F = f)$ is homotopic through liftings of f to a holomorphic lifting.

$$\begin{array}{ccc} & & E \\ & {\scriptstyle F}\nearrow & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\;f\;} & B \end{array}$$

Proof. Assume first that X is finite dimensional. Then there is a stratification $X = X_0 \supset X_1 \supset \cdots \supset X_l = \emptyset$ by closed complex subvarieties, with smooth differences $S_j = X_j \backslash X_{j+1}$, such that each connected component S of every S_j is mapped by f to a stratum $B_k \backslash B_{k+1}$ for some $k = k(j)$. The pull-back submersion

$$f^*E = \{(x,e) \in X \times E\colon f(x) = \pi(e)\} \longmapsto X$$

then satisfies the assumptions of Theorem 2.4 with respect to this stratification of X. Note that liftings $X \to E$ of $f\colon X \to B$ correspond to sections $X \to f^*E$, and hence the result follows from Theorem 2.4. The general case follows by induction over an exhaustion of X by an increasing sequence of relatively compact Stein Runge domains in X. □

A fascinating application of Theorem 2.6 has recently been found by Ivarsson and Kutzschebauch [19, 20] who solved the following *Gromov's Vaserstein problem*:

Theorem 2.7. (Ivarsson and Kutzschebauch [19, 20]) *Let X be a finite-dimensional reduced Stein space and let $f\colon X \to \mathrm{SL}_m(\mathbb{C})$ be a null-homotopic holomorphic mapping. Then there exist a natural number N and holomorphic mappings $G_1, \ldots, G_N\colon X \to \mathbb{C}^{m(m-1)/2}$ (thought of as lower resp. upper triangular matrices) such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices. (For odd N the last matrix has $G_N(x)$ in the lower left corner.)

In this application one takes $B = SL_m(\mathbb{C})$, $E = (\mathbb{C}^{m(m-1)/2})^N$, and $\pi\colon E \to B$ is the map

$$\pi(G_1, G_2, \ldots, G_N) = \begin{pmatrix} 1 & 0 \\ G_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N \\ 0 & 1 \end{pmatrix}.$$

Every null-homotopic holomorphic map $f\colon X \to B = SL_m(\mathbb{C})$ admits a continuous lifting $F\colon X \to E$ for a suitably chosen $N \in \mathbb{N}$ (Vaserstein [31]), and the goal is to deform F to a holomorphic lifting $G = (G_1, \ldots, G_N)\colon X \to E$. This is done inductively by applying Theorem 2.6 to auxiliary submersions obtained by composing π with certain row projections. Stratified elliptic submersions naturally appear in their proof.

3. Convex approximation property

In this section we recall from [8] a characterization of Oka properties in terms of an Oka-Weil approximation property for entire maps $\mathbb{C}^n \to Y$.

Let $z = (z_1, \ldots, z_n)$, $z_j = x_j + \mathrm{i}\, y_j$, be complex coordinates on $\mathbb{C}^n$. Given numbers $a_j, b_j > 0$ $(j = 1, \ldots, n)$ we set

$$Q = \{z \in \mathbb{C}^n \colon |x_j| \le a_j,\ |y_j| \le b_j,\ j = 1, \ldots, n\}. \tag{3.1}$$

Definition 3.1. A *special convex set* in $\mathbb{C}^n$ is a compact convex set of the form

$$K = \{z \in Q \colon y_n \le \phi(z_1, \ldots, z_{n-1}, x_n)\}, \tag{3.2}$$

where Q is a cube (3.1) and ϕ is a continuous concave function with values in $(-b_n, b_n)$. Such (K, Q) is called a *special convex pair* in $\mathbb{C}^n$.

Definition 3.2. A complex manifold Y enjoys the *Convex Approximation Property* (CAP) if every holomorphic map $f \colon K \to Y$ on a special convex set $K \subset Q \subset \mathbb{C}^n$ (3.2) can be approximated, uniformly on K, by holomorphic maps $Q \to Y$.

Y enjoys the *Parametric Convex Approximation Property* (PCAP) if for every special convex pair (K, Q) and for every pair of parameter spaces $P_0 \subset P$ as in Def. 1.1, a map $f \colon Q \times P \to Y$ such that $f_p = f(\cdot\,, p) \colon Q \to Y$ is holomorphic for every $p \in P_0$, and is holomorphic on K for every $p \in P$, can be approximated uniformly on $K \times P$ by maps $\widetilde{f} \colon Q \times P \to Y$ such that $\widetilde{f}_p$ is holomorphic on Q for all $p \in P$, and $\widetilde{f}_p = f_p$ for all $p \in P_0$.

The following characterization of the Oka property was found in [8, 9] (for Stein source manifolds), thereby answering a question of Gromov [16, p. 881, 3.4.(D)]. For the extension to Stein source spaces see [10].

Theorem 3.3. *For every complex manifold we have*

$$\mathrm{BOP} \iff \mathrm{CAP}, \qquad \mathrm{POP} \iff \mathrm{PCAP}.$$

Remark 3.4. The implication PCAP $\Longrightarrow$ POP also holds for a specific pair of (compact, Hausdorff) parameter spaces as is seen from the proof in [8]. More precisely, if a complex manifold Y enjoys PCAP for a certain pair $P_0 \subset P$, then it also satisfies POP for that same pair. □

4. A parametric Oka principle for liftings

In this section we prove the main result of this paper, Theorem 4.2, which generalizes Theorem 2.6 to families of holomorphic maps. We begin by recalling the relevant terminology from [13].

Definition 4.1. Let $h \colon Z \to X$ be a holomorphic map of complex spaces, and let $P_0 \subset P$ be topological spaces.

(a) A P-*section* of $h \colon Z \to X$ is a continuous map $f \colon X \times P \to Z$ such that $f_p = f(\cdot\,, p) \colon X \to Z$ is a section of h for each $p \in P$. Such f is *holomorphic* if f_p is holomorphic on X for each fixed $p \in P$. If K is a compact set in X and if X' is a closed complex subvariety of X, then f is *holomorphic on* $K \cup X'$ if there is an open set $U \subset X$ containing K such that the restrictions $f_p|_U$ and $f_p|_{X'}$ are holomorphic for every $p \in P$.

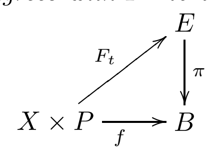
(b) A *homotopy of P-sections* is a continuous map $H\colon X \times P \times [0,1] \to Z$ such that $H_t = H(\cdot\,,\cdot\,,t)\colon X \times P \to Z$ is a P-section for each $t \in [0,1]$.

(c) A (P, P_0)*-section* of h is a P-section $f\colon X \times P \to Z$ such that $f_p = f(\cdot\,,p)\colon X \to Z$ is holomorphic on X for each $p \in P_0$. A (P, P_0)-section is holomorphic on a subset $U \subset X$ if $f_p|_U$ is holomorphic for every $p \in P$.

(d) A P-map $X \to Y$ to a complex space Y is a map $X \times P \to Y$. Similarly one defines (P, P_0)-maps and their homotopies.

Theorem 4.2. *Assume that E and B are complex spaces and $\pi\colon E \to B$ is a subelliptic submersion* (*Def.* 2.3), *or a holomorphic fiber bundle with* POP *fiber* (*Def.* 1.1). *Let P be a compact Hausdorff space and P_0 a closed subspace of P. Given a Stein space X, a compact $\mathcal{O}(X)$-convex subset K of X, a closed complex subvariety X' of X, a holomorphic P-map $f\colon X \times P \to B$, and a (P, P_0)-map $F\colon X \times P \to E$ that is a π-lifting of f ($\pi \circ F = f$) and is holomorphic on* (*a neighborhood of*) *K and on the subvariety X', there exists a homotopy of liftings $F^t\colon X \times P \to E$ of f ($t \in [0,1]$) that is fixed on $(X \times P_0) \cup (X' \times P)$, that approximates $F = F^0$ uniformly on $K \times P$, and such that F^1_p is holomorphic on X for all $p \in P$.*

If in addition F is holomorphic in a neighborhood of $K \cup X'$ then the homotopy F^t can be chosen such that it agrees with F^0 to a given finite order along X'.



Definition 4.3. A map $\pi\colon E \to B$ satisfying the conclusion of Theorem 4.2 is said to enjoy the parametric Oka property (c.f. Lárusson [22, 23, 24]). ☐

Proof. The first step is a reduction to the graph case. Set $Z = X \times E$, $\widetilde{Z} = X \times B$, and let $\widetilde{\pi}\colon Z \to \widetilde{Z}$ denote the map

$$\widetilde{\pi}(x, e) = (x, \pi(e)), \quad x \in X,\ e \in E.$$

Then $\widetilde{\pi}$ is a subelliptic submersion, resp. a holomorphic fiber bundle with POP fiber. Let $\widetilde{h}\colon \widetilde{Z} = X \times B \to X$ denote the projection onto the first factor, and let $h = \widetilde{h} \circ \widetilde{\pi}\colon Z \to X$. To a P-map $f\colon X \times P \to B$ we associate the P-section $\widetilde{f}(x,p) = (x, f(x,p))$ of $\widetilde{h}\colon \widetilde{Z} \to X$. Further, to a lifting $F\colon X \times P \to E$ of f we associate the P-section $\widetilde{F}(x,p) = (x, F(x,p))$ of $h\colon Z \to X$. Then $\widetilde{\pi} \circ \widetilde{F} = \widetilde{f}$. This allows us to drop the tilde's on π, f and F and consider from now on the following situation:

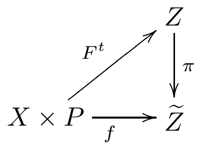
(i) Z and $\widetilde{Z}$ are complex spaces,

(ii) $\pi\colon Z \to \widetilde{Z}$ is a subelliptic submersion, or a holomorphic fiber bundle with POP fiber,

(iii) $\widetilde{h} \colon \widetilde{Z} \to X$ is a holomorphic map onto a Stein space X,
(iv) $f \colon X \times P \to \widetilde{Z}$ is a holomorphic P-section of $\widetilde{h}$,
(v) $F \colon X \times P \to Z$ is a holomorphic (P, P_0)-section of $h = \widetilde{h} \circ \pi \colon Z \to X$ such that $\pi \circ F = f$, and F is holomorphic on $K \cup X'$.

We need to find a homotopy $F^t \colon X \times P \to Z$ $(t \in [0,1])$ consisting of (P, P_0)-sections of $h \colon Z \to X$ such that $\pi \circ F^t = f$ for all $t \in [0,1]$, and

(α) $F^0 = F$,
(β) F^1 is a holomorphic P-section, and
(γ) for every $t \in [0,1]$, F^t is holomorphic on K, it is uniformly close to F^0 on $K \times P$, and it agrees with F^0 on $(X \times P_0) \cup (X' \times P)$.



Set $f_p = f(\cdot\,, p) \colon X \to \widetilde{Z}$ for $p \in P$. The image $f_p(X)$ is a closed Stein subspace of $\widetilde{Z}$ that is biholomorphic to X (since $\widetilde{h} \circ f$ is the identity on X).

When $P = \{p\}$ is a singleton, there is only one section $f = f_p$, and the desired conclusion follows by applying Theorem 2.4 to the restricted submersion $\pi \colon Z|_{f(X)} \to f(X)$.

In general we consider the family of restricted submersions $Z|_{f_p(X)} \to f_p(X)$ $(p \in P)$. The proof of the parametric Oka principle [12, Theorem 1.4] requires certain modifications that we now explain. It suffices to obtain a homotopy F^t of liftings of f over a relatively compact subset D of X with $K \subset D$; the proof is then finished by induction over an exhaustion of X. The initial step is provided by the following proposition. (No special assumption is needed on the submersion $\pi \colon Z \to \widetilde{Z}$ for this result.)

Proposition 4.4. (Assumptions as above) *Let D be an open relatively compact set in X with $K \subset D$. There exists a homotopy of liftings of f over D from $F = F^0|_{D \times P}$ to a lifting F' such that properties* (α) *and* (γ) *hold for F', while* (β) *is replaced by*

(β') *F'_p is holomorphic on D for all p in a neighborhood $P'_0 \subset P$ of P_0.*

The existence of such local holomorphic extension F' is used at several subsequent steps. We postpone the proof of the proposition to the end of this section and continue with the proof of Theorem 4.2. Replacing F by F' and X by D, we assume from now on that F_p is holomorphic on X for all $p \in P'_0$ (a neighborhood of P_0).

Assume for the sake of discussion that X is a Stein manifold, that $X' = \emptyset$, and that $\pi \colon Z \to \widetilde{Z}$ is a subelliptic submersion. (The proof in the fiber bundle case is simpler and will be indicated along the way. The case when X has singularities or $X' \neq \emptyset$ uses the induction scheme from [10], but the details presented here remain unchanged.) It suffices to explain the following:

Main step: Let $K \subset L$ be compact strongly pseudoconvex domains in X that are $\mathcal{O}(X)$-convex. Assume that $F^0 = \{F^0_p\}_{p\in P}$ is a π-lifting of $f = \{f_p\}_{p\in P}$ such that F^0_p is holomorphic on K for all $p \in P$, and F^0_p is holomorphic on X when $p \in P'_0$. Find a homotopy of liftings $F^t = \{F^t_p\}_{p\in P}$ ($t \in [0, 1]$) that are holomorphic on K, uniformly close to F^0 on $K \times P$, the homotopy is fixed for all p in a neighborhood of P_0, and F^1_p is holomorphic on L for all $p \in P$.

Granted the Main Step, a solution satisfying the conclusion of Theorem 4.2 is then obtained by induction over a suitable exhaustion of X.

Proof of the Main Step. We cover the compact set $\bigcup_{p\in P} f_p(\overline{L\backslash K}) \subset \widetilde{Z}$ by open sets $U_1, \ldots, U_N \subset \widetilde{Z}$ such that every restricted submersion $\pi\colon Z|_{U_j} \to U_j$ admits a finite dominating family of π-sprays. In the fiber bundle case we choose the sets U_j such that $Z|_{U_j}$ is isomorphic to the trivial bundle $U_j \times Y \to U_j$ with POP fiber Y.

Choose a Cartan string $\mathcal{A} = (A_0, A_1, \ldots, A_n)$ in X [12, Def. 4.2] such that $K = A_0$ and $L = \bigcup_{j=0}^{n} A_j$. The construction is explained in [12, Corollary 4.5]: It suffices to choose each of the compact sets A_k to be a strongly pseudoconvex domain such that $\left(\bigcup_{j=0}^{k-1} A_j, A_k\right)$ is a Cartan pair for all $k = 1, \ldots, n$. In addition, we choose the sets $A_1, \ldots, A_n$ small enough such that $f_p(A_j)$ is contained in one of the sets U_l for every $p \in P$ and $j = 1, \ldots, n$.

We cover P by compact subsets $P_1, \ldots, P_m$ such that for every $j = 1, \ldots, m$ and $k = 1, \ldots, n$, there is a neighborhood $P'_j \subset P$ of P_j such that the set $\bigcup_{p\in P'_j} f_p(A_k)$ is contained in one of the sets U_l.

As in [12] we denote by $\mathcal{K}(\mathcal{A})$ the *nerve complex* of $\mathcal{A} = (A_0, A_1, \ldots, A_n)$, i.e., a combinatorial simplicial complex consisting of all multiindices $J = (j_0, j_1, \ldots, j_k)$, with $0 \le j_0 < j_1 < \cdots < j_k \le n$, such that

$$A_J = A_{j_0} \cap A_{j_1} \cap \cdots \cap A_{j_k} \neq \emptyset.$$

Its *geometric realization*, $K(\mathcal{A})$, is a finite polyhedron in which every multiindex $J = (j_0, j_1, \ldots, j_k) \in \mathcal{K}(\mathcal{A})$ of length $k + 1$ determines a closed k-dimensional face $|J| \subset K(\mathcal{A})$, homeomorphic to the standard k-simplex in $\mathbb{R}^k$, and every k-dimensional face of $K(\mathcal{A})$ is of this form. The face $|J|$ is called the *body* (or *carrier*) of J, and J is the *vertex scheme* of $|J|$. Given $I, J \in \mathcal{K}(\mathcal{A})$ we have $|I|\cap|J| = |I\cup J|$. The vertices of $K(\mathcal{A})$ correspond to the individual sets A_j in $\mathcal{A}$, i.e., to singletons $(j) \in \mathcal{K}(\mathcal{A})$. (See [18] or [30] for simplicial complexes and polyhedra.)

Given a compact set A in X, we denote by $\Gamma_{\mathcal{O}}(A, Z)$ the space of all sections of $h\colon Z \to X$ that are holomorphic over some unspecified open neighborhood A in Z, in the sense of germs at A.

Recall that a *holomorphic* $\mathcal{K}(\mathcal{A}, Z)$*-complex* [12, Def. 3.2] is a continuous family of holomorphic sections

$$F_* = \{F_J \colon |J| \to \Gamma_{\mathcal{O}}(A_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\}$$

satisfying the following compatibility conditions:

$$I, J \in \mathcal{K}(\mathcal{A}),\ I \subset J \Longrightarrow F_J(t) = F_I(t)|_{A_J}\ (\forall t \in |I|).$$

Note that

- $F_{(k)}$ a holomorphic section over (a neighborhood of) A_k,
- $F_{(k_0,k_1)}$ is a homotopy of holomorphic sections over $A_{k_0} \cap A_{k_1}$ connecting $F_{(k_0)}$ and $F_{(k_1)}$,
- $F_{(k_0,k_1,k_2)}$ is a triangle of homotopies with vertices $F_{(k_0)}, F_{(k_1)}, F_{(k_2)}$ and sides $F_{(k_0,k_1)}, F_{(k_0,k_2)}, F_{(k_1,k_2)}$, etc.

Similarly one defines a continuous $\mathcal{K}(\mathcal{A}, Z)$-complex.

A $\mathcal{K}(\mathcal{A}, Z; P)$-complex is defined in an obvious way by adding the parameter $p \in P$. It can be viewed as a $\mathcal{K}(\mathcal{A}, Z)$-complex of P-sections of $Z \to X$, or as a family of $\mathcal{K}(\mathcal{A}, Z)$-complexes depending continuously on the parameter $p \in P$. Similarly, a $\mathcal{K}(\mathcal{A}, Z; P, P_0)$-complex is a $\mathcal{K}(\mathcal{A}, Z; P)$-complex consisting of holomorphic sections (over the set $L = \bigcup_{j=0}^n A_j$) for the parameter values $p \in P_0$. The terminology of Def. 4.1 naturally applies to complexes of sections.

By choosing the sets $A_1, \ldots, A_n$ sufficiently small and by shrinking the neighborhood P_0' (furnished by Proposition 4.4) around P_0 if necessary we can deform $F = F^0$ to a holomorphic $\mathcal{K}(\mathcal{A}, Z; P, P_0')$-complex $F_{*,*} = \{F_{*,p}\}_{p\in P}$ such that

- every section in $F_{*,p}$ projects by $\pi\colon Z \to \widetilde{Z}$ to the section f_p (such $F_{*,*}$ is called a *lifting* of the holomorphic P-section $f = \{f_p\}_{p\in P}$),
- $F_{(0),p}$ is the restriction to $A_0 = K$ of the initial section F_p^0, and
- for $p \in P_0'$, every section in $F_{*,p}$ is the restriction of F_p^0 to the appropriate subdomain (i.e., the deformation from F^0 to $F_{*,*}$ is fixed over P_0').

A completely elementary construction of such *initial holomorphic complex* $F_{*,*}$ can be found in [12, Proposition 4.7].

Remark 4.5. We observe that, although the map $h = \widetilde{h} \circ \pi\colon Z \to X$ is not necessarily a submersion (since the projection $\widetilde{h}\colon \widetilde{Z} \to X$ may have singular fibers), the construction in [12] still applies since we only work with the fiber component of F_p (over f_p) with respect to the submersion $\pi\colon Z \to \widetilde{Z}$. All lifting problems locally reduce to working with functions. □

The rest of the construction amounts to finitely many homotopic modifications of the complex $F_{*,*}$. At every step we collapse one of the cells in the complex and obtain a family (parametrized by P) of holomorphic sections over the union of the sets that determine the cell. In finitely many steps we obtain a family of *constant complexes* $F^1 = \{F_p^1\}_{p\in P}$, that is, F_p^1 is a holomorphic section of $Z \to X$ over L. This procedure is explained in [12, Sect. 5] (see in particular Proposition 5.1.). The additional lifting condition is easily satisfied at every step of the construction. In the end, the homotopy of complexes from F^0 to F^1 is replaced by a homotopy of constant complexes, i.e., a homotopy of liftings F^t of f that consist of sections over L (see the conclusion of proof of Theorem 1.5 in [12, p. 657]).

Let us describe more carefully the main step – collapsing a segment in a holomorphic complex. (All substeps in collapsing a cell reduce to collapsing a segment, each time with an additional parameter set.)

We have a special pair (A, B) of compact sets contained in $L \subset X$, called a *Cartan pair* [13, Def. 4], with B contained in one of the sets $A_1, \ldots, A_n$ in our Cartan string $\mathcal{A}$. (Indeed, B is the intersection of some of these sets.) Further, we have an additional compact parameter set $\widetilde{P}$ (which appears in the proof) and families of holomorphic sections of $h\colon Z \to X$, $a_{(p,\tilde{p})}$ over A and $b_{(p,\tilde{p})}$ over B, depending continuously on $(p, \tilde{p}) \in P \times \widetilde{P}$ and projecting by $\pi\colon Z \to \widetilde{Z}$ to the section f_p. For $p \in P'_0$ we have $a_{(p,\tilde{p})} = b_{(p,\tilde{p})}$ over $A \cap B$. These two families are connected over $A \cap B$ by a homotopy of holomorphic sections $b^t_{(p,\tilde{p})}$ $(t \in [0, 1])$ such that

$$b^0_{(p,\tilde{p})} = a_{(p,\tilde{p})}, \quad b^1_{(p,\tilde{p})} = b_{(p,\tilde{p})}, \quad \pi \circ b^t_{(p,\tilde{p})} = f_p$$

hold for each $p \in P$ and $t \in [0, 1]$, and the homotopy is fixed for $p \in P'_0$. These two families are joined into a family of holomorphic sections $\widetilde{a}_{(p,\tilde{p})}$ over $A \cup B$, projecting by π to f_p. The deformation consists of two substeps:

1. by applying the Oka-Weil theorem [11, Theorem 4.2] over the pair $A \cap B \subset B$ we approximate the family $a_{(p,\tilde{p})}$ sufficiently closely, uniformly on a neighborhood of $A \cap B$, by a family $\widetilde{b}_{(p,\tilde{p})}$ of holomorphic sections over B;
2. assuming that the approximation in (1) is sufficiently close, we glue the families $a_{(p,\tilde{p})}$ and $\widetilde{b}_{(p,\tilde{p})}$ into a family of holomorphic sections $\widetilde{a}_{(p,\tilde{p})}$ over $A \cup B$ such that $\pi \circ \widetilde{a}_{(p,\tilde{p})} = f_p$.

For Substep (2) we can use local holomorphic sprays as in [8, Proposition 3.1], or we apply [11, Theorem 5.5]. The projection condition $\pi \circ \widetilde{a}_{(p,\tilde{p})} = f_p$ is a trivial addition.

Substep (1) is somewhat more problematic as it requires a dominating family of π-sprays on $Z|_U$ over an open set $U \subset \widetilde{Z}$ to which the sections $b^t_{(p,\tilde{p})}$ project. (In the fiber bundle case we need triviality of the restricted bundle $Z|_U \to U$ and POP of the fiber.) Recall that B is contained in one of the sets A_k, and therefore

$$\bigcup_{p \in P'_j} f_p(B) \subset \bigcup_{p \in P'_j} f_p(A_k) \subset U_{l(j,k)}.$$

Since $\pi \circ b^t_{(p,\tilde{p})} = f_p$ and Z admits a dominating family of π-sprays over each set U_l, Substep (1) applies separately to each of the m families

$$\{b^t_{(p,\tilde{p})}\colon p \in P'_j,\ \tilde{p} \in \widetilde{P},\ t \in [0, 1]\}, \quad j = 1, \ldots, m.$$

To conclude the proof of the Main Step we use the *stepwise extension method*, similar to the one in [12, pp. 138–139]. In each step we make the lifting holomorphic for the parameter values in one of the sets P_j, keeping the homotopy fixed over the union of the previous sets.

We begin with P_1. The above shows that the Main Step can be accomplished in finitely many applications of Substeps (1) and (2), using the pair of parameter spaces $P_0 \cap P'_1 \subset P'_1$ (instead of $P_0 \subset P$). We obtain a homotopy of liftings

$\{F_p^t \colon p \in P_1', t \in [0,1]\}$ of f_p such that F_p^1 is holomorphic on L for all p in a neighborhood of P_1, and $F_p^t = F_p^0$ for all $t \in [0,1]$ and all p in a relative neighborhood of $P_0 \cap P_1'$ in P_1'. We extend this homotopy to all values $p \in P$ by replacing F_p^t by $F_p^{t\chi(p)}$, where $\chi \colon P \to [0,1]$ is a continuous function that equals one near P_1 and has support contained in P_1'. Thus F_p^1 is holomorphic on L for all p in a neighborhood V_1 of $P_0 \cup P_1$, and $F_p^1 = F_p^0$ for all p in a neighborhood of P_0.

We now repeat the same procedure with F^1 as the 'initial' lifting of f, using the pair of parameter spaces $(P_0 \cup P_1) \cap P_2' \subset P_2'$. We obtain a homotopy of liftings $\{F_p^t\}_{t\in[1,2]}$ of f_p for $p \in P_2'$ such that the homotopy is fixed for all p in a neighborhood of $(P_0 \cup P_1) \cap P_2'$ in P_2', and F_p^2 is holomorphic on L for all p in a neighborhood of $P_0 \cup P_1 \cup P_2$ in P.

In m steps of this kind we get a homotopy $\{F^t\}_{t\in[0,m]}$ of liftings of f such that F_p^m is holomorphic on L for all $p \in P$, and the homotopy is fixed in a neighborhood of P_0 in P. It remains to rescale the parameter interval $[0,m]$ back to $[0,1]$.

This concludes the proof in the special case when X is a Stein manifold and $X' = \emptyset$. In the general case we follow the induction scheme in the proof of the parametric Oka principle for stratified fiber bundles with POP fibers in [10]; Cartan strings are now used inside the smooth strata.

When $\pi \colon Z \to \widetilde{Z}$ is a fiber bundle, we apply the one-step approximation and gluing procedure as in [8], without having to deal with holomorphic complexes. The Oka-Weil approximation theorem in Substep (1) is replaced by POP of the fiber. $\square$

Proof of Proposition 4.4. We begin by considering the special case when $\pi \colon Z = \widetilde{Z} \times \mathbb{C} \to \widetilde{Z}$ is a trivial line bundle. We have $F_p = (f_p, g_p)$ where g_p is a holomorphic function on X for $p \in P_0$, and is holomorphic on $K \cup X'$ for all $p \in P$. We replace X by a relatively compact subset containing $\bar{D}$ and consider it as a closed complex subvariety of a Euclidean space $\mathbb{C}^N$. Choose bounded pseudoconvex domains $\Omega \Subset \Omega'$ in $\mathbb{C}^N$ such that $\bar{D} \subset \Omega \cap X$.

By [13, Lemma 3.1] there exist bounded linear extension operators

$$\begin{aligned} S \colon H^\infty(X \cap \Omega') &\longrightarrow H^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega), \\ S' \colon H^\infty(X' \cap \Omega') &\longrightarrow H^2(\Omega), \end{aligned}$$

such that $S(g)|_{X\cap\Omega} = g|_{X\cap\Omega}$, and likewise for S'. (In [13] we obtained an extension operator into $H^\infty(\Omega)$, but the Bergman space appeared as an intermediate step. Unlike the Ohsawa-Takegoshi extension theorem [26], this is a soft result depending on the Cartan extension theorem and some functional analysis; the price is shrinking of the domain.) Set

$$h_p = S(g_p|_{X\cap\Omega'}) - S'(g_p|_{X'\cap\Omega'}) \in H^2(\Omega), \qquad p \in P_0.$$

Then h_p vanishes on X', and hence it belongs to the closed subspace $H^2_{X'}(\Omega)$ consisting of all functions in $H^2(\Omega)$ that vanish on $X' \cap \Omega$. Since these are Hilbert spaces, the generalized Tietze extension theorem (a special case of Michael's convex selection theorem; see [28, Part C, Theorem 1.2, p. 232] or [3, 25]) furnishes a continuous extension of the map $P_0 \to H^2_{X'}(\Omega)$, $p \to h_p$, to a map $P \ni p \to \widetilde{h}_p \in H^2_{X'}(\Omega)$. Set

$$G_p = \widetilde{h}_p + S'(g_p|_{X' \cap \Omega'}) \in H^2(\Omega), \qquad p \in P.$$

Then

$$G_p|_{X' \cap \Omega} = g_p|_{X' \cap \Omega} \ (\forall p \in P), \quad G_p|_{X \cap \Omega} = g_p|_{X \cap \Omega} \ (\forall p \in P_0).$$

This solves the problem, except that G_p should approximate g_p uniformly on K. Choose holomorphic functions $\phi_1, \ldots, \phi_m$ on $\mathbb{C}^N$ that generate the ideal sheaf of the subvariety X' at every point in Ω'. A standard application of Cartan's Theorem B shows that in a neighborhood of K we have

$$g_p = G_p + \sum_{j=1}^{m} \phi_j \, \xi_{j,p}$$

for some holomorphic functions $\xi_{j,p}$ in a neighborhood of K, depending continuously on $p \in P$ and vanishing identically on X for $p \in P_0$. (See, e.g., [12, Lemma 8.1].)

Since the set K is $\mathcal{O}(X)$-convex, and hence polynomially convex in $\mathbb{C}^N$, an extension of the Oka-Weil approximation theorem (using a bounded linear solution operator for the $\overline{\partial}$-equation, given for instance by Hörmander's L^2-methods [17] or by integral kernels) furnishes functions $\widetilde{\xi}_{j,p} \in \mathcal{O}(\Omega)$, depending continuously on $p \in P$, such that $\widetilde{\xi}_{j,p}$ approximates $\xi_{j,p}$ as close as desired uniformly on K, and it vanishes on $X \cap \Omega$ when $p \in P_0$. Setting

$$\widetilde{g}_p = G_p + \sum_{j=1}^{m} \phi_j \, \widetilde{\xi}_{j,p}, \quad p \in P$$

gives the solution. This proof also applies to vector-valued maps by applying it componentwise.

The general case reduces to the special case by using that for every $p_0 \in P_0$, the Stein subspace $F_{p_0}(X)$ (resp. $f_{p_0}(X)$) admits an open Stein neighborhood in Z (resp. in $\widetilde{Z}$) according to a theorem of Siu [2, 29]. Embedding these neighborhoods in Euclidean spaces and using holomorphic retractions onto fibers of π (see [10, Proposition 3.2]), the special case furnishes neighborhoods $U_{p_0} \subset U'_{p_0}$ of p_0 in P and a P-section $F' \colon \bar{D} \times P \to Z$, homotopic to F through liftings of f, such that

(i) $\pi \circ F'_p = f_p$ for all $p \in P$,
(ii) F'_p is holomorphic on $\bar{D}$ when $p \in U_{p_0}$,
(iii) $F'_p = F_p$ for $p \in P_0 \cup (P \backslash U'_{p_0})$,
(iv) $F'_p|_{X' \cap D} = F_p|_{X' \cap D}$ for all $p \in P$, and
(v) F' approximates F on $K \times P$.

(The special case is first used for parameter values p in a neighborhood U'_{p_0} of p_0; the resulting family of holomorphic maps $\bar{D} \times U'_{p_0} \to \mathbb{C}^N$ is then patched with F by using a cut-off function $\chi(p)$ with support in U'_{p_0} that equals one on a neighborhood U_{p_0} of p_0, and applying holomorphic retractions onto the fibers of π.) In finitely many steps of this kind we complete the proof. □

Remark 4.6. One might wish to extend Theorem 4.2 to the case when $\pi\colon E \to B$ is a *stratified* subelliptic submersion, or a stratified fiber bundle with POP fibers. The problem is that the induced stratifications on the pull-back submersions $f_p^* E \to X$ may change discontinuously with respect to the parameter p. Perhaps one could get a positive result by assuming that the stratification of $E \to B$ is suitably compatible with the variable map $f_p\colon X \to B$. □

Recall (Def. 4.3) that a holomorphic map $\pi\colon E \to B$ satisfies POP if the conclusion of Theorem 4.2 holds. We show that this is a local property.

Theorem 4.7. (Localization principle for POP) *A holomorphic submersion $\pi\colon E \to B$ of a complex space E onto a complex space B satisfies* POP *if and only if every point $x \in B$ admits an open neighborhood $U_x \subset B$ such that the restricted submersion $\pi\colon E|_{U_x} \to U_x$ satisfies* POP.

Proof. If $\pi\colon E \to B$ satisfies POP then clearly so does its restriction to any open subset U of B.

Conversely, assume that B admits an open covering $\mathcal{U} = \{U_\alpha\}$ by open sets such that every restriction $E|_{U_\alpha} \to U_\alpha$ enjoys POP. When proving POP for $\pi\colon E \to B$, a typical step amounts to choosing small compact sets $A_1, \ldots, A_n$ in the source (Stein) space X such that, for a given compact set $A_0 \subset X$, $\mathcal{A} = (A_0, A_1, \ldots, A_n)$ is a Cartan string. We can choose the sets $A_1, \ldots, A_n$ sufficiently small such that each map $f_p\colon X \to B$ in the given family sends each A_j into one of the sets $U_\alpha \in \mathcal{U}$.

To the string $\mathcal{A}$ we associate a $\mathcal{K}(\mathcal{A}, Z; P, P_0)$-complex $F_{*,*}$ which is then inductively deformed into a holomorphic P-map $\widetilde{F}\colon \bigcup_{j=0}^n A_j \times P \to E$ such that $\pi \circ \widetilde{F} = f$. The main step in the inductive procedure amounts to patching a pair of liftings over a Cartan pair (A', B') in X, where the set B' is contained in one of the sets $A_1, \ldots, A_n$ in the Cartan string $\mathcal{A}$. This is subdivided into substeps (1) and (2) (see the proof of Theorem 4.2). Only the first of these substeps, which requires a Runge-type approximation property, is a nontrivial condition on the submersion $E \to B$. It is immediate from the definitions that this approximation property holds if there is an open set $U \subset B$ containing the image $f_p(B')$ (for a certain set of parameter values $p \in P$) such that the restricted submersion $E|_U \to U$ satisfies POP. In our case this is so since we have insured that $f_p(B') \subset f_p(A_j) \subset U_\alpha$ for some $j \in \{1, \ldots, n\}$ and $U_\alpha \in \mathcal{U}$. □

5. Ascent and descent of the parametric Oka property

In this section we prove Theorem 1.2 stated in Section 1.

Proof of (i): Assume that B enjoys POP (which is equivalent to PCAP). Let (K, Q) be a special convex pair in $\mathbb{C}^n$ (Def. 3.1), and let $F\colon Q \times P \to E$ be a (P, P_0)-map that is holomorphic on K (Def. 4.1).

Then $f = \pi \circ F\colon Q \times P \to B$ is a (P, P_0)-map that is holomorphic on K. Since B enjoys POP, there is a holomorphic P-map $g\colon Q \times P \to B$ that agrees with f on $Q \times P_0$ and is uniformly close to f on a neighborhood of $K \times P$ in $\mathbb{C}^n \times P$.

If the latter approximation is close enough, there exists a holomorphic P-map $G\colon K \times P \to E$ such that $\pi \circ G = g$, G approximates F on $K \times P$, and $G = F$ on $K \times P_0$. To find such lifting of g, we consider graphs of these maps (as in the proof of Theorem 4.2) and apply a holomorphic retraction onto the fibers of π [10, Proposition 3.2].

Since $G = F$ on $K \times P_0$, we can extend G to $(K \times P) \cup (Q \times P_0)$ by setting $G = F$ on $Q \times P_0$.

Since $\pi\colon E \to B$ is a Serre fibration and K is a strong deformation retract of Q (these sets are convex), G extends to a continuous (P, P_0)-map $G\colon Q \times P \to E$ such that $\pi \circ G = g$. The extended map remains holomorphic on K.

By Theorem 4.2 there is a homotopy of liftings $G^t\colon Q \times P \to E$ of g $(t \in [0, 1])$ which is fixed on $Q \times P_0$ and is holomorphic and uniformly close to $G^0 = G$ on $K \times P$. The holomorphic P-map $G^1\colon Q \times P \to E$ then satisfies the condition in Def. 3.2 relative to F. This proves that E enjoys PCAP and hence POP.

Proof of (ii): Assume that E enjoys POP. Let (K, Q) be a special convex pair, and let $f\colon Q \times P \to B$ be a (P, P_0)-map that is holomorphic on K. Assuming that P is contractible, the Serre fibration property of $\pi\colon E \to B$ insures the existence of a continuous P-map $F\colon Q \times P \to E$ such that $\pi \circ F = f$. (The subset P_0 of P does not play any role here.) Theorem 4.2 furnishes a homotopy $F^t\colon Q \times P \to E$ $(t \in [0, 1])$ such that

(a) $F^0 = F$,
(b) $\pi \circ F^t = f$ for each $t \in [0, 1]$, and
(c) F^1 is a (P, P_0)-map that is holomorphic on K.

This is accomplished in two steps: We initially apply Theorem 4.2 with $Q \times P_0$ to obtain a homotopy $F^t\colon Q \times P_0 \to E$ $(t \in [0, \frac{1}{2}])$, satisfying properties (a) and (b) above, such that $F_p^{1/2}$ is holomorphic on Q for all $p \in P_0$. For trivial reasons this homotopy extends continuously to all values $p \in P$. In the second step we apply Theorem 4.2 over $K \times P$, with $F^{1/2}$ as the initial lifting of f and keeping the homotopy fixed for $p \in P_0$ (where it is already holomorphic), to get a homotopy F^t $(t \in [\frac{1}{2}, 1])$ such that $\pi \circ F^t = f$ and F_p^1 is holomorphic on K for all $p \in P$.

Since E enjoys POP, F^1 can be approximated uniformly on $K \times P$ by holomorphic P-maps

$$\widetilde{F}\colon Q \times P \to E$$

such that $\widetilde{F} = F^1$ on $Q \times P_0$.

Then

$$\widetilde{f} = \pi \circ \widetilde{F} \colon Q \times P \to B$$

is a holomorphic P-map that agrees with f on $Q \times P_0$ and is close to f on $K \times P$.

This shows that B enjoys PCAP for any contractible (compact, Hausdorff) parameter space P and for any closed subspace P_0 of P. Since the implication PCAP$\Longrightarrow$POP in Theorem 3.3 holds for each specific pair (P_0, P) of parameter spaces, we infer that B also enjoys POP for such parameter pairs. This completes the proof of (ii).

Proof of (iii): Contractibility of P was used in the proof of (ii) to lift the map $f \colon Q \times P \to B$ to a map $F \colon Q \times P \to E$. Such a lift exists for every topological space if $\pi \colon E \to B$ is a weak homotopy equivalence. This is because a Serre fibration between smooth manifolds is also a Hurewicz fibration (by Cauty [1]), and a weak homotopy equivalence between them is a homotopy equivalence by the Whitehead Lemma. □

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Added in proofs

Since the completion of this paper, the author gave a positive answer to the question posed in Remark 1.4 for parameter spaces $P_0 \subset P$ that are compact sets in a Euclidean space $\mathbb{R}^m$ (C. R. Acad. Sci. Paris, Ser. I **347**, 1017–1020 (2009); C. R. Acad. Sci. Paris, Ser. I (2009)).

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