Illinois Journal of Mathematics
Volume 56, Number 1, Spring 2012, Pages 53–65
S 0019-2082

CHARACTERIZATIONS OF PROJECTIVE HULLS BY ANALYTIC DISCS

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Dedicated to John P. D'Angelo on the occasion of his 60th birthday

ABSTRACT. The notion of the projective hull of a compact set in a complex projective space $\mathbb{P}^n$ was introduced by Harvey and Lawson in 2006. In this paper, we describe the projective hull by Poletsky sequences of analytic discs, in analogy to the known descriptions of the holomorphic and the plurisubharmonic hull.

1. Projective hulls

Given a compact set K in the complex projective space $\mathbb{P}^n$, Harvey and Lawson [HL] introduced its *projective hull* $\widehat{K}_{\mathbb{P}^n}$ as the set of all points $x \in \mathbb{P}^n$ for which there exists a constant $C(x) < +\infty$ satisfying

$$|\mathcal{P}(x)| \le C(x)^d \sup_K |\mathcal{P}| \tag{1.1}$$

for all holomorphic sections $\mathcal{P}$ of the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ and all integers $d > 0$. Their principal motivation was to extend the notion of the polynomial hull and to generalize Wermer's classical theorem [Wer] that the polynomial hull of a closed real analytic curve $\gamma \subset \mathbb{C}^n$ is either γ itself, or a complex curve with boundary γ.

In this note, we give several descriptions of projective hulls by sequences of analytic discs, in analogy to Poletsky's description of the polynomial hull [Pol]. We also characterize the polynomial hull of a compact connected circular set K in $\mathbb{C}^n$ by analytic discs that have the entire boundary circle close to K (see Theorem 5.3).

We begin by recalling some of the main results from the paper [HL] by Harvey and Lawson, with emphasis on those that are used in this paper. In

Received January 3, 2012; received in final form April 2, 2012.
Research supported by grants P1-0291 and J1-2152, Republic of Slovenia.
2010 *Mathematics Subject Classification.* Primary 32U05. Secondary 32H02, 32E10.

the sequel, we let $C_K(x) \geq 1$ be the smallest constant satisfying (1.1) and set $C_K(x) = +\infty$ for $x \in \mathbb{P}^n \setminus \widehat{K}_{\mathbb{P}^n}$; this gives the *best constant function* $C_K : \mathbb{P}^n \to [1, +\infty]$.

There is a simple description of the projective hull in terms of the polynomial hull. Let $\pi : \mathbb{C}^{n+1}_* = \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the standard projection taking a point $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}_*$ onto the point $x = \pi(z) \in \mathbb{P}^n$ with homogeneous coordinates $[z_0 : \cdots : z_n]$. Denote by $L_x = \pi^{-1}(x) \cup \{0\}$ the complex line in $\mathbb{C}^{n+1}$ over x. Let $\mathbb{B} = \{z \in \mathbb{C}^{n+1} : |z| < 1\}$ and $S = b\mathbb{B} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$. Given a compact set $K \subset \mathbb{P}^n$, we define compact subsets $S_K \subset B_K$ of $\mathbb{C}^{n+1}$ by

$$B_K = \overline{\mathbb{B}} \cap \big(\pi^{-1}(K) \cup \{0\}\big), \qquad S_K = S \cap \pi^{-1}(K). \tag{1.2}$$

Clearly these two sets have the same polynomial hull in $\mathbb{C}^{n+1}$, $\widehat{B}_K = \widehat{S}_K$. The intersection of $\widehat{B}_K$ with the complex line L_x over any point $x \in \mathbb{P}^n$ is a closed disc $\triangle_x$ in L_x of radius $r(x) \geq 0$ centered at 0. Then we have (see [HL, Proposition 5.2])

$$\widehat{K}_{\mathbb{P}^n} = \big\{x \in \mathbb{P}^n : r(x) > 0\big\} \quad \text{and} \quad C_K(x) = \frac{1}{r(x)}, \tag{1.3}$$

where C_K is the best constant function from (1.1). Equivalently,

$$\widehat{K}_{\mathbb{P}^n} = \pi\big(\widehat{S}_K \setminus \{0\}\big). \tag{1.4}$$

This follows by observing that sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ correspond to homogeneous polynomials of degree d on $\mathbb{C}^{n+1}$. Indeed, let us denote by $L = \mathcal{O}_{\mathbb{P}^n}(-1)$ the universal line bundle over $\mathbb{P}^n$. The total space of L is $\mathbb{C}^{n+1}$ blown up at the origin, with the zero section $L_0 \cong \mathbb{P}^n$ corresponding to the exceptional fiber, and $L \setminus L_0 \cong \mathbb{C}^{n+1}_*$. A homogeneous polynomial of degree d on $\mathbb{C}^{n+1}$ defines a linear functional on the dth tensor power $L^{\otimes d} \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ of L, and hence a holomorphic section of the dual bundle $(L^{\otimes d})^* \cong \mathcal{O}_{\mathbb{P}^n}(d)$. This shows that the projective hull equals the set (1.4) if we replace the polynomial hull of S_K by the (ostensibly larger) hull obtained by homogeneous polynomials. Since S_K is circular, these two hulls coincide [HL, Proposition 5.4].

Another characterization of the projective hull is given in terms of the extremal function of K with respect to the family $\mathrm{Psh}_\omega(\mathbb{P}^n)$ of all upper semicontinuous functions $v : \mathbb{P}^n \to \mathbb{R} \cup \{-\infty\}$ satisfying $\mathrm{dd}^c v + \omega = 2\mathrm{i}\partial\bar\partial v + \omega \geq 0$, where $\omega = \omega_{FS}$ is the Fubini–Study form on $\mathbb{P}^n$. (Here $\mathrm{d}^c = \mathrm{i}(\bar\partial - \partial)$.) Setting

$$V_K^\omega = \sup\big\{v \in \mathrm{Psh}_\omega\big(\mathbb{P}^n\big) : v \leq 0 \text{ on } K\big\},$$

we have that

$$\widehat{K}_{\mathbb{P}^n} = \big\{x \in \mathbb{P}^n : V_K^\omega(x) < +\infty\big\}, \quad V_K^\omega(x) = \log C_K(x). \tag{1.5}$$

In particular, $\widehat{K}_{\mathbb{P}^n} \neq \mathbb{P}^n$ if and only if K is ω-pluripolar, that is, there exists $v \in \mathrm{Psh}_\omega(\mathbb{P}^n)$ with $v \not\equiv -\infty$ and $K \subset \{v = -\infty\}$.

The *Lelong class* $\mathcal{L}_{\mathbb{C}^n}$ on $\mathbb{C}^n$ is the set of all plurisubharmonic functions $v : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ for which there exist constants $r > 0$ and $C \in \mathbb{R}$ (depending on v) such that

$$v(z) \le \log|z| + C, \quad z \in \mathbb{C}^n, \ |z| > r.$$

Given a subset $E \subset \mathbb{C}^n$, the *Siciak–Zaharyuta extremal function* $V_E : \mathbb{C}^n \to \mathbb{R} \cup \{\infty\}$ (see [Kli]) is defined by

$$V_E(z) = \sup\{v(z) : v \in \mathcal{L}, v|_E \le 0\}.$$

If K is a compact set in an affine chart $\mathbb{C}^n \subset \mathbb{P}^n$, then a point $z \in \mathbb{C}^n$ belongs to $\widehat{K}_{\mathbb{P}^n}$ if and only if $V_K(z) < +\infty$. This is seen by considering $\mathbb{C}^n$ as the hyperplane $\{z_0 = 1\} \subset \mathbb{C}^{n+1}$ and observing that a polynomial of degree d on $\mathbb{C}^n$ corresponds to a homogeneous polynomial of degree d on $\mathbb{C}^{n+1}$. One also has a bijective correspondence between the Lelong class $\mathcal{L}_{\mathbb{C}^n}$ and $\mathrm{Psh}_\omega(\mathbb{P}^n)$ (see [BT] and [GZ, Example 1.2]). Explicitly, given $v \in \mathcal{L}_{\mathbb{C}^n}$, the function

$$\widetilde{v}(z_0, \ldots, z_n) = \log|z_0| + v\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)$$

is plurisubharmonic on $\mathbb{C}^{n+1}_*$; its restriction to the unit sphere S is circle invariant and hence defines a function v' on $\mathbb{P}^n$. It is easily seen that $v \mapsto v'$ is a bijective map of $\mathcal{L}_{\mathbb{C}^n}$ onto $\mathrm{Psh}_\omega(\mathbb{P}^n)$.

Since the polynomial hull $\widehat{K} := \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ of a compact set $K \subset \mathbb{C}^n \subset \mathbb{P}^n$ equals $\{V_K = 0\}$, it is contained in the projective hull. Conversely, if $\widehat{K}_{\mathbb{P}^n}$ lies in the complement $\Omega = \mathbb{P}^n \setminus \Lambda$ of an algebraic hypersurface $\Lambda \subset \mathbb{P}^n$, then $\widehat{K}_{\mathbb{P}^n} = \widehat{K}_{\mathcal{O}(\Omega)}$ [HL, Corollary 12.7]. However, it is in general impossible to describe the projective hull of a compact affine set $K \subset \mathbb{P}^n$ in terms of its polynomial hulls in affine subsets of $\mathbb{P}^n$ containing K. For example, there exists a smooth closed curve in $\mathbb{C}^2$ whose polynomial hull is a holomorphic disc bounded by the curve, but whose projective hull is $\mathbb{P}^2$ [HL, Remark 4.5].

We now describe our main results. Let $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc in $\mathbb{C}$ and let $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be its boundary circle. An *analytic disc* in a complex space X is a continuous map $f : \overline{\mathbb{D}} \to X$ which is holomorphic in $\mathbb{D}$. The point $f(0)$ is called the *center* of the disc. We denote the space of all such discs by $\mathcal{A}_X$.

Our first characterization of the projective hull, which applies to any compact set in $\mathbb{P}^n$, is in terms of Poletsky sequences of discs in $\mathbb{P}^n$ that have a bounded lifting property with respect to the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ (see Theorem 3.2). By a *Poletsky sequence of discs* f_j (for a given compact set K), we mean that the set of points $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$ for which $\mathrm{dist}(f_j(\mathrm{e}^{\mathrm{i}t}), K) < 1/j$ has measure $> 2\pi - 1/j$. The proof uses Poletsky's theorem characterizing polynomial hulls by analytic discs.

The second characterization is motivated by a result of Lawson and Wermer [LW] in the case when K is a simple closed curve; we simplify their proof and extend it to any compact connected set K contained in an affine chart $\mathbb{C}^n = \mathbb{P}^n \setminus H$ of $\mathbb{P}^n$. The projective hull of K is the set of all centers of sequences of analytic discs f_j in $\mathbb{P}^n$, with boundary circles $f_j(\mathbb{T})$ converging to K, such that a certain disc functional J is uniformly bounded on the sequence (see Theorem 4.1). The functional $J(f)$, which was first introduced by Lárusson and Sigurdsson in [LS], is determined by the intersection divisor of the disc with the hyperplane at infinity; see (4.1) for the explicit formula.

Perhaps the most interesting is our third characterization which applies to any compact *connected* set K in $\mathbb{P}^n$ (see Theorem 5.1). Let S_K be defined by (1.2). Then a point $x \in \mathbb{P}^n$ belongs to the projective hull $\widehat{K}_{\mathbb{P}^n}$ if and only if there is a point $0 \neq p \in L_x = \pi^{-1}(x)$ over x and a sequence of analytic discs $F_j : \overline{\mathbb{D}} \to \mathbb{C}^{n+1} \setminus \{0\}$ satisfying

$$F_j(0) = p \quad (\forall j \in \mathbb{N}), \qquad \lim_{j\to\infty} \max_{t\in[0,2\pi]} \operatorname{dist}\big(F_j(\mathrm{e}^{\mathrm{i}t}), S_K\big) = 0.$$

The projected sequence of analytic discs $f_j = \pi \circ F_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ clearly enjoys the bounded lifting property and also

$$f_j(0) = x \quad (\forall j \in \mathbb{N}), \qquad \lim_{j\to\infty} \max_{t\in[0,2\pi]} \operatorname{dist}\big(f_j(\mathrm{e}^{\mathrm{i}t}), K\big) = 0.$$

The nontrivial addition when compared to Theorem 3.2 is that the entire circle $\mathbb{T}$ is mapped arbitrarily close to the set K (resp. to S_K).

Both the second and the third characterization mentioned above are based on a result of Lárusson and Sigurdsson [LS] which expresses the Siciak–Zaharyuta extremal function of a connected open set in $\mathbb{C}^n$ as the envelope of a certain disc functional.

2. Plurisubharmonic hulls and analytic discs

Recall that the plurisubharmonic hull of a compact set K in a complex space X is defined by

$$\widehat{K}_{\mathrm{Psh}(X)} = \Big\{ x \in X : v(x) \le \sup_K v, \forall v \in \mathrm{Psh}(X) \Big\}.$$

If X is a Stein space, then $\widehat{K}_{\mathrm{Psh}(X)}$ coincides with the holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$; in particular, $\widehat{K}_{\mathrm{Psh}(\mathbb{C}^n)} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is the polynomial hull of K. The following result is due to Poletsky [Pol] in the basic case when X is a domain in $\mathbb{C}^n$, to Rosay [Ro1], [Ro2] when X is a complex manifold, and to the authors [DF1, Corollary 1.4] in the general case stated here. (We give a somewhat more precise formulation that will be used in the sequel. Additional references can be found in [DF1].)

THEOREM 2.1. *Assume that X is a locally irreducible complex space and dist_X is a distance function on X inducing the standard topology. Let K be a compact set in X whose plurisubharmonic hull $\widehat{K}_{\mathrm{Psh}(X)}$ is compact. Then a* *point $x \in X$ belongs to $\widehat{K}_{\mathrm{Psh}(X)}$ if and only if for every open relatively compact set $\Omega \Subset X$ containing $\widehat{K}_{\mathrm{Psh}(X)}$ there is a sequence of analytic discs $f_j \in \mathcal{A}_\Omega$ satisfying $f_j(0) = x$ $(j = 1, 2, \ldots)$ and*

$$\Big|\big\{t \in [0, 2\pi] : \operatorname{dist}_X\big(f_j\big(\mathrm{e}^{\mathrm{i}t}\big), K\big) < 1/j\big\}\Big| > 2\pi - 1/j, \quad j \in \mathbb{N}. \tag{2.1}$$

Here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

DEFINITION 2.2. A sequence of analytic discs f_j satisfying the conditions in the above theorem, with supports contained in a compact subset of X, is called a *P-sequence* for the pair (K, x).

One can replace $1/j$ in (2.1) by any sequence $\varepsilon_j > 0$ decreasing to 0 without changing the conclusion of the theorem. The existence of a P-sequence for (K, x) trivially implies that $x \in \widehat{K}_{\mathrm{Psh}(X)}$, but the converse is nontrivial. Applying this theorem to a family of neighborhoods shrinking down to the hull $\widehat{K}_{\mathrm{Psh}(X)}$ we obtain the following corollary.

COROLLARY 2.3. *Assume that the sets $K \subset X$ satisfy the hypotheses of Theorem 2.1. For every point $x \in \widehat{K}_{\mathrm{Psh}(X)}$ there exists a sequence of analytic discs $f_j \in \mathcal{A}_X$ satisfying $f_j(0) = x$, the condition (2.1), and also*

$$\max_{\zeta \in \overline{D}} \operatorname{dist}_X\big(f_j(\zeta), \widehat{K}_{\mathrm{Psh}(X)}\big) < 1/j, \quad j = 1, 2, \ldots.$$

By Wold [Wol], Corollary 2.3 implies the following theorem of Duval and Sibony [DS] with additional control of the support of the current.

COROLLARY 2.4. *Let K be a compact set in $\mathbb{C}^n$. For every point $p \in \widehat{K}$ there exists a positive $(1,1)$-current T on $\mathbb{C}^n$ (acting on the space of $(1,1)$-forms with continuous coefficients) satisfying $p \in \operatorname{supp} T \subset \widehat{K}$ and $\mathrm{dd}^c T = \mu - \delta_p$, where μ is a probability measure on K and δ_p is the Dirac mass at p.*

The converse is trivial: If a current T with these properties exists, then for any plurisubharmonic function $u \in \mathcal{C}^2(\mathbb{C}^n)$ we have

$$0 \le T\big(\mathrm{dd}^c u\big) = \int_K u \,\mathrm{d}\mu - u(p) \le \sup_K u - u(p)$$

and hence $u(p) \le \sup_K u$.

In a Stein space X, we get the same description for the part of the hull in the regular locus even if X is not locally irreducible.

PROPOSITION 2.5. *Let K be a compact set in Stein space X such that $K \not\subset X_{\mathrm{sing}}$. Choose a relatively compact pseudoconvex Runge domain $V \Subset X$ containing $\widehat{K}_{\mathcal{O}(X)}$. Then a point $x \in X_{\mathrm{reg}} \cap V$ belongs to $\widehat{K}_{\mathcal{O}(X)}$ if and only if for every open set $U \supset K$ and $\varepsilon > 0$ there exist a disc $f \in \mathcal{A}_V$ and a set $E_f \subset [0, 2\pi]$ of Lebesgue measure $|E_f| < \varepsilon$ such that*

$$f(0) = x \quad \text{and} \quad f\big(\mathrm{e}^{\mathrm{i}t}\big) \in U \quad \text{for all } t \in [0, 2\pi] \backslash E_f.$$

Remark 2.6. We do not know whether the same conclusion holds for points $x \in \widehat{K}_{\mathcal{O}(X)} \cap X_{\mathrm{sing}}$. However, if K is entirely contained in the singular locus X_{sing}, then its holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ also lies in X_{sing} and $\widehat{K}_{\mathcal{O}(X)} = \widehat{K}_{\mathcal{O}(X_{\mathrm{sing}})}$, so we may apply Proposition 2.5 to the Stein space X_{sing}.

Proof of Proposition 2.5. Assume first that a point $x \in X$ satisfies the stated conditions; we shall prove that $x \in \widehat{K}_{\mathcal{O}(X)}$. Choose a function $\rho \in \mathrm{Psh}(X)$. Set $M = \sup_K \rho$ and $M' = \sup_V \rho$. Pick a number $\varepsilon > 0$ and an open set U with $K \subset U \Subset V$ such that $\sup_U \rho < M + \varepsilon$. Let the disc $f \in \mathcal{A}_V$ and the set $E_f \subset [0, 2\pi]$ satisfy the hypotheses of the proposition. Recall that the Poisson functional $P_u(f)$ associated to an upper semicontinuous function $u : X \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$P_u(f) = \int_0^{2\pi} u\big(f\big(\mathrm{e}^{\mathrm{i}t}\big)\big)\,\frac{\mathrm{d}t}{2\pi}, \quad f \in \mathcal{A}_X. \tag{2.2}$$

Since ρ is plurisubharmonic, we have

$$\rho(x) \le P_\rho(f) = \int_{E_f} \rho\big(f\big(\mathrm{e}^{\mathrm{i}t}\big)\big)\,\frac{\mathrm{d}t}{2\pi} + \int_{[0,2\pi]\setminus E_f} \rho\big(f\big(\mathrm{e}^{\mathrm{i}t}\big)\big)\,\frac{\mathrm{d}t}{2\pi} < M'\varepsilon + M + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we get that $\rho(x) \le M$. As $\rho \in \mathrm{Psh}(X)$ was arbitrary, we conclude that $x \in \widehat{K}_{\mathrm{Psh}(X)} = \widehat{K}_{\mathcal{O}(X)}$.

Conversely, assume that $x \in \widehat{K}_{\mathcal{O}(X)} \cap X_{\mathrm{reg}}$. Since V is a Runge pseudoconvex domain in a Stein space X, we have $\widehat{K}_{\mathcal{O}(X)} = \widehat{K}_{\mathcal{O}(V)} = \widehat{K}_{\mathrm{Psh}(V)}$. The function $u : V \to [-1, 0]$ which equals -1 on the open set $U \supset K$ and equals 0 on $V \backslash U$ is upper semicontinuous. Let $v : V \to \mathbb{R}$ be the envelope of the Poisson functional P_u (2.2) corresponding to u. Then clearly $-1 \le v \le 0$ on V, and $v = -1$ on U. According to [DF2, Theorem 1.1], the function v is plurisubharmonic on $V \cap X_{\mathrm{reg}}$.

The singularity locus X_{sing} is closed and locally complete pluripolar. Since X is Stein, it is also complete pluripolar (see [Col], [Dem]). Pick a plurisubharmonic function ρ on X such that $\rho^{-1}(-\infty) = X_{\mathrm{sing}}$. The function $v + \varepsilon\rho$ is then plurisubharmonic on V for every $\varepsilon > 0$. Therefore,

$$(v + \varepsilon\rho)(x) \le \sup_K (v + \varepsilon\rho) = -1 + \varepsilon \sup_K \rho$$

for every $\varepsilon > 0$. Since $x \in X_{\mathrm{reg}}$ and $K \not\subset X_{\mathrm{sing}}$, we get by letting $\varepsilon \to 0$ that $v(x) = -1$. The definition of v implies that for every $\varepsilon > 0$ there is a disc $f \in \mathcal{A}_V$ with $f(0) = x$ and $P_u(f) < -1 + \varepsilon/2\pi$. Hence, the set $E_f = \{t \in [0, 2\pi] : f(\mathrm{e}^{\mathrm{i}t}) \notin U\}$ has measure at most ε. $\square$

3. Sequences of analytic discs with the bounded lifting property

Let $\pi : \mathbb{C}^{n+1}_* = \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the standard projection; this is a holomorphic fiber bundle with fiber $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, obtained by removing the zero section from the universal line bundle $L \to \mathbb{P}^n$. Every continuous map $f : \overline{\mathbb{D}} \to \mathbb{P}^n$ from the closed disc lifts to a continuous map $F : \overline{\mathbb{D}} \to \mathbb{C}^{n+1}_*$ as illustrated in the following diagram:

$$\begin{tikzcd} & \mathbb{C}^{n+1}_* \arrow[d, "\pi"] \\ \overline{\mathbb{D}} \arrow[ur, "F"] \arrow[r, "f"] & \mathbb{P}^n \end{tikzcd} \tag{3.1}$$

The Oka principle shows that a holomorphic map f can be lifted to a holomorphic map F (see Corollary 5.4.11 in [For, p. 196]). In fact, lifting f is equivalent to finding a nowhere vanishing section, that is, a trivialization, of the pullback f^*L of the universal bundle $L \to \mathbb{P}^n$. Since every holomorphic line bundle over the disc $\mathbb{D}$ (in fact, over any open Riemann surface) is holomorphically trivial, a lifting exists.

Definition 3.1. A sequence of analytic discs $f_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ enjoys the *bounded lifting property* if there exist a constant $C > 0$ and a sequence of analytic discs $F_j : \overline{\mathbb{D}} \to \mathbb{C}^{n+1}_*$ satisfying $\pi \circ F_j = f_j$ and

$$\sup_{t\in[0,2\pi]} \left|F_j\left(\mathrm{e}^{\mathrm{i}t}\right)\right| \le C\left|F_j(0)\right|, \quad j = 1, 2, \ldots. \tag{3.2}$$

The following result describes the projective hull in terms of Poletsky sequences of discs in $\mathbb{P}^n$ with the bounded lifting property. For the notion of a P-sequence, see Definition 2.2 above.

Theorem 3.2. *Let K be a compact set in $\mathbb{P}^n$. A point $x \in \mathbb{P}^n$ belongs to the projective hull $\widehat{K}_{\mathbb{P}^n}$ if and only if there is a P-sequence $f_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ for (K, x) with the bounded lifting property.*

Remark 3.3. The bounded lifting property in Theorem 3.2 is crucial. Indeed, taking $K = \{p\}$ to be a singleton and x to be any point of $\mathbb{P}^n$, we consider a projective line $\mathbb{P}^1 \cong \Lambda \subset \mathbb{P}^n$ through x and p. By removing from Λ a small disc around p, we obtain an analytic disc through x that has all of its boundary as close as desired to K.

Proof of Theorem 3.2. Fix a point $x \in \widehat{K}_{\mathbb{P}^n}$. Let L_x be the complex line through 0 in $\mathbb{C}^{n+1}$ determined by x. By (1.3) the disc $\triangle_x = L_x \cap \widehat{S}_K$ has positive radius $r(x) > 0$. Pick a point $p \in \triangle_x$ with $|p| = r(x)$. Theorem 2.1 furnishes a sequence of analytic discs

$$F_j : \overline{\mathbb{D}} \to \mathbb{B}_{1+1/j} = \left\{z \in \mathbb{C}^{n+1} : |z| < 1 + 1/j\right\}$$

such that for all $j \in \mathbb{N}$ we have $F_j(0) = p$ and

$$\left|\left\{t \in [0, 2\pi] : \operatorname{dist}_{\mathbb{C}^{n+1}}\left(F_j\left(\mathrm{e}^{\mathrm{i}t}\right), S_K\right) < 1/j\right\}\right| > 2\pi - 1/j. \tag{3.3}$$

By a small deformation of F_j, keeping the centers $F_j(0) = p$ fixed, we may assume that none of the image discs $F_j(\overline{\mathbb{D}})$ contains the origin, so $F_j(\overline{\mathbb{D}}) \subset$

$\mathbb{C}^{n+1}_*$ for all $j \in \mathbb{N}$. Set $f_j = \pi \circ F_j : \overline{\mathbb{D}} \to \mathbb{P}^n$; hence $f_j(0) = x$ for all j. We endow the sphere $S \subset \mathbb{C}^{n+1}$ with the Riemannian metric induced from $\mathbb{C}^{n+1}$, and $\mathbb{P}^n$ is endowed with the Fubini–Study metric. In this pair of metrics the projection $\pi|_S : S \to \mathbb{P}^n$ has Lipschitz constant one; hence π has Lipschitz constant at most 2 in some neighborhood of S. This implies that

$$\left|\left\{t \in [0,2\pi] :\ \mathrm{dist}_{\mathbb{P}^n}\left(f_j\left(\mathrm{e}^{\mathrm{i}t}\right), K\right) < 2/j\right\}\right| > 2\pi - 1/j.$$

Thus the sequence of discs f_{2j} in $\mathbb{P}^n$ is a P-sequence for the pair (K,x). By the construction, the sequence f_j has the bounded lifting property; indeed, (3.2) holds for the constant $C = 1/r(x) + \varepsilon$ for any $\varepsilon > 0$ (but we can not take $\varepsilon = 0$). This establishes one of the implications.

To prove the converse implication, assume that there exists a P-sequence $f_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ for (K,x) with the bounded lifting property. Let $C > 0$ be a constant satisfying (3.2). Pick a point $p \in \pi^{-1}(x)$ with $|p| = 1/C$. Let $F_j : \overline{\mathbb{D}} \to \mathbb{C}^{n+1}_*$ be a lifting of f_j with $F_j(0) = p$ (this can be achieved by a rescaling). Then $\sup_{t\in[0,2\pi]} |F_j(\mathrm{e}^{\mathrm{i}t})| \le C|p| = 1$ for all j. Inside the unit ball $\overline{\mathbb{B}} \subset \mathbb{C}^{n+1}$ the set B_K (1.2) is a union of fibers of π, and the map π is expanding in directions orthogonal to the complex lines L_x. Hence, we have

$$\mathrm{dist}_{\mathbb{C}^{n+1}}\left(F_j\left(\mathrm{e}^{\mathrm{i}t}\right), B_K\right) \le \mathrm{dist}_{\mathbb{P}^n}\left(f_j\left(\mathrm{e}^{\mathrm{i}t}\right), K\right)$$

for all $t \in [0,2\pi]$ and $j \in \mathbb{N}$. Therefore the estimate (3.3) holds, which means that F_j is a P-sequence in $\mathbb{C}^{n+1}$ for the pair (S_K, p). Hence, the point p belongs to the polynomial hull of S_K, and so the point $x = \pi(p) \in \mathbb{P}^n$ belongs to the projective hull of K. □

It would be interesting to understand conditions on a sequence of analytic discs in $\mathbb{P}^n$ implying the bounded lifting property. Here is a simple observation: If Ω is an open simply connected Stein domain in $\mathbb{P}^n$ then the $\mathbb{C}^*$-bundle $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is holomorphically trivial over Ω, and hence any sequence of discs contained in a relatively compact subset of Ω has the bounded lifting property. This holds in particular if Ω is an affine chart $\mathbb{C}^n \subset \mathbb{P}^n$.

4. Projective hulls of compact connected sets in affine charts

In this section, we extend a theorem of Lawson and Wermer [LW], which pertains to projective hulls of closed real curves in $\mathbb{P}^n$, to an arbitrary compact connected set contained in an affine chart of $\mathbb{P}^n$.

Assume that $\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n$ is a projective hyperplane and Ω is a nonempty open subset of $\mathbb{C}^n = \mathbb{P}^n \setminus H$. Let $f : \overline{\mathbb{D}} \to \mathbb{P}^n$ be an analytic disc with $f(\mathbb{T}) \subset \Omega$. Then f intersects H in at most finitely many points of $\mathbb{D}$. The following quantity $J(f)$ (a disc functional) was first introduced by Lárusson and Sigurdsson in [LS]:

$$J(f) = -\sum_{\zeta\in\mathbb{D}} m_\zeta \log|\zeta| \ge 0. \tag{4.1}$$

Here $m_\zeta \in \mathbb{Z}_+$ denotes the intersection number of f with H at ζ; so $m_\zeta = 0$ if $f(\zeta) \notin H$. The number $J(f)$ equals the value at the origin of the (positive) Green function on $\mathbb{D}$ that equals zero on $b\mathbb{D}$ and has logarithmic poles at the finitely many points $\zeta_j \in \mathbb{D}$ for which $f(\zeta_j) \in H$ (see [DF2, §4]).

The following result is due to Lawson and Wermer [LW, Theorem 1] in the case when K is a connected closed curve. Our proof uses the same ingredients, but is simpler due to a more systematic use of inequalities involving the Siciak–Zaharyuta extremal functions.

THEOREM 4.1. *Assume that K is a compact connected set in $\mathbb{P}^n$ and $H \cong \mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^n$ such that $K \cap H = \emptyset$. Then a point $p \in \mathbb{C}^n = \mathbb{P}^n \setminus H$ belongs to the projective hull of K if and only if there exist a constant $0 \le C < +\infty$ and a sequence of analytic discs $f_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ $(j \in \mathbb{N})$ satisfying the following three properties*:

(a) $f_j(0) = p$ *for all* $j = 1, 2, \ldots,$
(b) $\lim_{j\to\infty} \max_{t\in[0,2\pi]} \mathrm{dist}(f_j(\mathrm{e}^{\mathrm{i}t}), K) = 0,$ *and*
(c) $J(f_j) \le C$ *for all* $j = 1, 2, \ldots.$

If this holds, then there exists a sequence of analytic discs f_j with simple poles satisfying conditions (a), (b) *and*

(c′) $\lim_{j\to\infty} J(f_j) = V_K(p)$, *the value at p of the Siciak–Zaharyuta extremal function of the set $K \subset \mathbb{C}^n$.*

Proof. For every analytic disc $f : \overline{\mathbb{D}} \to \mathbb{P}^n$ with center $f(0) \in \mathbb{C}^n$ and boundary $f(\mathbb{T})$ contained in an open set $\Omega \subset \mathbb{C}^n$ we have the inequality

$$V_\Omega\big(f(0)\big) \le J(f).$$

(See the proof of Theorem 4.2 in [DF2]. The point is simply that $V_\Omega \circ f$ is a subharmonic function on $\overline{\mathbb{D}} \setminus f^{-1}(H)$ that vanishes on $\mathbb{T}$ and has a logarithmic pole with weight m_j at every point ζ_j of the divisor $f^{-1}(H) = \sum m_j \zeta_j$. Hence, it is bounded above by the Green function on $\mathbb{D}$ with the same poles. Comparing the values at the origin gives the stated inequality.)

Assume now that $\Omega_1 \supset \Omega_2 \supset \cdots$ are open sets with $\bigcap_{j=1}^\infty \Omega_j = K$, and $f_j : \overline{\mathbb{D}} \to \mathbb{P}^n$ is a sequence of analytic discs with $f_j(0) = p$, $f_j(\mathbb{T}) \subset \Omega_j$, and $J(f_j) \le C < \infty$ for all $j \in \mathbb{N}$. Then $V_{\Omega_j}(p) \le J(f_j) \le C$ for all j. As $j \to \infty$, the numbers $V_{\Omega_j}(p)$ increase to $V_K(p)$, so we get $V_K(p) \le C < +\infty$. Thus, p belongs to the projective hull of K.

Conversely, assume that $p \in \widehat{K}_{\mathbb{P}^n}$; hence $V_K(p) < +\infty$. Since K is connected, we can choose a decreasing sequence of connected open neighborhoods $\Omega_j \supset K$ as above. Pick a decreasing sequence of numbers $\varepsilon_j > 0$ converging to zero. By Lárusson and Sigurdsson [LS], we have for every connected open set $\Omega \subset \mathbb{C}^n \subset \mathbb{P}^n$ that

$$V_\Omega(p) = \inf_f J(f),$$

the infimum being taken over all discs in $\mathbb{P}^n$ with $f(0)=p$ and $f(\mathbb{T})\subset\Omega$. Hence there exists for every $j\in\mathbb{N}$ an analytic disc $f_j:\overline{\mathbb{D}}\to\mathbb{P}^n$ such that $f_j(0)=p$, $f_j(\mathbb{T})\subset\Omega_j$, and

$$V_{\Omega_j}(p)\le J(f_j)<V_{\Omega_j}(p)+\varepsilon_j.$$

By the transversality theorem, we may assume that each f_j has simple poles, that is, it intersects the hyperplane H transversely. As $j\to\infty$, the numbers $V_{\Omega_j}(p)$ increase monotonically to $V_K(p)$. It follows that the sequence f_j satisfies properties (a), (b) and (c′). □

REMARK 4.2. At this point, one can proceed as in [LW] to write $f_j=G_j/B_j$, where $G_j:\overline{\mathbb{D}}\to\mathbb{C}^n$ is a holomorphic disc in $\mathbb{C}^n$ and B_j is a finite Blaschke product whose zeros are precisely the poles of f_j (i.e., the points in $f^{-1}(H)$). The condition $J(f_j)\le C$ implies that $|B_j(0)|\ge \mathrm{e}^{-C}>0$ for all j. Furthermore, on $\mathbb{T}$ we have $|G_j|=|f_jB_j|=|f_j|$ which is uniformly bounded, and hence the sequence $|G_j|$ is uniformly bounded on $\overline{\mathbb{D}}$ by the maximum principle. Passing to subsequences we may assume that the sequence B_j converges uniformly on compacts in $\mathbb{D}$ to a nonzero Blaschke product B, and the sequence G_j converges to a bounded holomorphic map $G:\mathbb{D}\to\mathbb{C}^n$. Lawson and Wermer then show that the holomorphic map $f=G/B:\mathbb{D}\to\mathbb{P}^n$ satisfies $f(\mathbb{D})\subset\widehat{K}_{\mathbb{P}^n}$. However, since nothing in this argument prevents f from being the constant map $f\equiv f(0)$, the limit discs obtained in this way do not seem to give a satisfactory description of the projective hull.

5. Characterization of the projective hull of a compact connected set by analytic discs

In this section, we obtain the following result which improves Theorem 3.2 in the case when the compact set $K\subset\mathbb{P}^n$ is also connected.

THEOREM 5.1. *Let K be a compact connected set in $\mathbb{P}^n$. A point $p\in\mathbb{C}^{n+1}\setminus\{0\}$ belongs to the polynomial hull of the set $S_K\subset\mathbb{C}^{n+1}$ (1.2), and hence $x=\pi(p)\in\mathbb{P}^n$ belongs to the projective hull of K, if and only if there exists a sequence of analytic discs $F_j:\overline{\mathbb{D}}\to\mathbb{C}^{n+1}\setminus\{0\}$ such that*

$$F_j(0)=p\quad(\forall j\in\mathbb{N}),\qquad \lim_{j\to\infty}\max_{t\in[0,2\pi]}\mathrm{dist}\big(F_j\big(\mathrm{e}^{\mathrm{i}t}\big),S_K\big)=0. \tag{5.1}$$

REMARK 5.2. By the maximum principle, the images $F_j(\overline{\mathbb{D}})$ are contained in balls $(1+\varepsilon_j)\mathbb{B}$ with $\varepsilon_j=\max_{t\in[0,2\pi]}\mathrm{dist}(F_j(\mathrm{e}^{\mathrm{i}t}),S_K)$. However, it does not seem possible to control from below the distance of $F_j(\overline{\mathbb{D}})$ to the origin. The projected sequence $f_j=\pi\circ F_j:\overline{\mathbb{D}}\to\mathbb{P}^n$ then clearly enjoys the bounded lifting property (see Definition 3.1) and satisfies

$$f_j(0)=x\quad(\forall j\in\mathbb{N}),\qquad \lim_{j\to\infty}\max_{t\in[0,2\pi]}\mathrm{dist}\big(f_j\big(\mathrm{e}^{\mathrm{i}t}\big),K\big)=0.$$

The advantage over the P-sequence found in Theorem 3.2 is that the entire boundary circle $f_j(\mathbb{T})$ is mapped close to the set K, similarly to what happened in Theorem 4.1. However, the set K in Theorem 5.1 need not be contained in any affine chart of $\mathbb{P}^n$.

Proof of Theorem 5.1. The existence of a sequence of discs satisfying condition (5.1) clearly implies that the point p belongs to the polynomial hull of S_K.

Assume now that a point $p \in \mathbb{C}^{n+1} \setminus \{0\}$ belongs to $\widehat{S_K}$; we shall find a sequence F_j satisfying (5.1). To this end, we consider analytic discs as in the diagram (3.1). We compactify $\mathbb{C}^{n+1}$ by adding the hyperplane at infinity and obtain $\mathbb{P}^{n+1} = \mathbb{C}^{n+1} \cup H_\infty$. Given a set $\Omega \subset \mathbb{C}^{n+1}$ we recall that V_Ω is the Siciak–Zaharyuta extremal function with logarithmic pole at H_∞. Since the point p belongs to the polynomial hull $\widehat{S}_K$, we have $V_{S_K}(p) = 0$. Choose a decreasing sequence of connected open sets $\Omega_j \subset \mathbb{C}^{n+1}$,

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \bigcap_{j=1}^{\infty} \Omega_j = S_K,$$

such that every Ω_j is circular (invariant with respect to the circle action $(t, z) \mapsto \mathrm{e}^{\mathrm{i}t} z$). For every j, we pick a smaller circular neighborhood Ω'_j of the set S_K and a number $\varepsilon_j > 0$ such that $\mathrm{e}^{\varepsilon_j} z \in \Omega_j$ for every $z \in \Omega'_j$.

Since $V_{S_K}(p) = 0$, we have $V_{\Omega'_j}(p) = 0$ for all j. By Lárusson and Sigurdsson [LS], there exists for every j an analytic disc $G_j : \overline{\mathbb{D}} \to \mathbb{P}^{n+1}$ such that $G_j(0) = p$, $G_j(\mathbb{T}) \subset \Omega'_j$, and $J(G_j) < \varepsilon_j$. Here $J(G_j) = -\sum_k m_{j,k} \log|\zeta_{j,k}|$, where $\sum_k m_{j,k}\zeta_{j,k} = G_j^{-1}(H_\infty)$ is the intersection divisor of G_j with the hyperplane H_∞. By general position, we may assume that $G_j(\overline{\mathbb{D}})$ does not contain the origin $0 \in \mathbb{C}^{n+1}$ for any j.

Let $B_j(\zeta)$ denote the Blaschke product with the zeros $\zeta_{j,k}$ of multiplicity $m_{j,k}$. A calculation gives

$$\left|B_j(0)\right| = \mathrm{e}^{-J(G_j)} > \mathrm{e}^{-\varepsilon_j}.$$

Define new analytic discs by

$$F_j(\zeta) = \frac{B_j(\zeta)}{B_j(0)} G_j(\zeta), \quad j = 1, 2, \ldots.$$

Then F_j is an analytic disc in $\mathbb{C}^{n+1} \setminus \{0\}$ (since the poles of G_j are exactly cancelled off by the zeros of B_j and no additional zeros appear), and $F_j(0) = G_j(0) = p$. Since $|B_j| = 1$ on $\mathbb{T}$ and the sets $\Omega'_j \subset \Omega_j$ are circular, our choice of the number ε_j implies that $F_j(\mathbb{T}) \subset \Omega_j$. Hence, the sequence F_j satisfies the stated properties. $\square$

The above proof does not use any special hypothesis of the set S_K other that it is connected and circular. Hence, we get the following result of possible independent interest. It vaguely resembles the description of the polynomial hulls of sets in $\mathbb{C}^2$ fibered over the unit circle, with disc fibers, due to Alexander and Wermer [AW].

THEOREM 5.3. *Assume that K is a compact connected set in $\mathbb{C}^n$ which is invariant with respect to the circle action $(t,z)\mapsto \mathrm{e}^{\mathrm{i}t}z$. Then a point $p\in\mathbb{C}^n$ belongs to the polynomial hull of K if and only if there exists a sequence of analytic discs $f_j:\overline{\mathbb{D}}\to\mathbb{C}^n$ such that*

$$f_j(0)=p \quad (\forall j\in\mathbb{N}), \qquad \lim_{j\to\infty}\max_{t\in[0,2\pi]}\operatorname{dist}\bigl(f_j(\mathrm{e}^{\mathrm{i}t}),K\bigr)=0.$$

Theorem 5.3 fails in general for a disconnected circular set K. A simple example for which the conclusion fails is the union $K=T_1\cup T_2$ of two disjoint totally real tori T_1,T_2 in the unit sphere of $\mathbb{C}^2$ such that K bounds an embedded complex annulus $A\subset\mathbb{C}^2$, with the two boundary circles of A contained in different connected components of K. However, we do not know the answer to the following question.

PROBLEM 5.4. Does Theorem 5.3 still hold if the set K is not circular?

Acknowledgments. We wish to thank Norman Levenberg and Finnur Lárusson for their remarks on an earlier version of the paper. The question whether the projective hull can be characterized by sequences of analytic discs was communicated to us by Levenberg.

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