The Poletsky-Rosay Theorem on Singular Complex Spaces

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ABSTRACT. In this paper we extend the Poletsky-Rosay theorem, concerning plurisubharmonicity of the Poisson envelope of an upper semicontinuous function, to locally irreducible complex spaces.

1. INTRODUCTION

Plurisubharmonic functions were introduced by K. Oka [Oka] and P. Lelong [Lel] in 1942; ever since then they have been playing a major role in complex analysis. The minimum of two plurisubharmonic functions is not plurisubharmonic in general. There has been a considerable amount of interest in studying situations where the infimum actually is plurisubharmonic. The first major result of this kind was Kiselman's *minimum principle* [Kis].

In the early 1990s E. Poletsky [Po1, Po2] found a novel way of constructing plurisubharmonic functions as pointwise infima of upper semicontinuous functions. Set $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Let $\mathcal{A}(\mathbb{D}, X)$ denote the set of all *analytic discs* in a complex space X, that is, continuous maps $\bar{\mathbb{D}} \to X$ that are holomorphic in $\mathbb{D}$. Then for $x \in X$ let $\mathcal{A}(\mathbb{D}, X, x) = \{f \in \mathcal{A}(\mathbb{D}, X) : f(0) = x\}$. Our main result is the following.

Theorem 1.1. *Let* $(X, \mathcal{O}_X)$ *be an irreducible and locally irreducible* (*reduced, paracompact*) *complex space, and let* $u\colon X \to \mathbb{R} \cup \{-\infty\}$ *be an upper semicontinuous function on X. Then the function*

$$\hat{u}(x) = \inf\left\{ \int_0^{2\pi} u(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} : f \in \mathcal{A}(\mathbb{D}, X, x)\right\}, \quad x \in X, \tag{1.1}$$

is plurisubharmonic on X or identically $-\infty$*; moreover,* $\hat{u}$ *is the supremum of the plurisubharmonic functions on X which are not greater than* u.

In the basic case when $X = \mathbb{C}^n$, this was proved by Poletsky [Po1, Po2] and by Bu and Schachermayer [BS]. The result was extended to some complex manifolds (and to certain other disc functionals) by Lárusson and Sigurdsson [LS1, LS2, LS3] and to all complex manifolds by Rosay [Ro1, Ro2]. In this paper we use the method of gluing sprays, developed in [DF, For], to give a new proof which applies to any normal complex space X. For a locally irreducible space the result follows easily by using seminormalization to reduce to the case of a normal space.

A reduction to the Poletsky-Rosay theorem on manifolds is also possible by appealing to the Hironaka desingularization theorem, but this approach seems unreasonable since our proof is no more difficult than the original proofs given for manifolds.

Example 1.2. The theorem fails in general if X is not irreducible; here is a trivial example. Let $X = \{(z,w) \in \mathbb{C}^2 \colon zw = 0\}$ be the union of two complex lines. Let $u \colon X \to \mathbb{R}$ be defined by $u(z,0) = 0$ for $z \in \mathbb{C}^*$ and $u(0,w) = 1$ for all $w \in \mathbb{C}$. Clearly u is upper semicontinuous. We have $\hat{u}(z,0) = 0$ for $z \in \mathbb{C}$ and $\hat{u}(0,w) = 1$ for $w \in \mathbb{C}^*$; hence $\hat{u}$ fails to be upper semicontinuos at the point $(0,0)$. On the other hand, local irreducibility is not always necessary. For example, if X is the Riemann sphere with one simple double point, then the above example does not work since the boundaries of analytic discs through any of the two local branches at the double point also reach the other branch.

The operator $P_u \colon \mathcal{A}(\mathbb{D}, X) \to \mathbb{R} \cup \{-\infty\}$ appearing in (1.1),

$$P_u(f) = \int_0^{2\pi} u(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi}, \quad f \in \mathcal{A}(\mathbb{D}, X),$$

is called the *Poisson functional*. Note that $P_u(f)$ is the value at $0 \in \mathbb{D}$ of the harmonic function on $\mathbb{D}$ with boundary values $u(f(\mathrm{e}^{\mathrm{i}t}))$. For this reason, the function $\hat{u}$ defined by (1.1) is also called the *Poisson envelope* of u.

Let us first justify the last statement in the theorem. Recall that an upper semicontinuous function $v \colon X \to \mathbb{R} \cup \{-\infty\}$ on a complex space that is not identically $-\infty$ on any irreducible component of X is plurisubharmonic if every point $x \in X$ admits a neighborhood $U \subset X$, embedded as a closed complex subvariety in a domain $\Omega \subset \mathbb{C}^N$, such that $v|_U$ is the restriction to U of a plurisubharmonic function $\tilde{v}$ on Ω. By [FN] a function v as above is plurisubharmonic if and only if the composition $v \circ f$ with any holomorphic disc $f \colon \mathbb{D} \to X$ is subharmonic on $\mathbb{D}$. This holds if and only if v satisfies the submeanvalue property on analytic discs, which precisely means that $v(x) \le P_v(f)$ for every $x \in X$ and $f \in \mathcal{A}(\mathbb{D}, X, x)$. Since for the constant disc $f(\zeta) = x$ we have $P_v(f) = v(x)$, we conclude that the following statement holds true.

Lemma 1.3. *An upper semicontinuous function $v \colon X \to \mathbb{R} \cup \{-\infty\}$ on a complex space X that is not identically $-\infty$ on any irreducible component of X is plurisubharmonic if and only if $v = \hat{v}$, where $\hat{v}$ is defined by* (1.1).

If $v \leq u$, then clearly $P_v(f) \leq P_u(f)$ for every $f \in \mathcal{A}(\mathbb{D}, X)$, and hence $\hat{v} \leq \hat{u}$. It follows that any plurisubharmonic function v for which $v \leq u$ satisfies $v = \hat{v} \leq \hat{u} \leq u$, so $\hat{u}$ is indeed the largest such function.

One of the main applications of Theorem 1.1 in the classical case $X = \mathbb{C}^n$ is the characterization of the polynomially convex hull $\hat{K}$ of a compact set $K \subset \mathbb{C}^n$ by analytic discs, due to Poletsky [Po1, Po2] and Bu and Schachermayer [BS]. (See also Remark 1.5 below.) In the situation considered in this paper we obtain a characterization of *plurisubharmonic hulls* in terms of analytic discs, a fact that was already observed (for complex manifolds) by Lárusson and Sigurdsson and by Rosay. Let $\mathrm{Psh}(X)$ denote the set of all plurisubharmonic functions on X. Given a compact set K in X, its plurisubharmonic hull is defined by

$$\hat{K}_{\mathrm{Psh}(X)} = \{x \in X : u(x) \leq \sup_K u, \ \forall\, u \in \mathrm{Psh}(X)\}.$$

Since the modulus $|f|$ of a holomorphic function is a plurisubharmonic function, we always have $\hat{K}_{\mathrm{Psh}(X)} \subset \hat{K}_{\mathcal{O}(X)}$. It is a much deeper result of Grauert [Gra] and Narasimhan [Nar] that the two hulls coincide if X is a Stein space or, more generally, a 1-convex complex space. (See also the papers [FN] and [GR2].) In the case when $X = \mathbb{C}^n$, the equality of the two hulls also follows from Poletsky's theorem, as was pointed out in [Po3, Theorem 5.1]. Related results concerning *pluripolar hulls* of compact sets in $\mathbb{C}^n$ were obtained in [LP].

Corollary 1.4. *Let K be a compact set in a locally irreducible complex space X such that $\hat{K}_{\mathrm{Psh}(X)}$ is compact. Choose an open set $V \Subset X$ containing $\hat{K}_{\mathrm{Psh}(X)}$. Then a point $x \in X$ belongs to $\hat{K}_{\mathrm{Psh}(X)}$ if and only if for every open set $U \supset K$ and every number $\varepsilon > 0$ there exist a disc $f \in \mathcal{A}(\mathbb{D}, V)$ and a set $E_f \subset [0, 2\pi]$ of Lebesgue measure $|E_f| < \varepsilon$ such that*

$$f(0) = x \quad \text{and} \quad f(\mathrm{e}^{\mathrm{i}t}) \in U \text{ for all } t \in [0, 2\pi] \setminus E_f.$$

Proof. Assume first that a point $x \in X$ satisfies the stated conditions; we shall prove that $x \in \hat{K}_{\mathrm{Psh}(X)}$. Choose a function $\rho \in \mathrm{Psh}(X)$. Set $M = \sup_K \rho$ and $M' = \sup_V \rho$. Pick a number $\varepsilon > 0$ and an open set U with $K \subset U \Subset V$ such that $\sup_U \rho < M + \varepsilon$. Let the disc $f \in \mathcal{A}(\mathbb{D}, V, x)$ and the set $E_f \subset [0, 2\pi]$ satisfy the hypotheses of the corollary. Then

$$\rho(x) \leq P_\rho(f) = \int_{E_f} \rho(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} + \int_{[0,2\pi]\setminus E_f} \rho(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < M'\varepsilon + M + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we get $\rho(x) \leq M$. As $\rho \in \mathrm{Psh}(X)$ was arbitrary, we conclude that $x \in \hat{K}_{\mathrm{Psh}(X)}$.

Conversely, assume that $x \in \hat{K}_{\mathrm{Psh}(X)}$. The function $u : V \to \mathbb{R}$ which equals -1 on the open set $U \supset K$ and equals 0 on $V \setminus U$ is upper semicontinuous.

Let $v = \hat{u}$ be the associated plurisubharmonic function defined by (1.1). Then $-1 \le v \le 0$ on V, and $v(x) = -1$. Theorem 1.1 furnishes a disc $f \in \mathcal{A}(\mathbb{D}, V, x)$ with $P_u(f) < -1 + \varepsilon/2\pi$. By the definition of u this implies that the set $E_f = \{t \in [0, 2\pi]\colon f(e^{it}) \notin U\}$ has measure at most ε. ◻

Remark 1.5. Let K be a compact subset of $\mathbb{C}^n$. It was shown by Duval and Sibony [DS1, DS2] that for any point $p \in \hat{K}$ and Jensen measure σ representing p there exists a positive current T of bidimension $(1,1)$ such that $dd^c T = \sigma - \delta_p$, where δ_p denotes the point evaluation at p; thus the hull $\hat{K}$ is the union of supports of such currents. Recently Wold [Wol] showed that every Duval-Sibony current T is a weak limit of currents $T_j = (f_j)_* G$, where G is the *Green current* on the unit disc $\mathbb{D}$, given by

$$G(\omega) = -\int_{\mathbb{D}} \log|\zeta| \cdot \omega, \quad \omega \in \mathcal{E}^{1,1}(\bar{\mathbb{D}}),$$

and $f_j\colon \bar{\mathbb{D}} \to \mathbb{C}^n$ is a sequence of Poletsky discs. On the other hand, it has been known since the classical examples of Stolzenberg and Alexander that the polynomial hull of K can not be explained in general by analytic varieties with boundaries in K. (In this direction see the recent paper of Dujardin [Duj].) Hence Poletsky's characterization of the polynomial hull remains the most universal one that we have at the moment. For more about hulls we refer the interested reader to Stout's monographs [St1, St2].

Remark 1.6. Another immediate implication of Theorem 1.1 is the following result, which was observed in the smooth case by J.-P. Rosay [Ro1, Corollary 0.2]: If X is as in Theorem 1.1 and if every bounded plurisubharmonic function on X is constant, then for every point $p \in X$, nonempty open set $U \subset X$, and number $\varepsilon > 0$, there exists an analytic disc $f\colon \bar{\mathbb{D}} \to X$ such that $f(0) = p$ and the set $\{t \in [0, 2\pi)\colon f(e^{it}) \in U\}$ has measure at least $2\pi - \varepsilon$. This follows by observing that the envelope $\hat{u}$ defined by (1.1) of the negative characteristic function $u = -\chi_U$ of the set U is bounded from above by 0, and hence it is constantly equal to -1.

We expect that the methods of our proof can be used to extend the main theorem of Rosay in [Ro3] as follows: A locally irreducible complex space X does not admit any nonconstant bounded plurisubharmonic function (such space is said to be Liouville) if and only if every closed loop in X can be approximated on a set of almost full linear measure in the circle by the boundary values of holomorphic discs in X. We hope to return to these questions in a future publication.

2. A Nonlinear Cousin-I Problem

We recall from [DF, For] the relevant results concerning holomorphic sprays, adjusting them to the applications in this paper.

Definition 2.1. Let $\ell \ge 2$ and $r \in \{0, \dots, \ell\}$ be integers. Assume that X is a complex space, D is a relatively compact domain with C^ℓ boundary in $\mathbb{C}$, and σ

is a finite set of points in D. A *spray of maps* of class $\mathcal{A}^r(D)$ with the exceptional set σ and with values in X is a map $f\colon P\times\bar D\to X$, where P (the *parameter set* of the spray) is an open subset of a Euclidean space $\mathbb{C}^m$ containing the origin, such that the following hold:

(i) f is holomorphic on $P\times D$ and of class C^r on $P\times\bar D$,
(ii) the maps $f(0,\cdot)$ and $f(t,\cdot)$ agree on σ for all $t\in P$, and
(iii) if $z\in\bar D\setminus\sigma$ and $t\in P$, then $f(t,z)\in X_{\mathrm{reg}}$ and

$$\partial_t f(t,z)\colon T_t\mathbb{C}^m\to T_{f(t,z)}X$$

is surjective (the *domination property*).

We call $f_0=f(0,\cdot)$ the *core* (or *central*) map of the spray f.

The following lemma is a special case of [DF, Lemma 4.2].

***Lemma 2.2* (Existence of sprays).** *Assume that ℓ, r, D, σ, and X are in accordance with Definition 2.1. Given a map $f_0\colon\bar D\to X$ of class $\mathcal{A}^r(D)$ such that the set $\{z\in\bar D\colon f_0(z)\in X_{\mathrm{sing}}\}$ is contained in σ, there exists a spray $f\colon P\times\bar D\to X$ of class $\mathcal{A}^r(D)$, with the exceptional set σ, such that $f(0,\cdot)=f_0$.*

Definition 2.3. Let $\ell\geq 2$ be an integer. A pair of open subsets $D_0,D_1\Subset\mathbb{C}$ is said to be a *Cartan pair of class C^ℓ* if

(i) D_0, D_1, $D=D_0\cup D_1$, and $D_{0,1}=D_0\cap D_1$ are domains with C^ℓ smooth boundaries, and
(ii) $\overline{D_0\setminus D_1}\cap\overline{D_1\setminus D_0}=\varnothing$ (the separation property).

The following is the main result on gluing sprays in the particular situation that we are considering (see [DF, Proposition 4.3] or [For, Lemma 3.2]). This is in fact a solution of a nonlinear Cousin-I problem.

***Proposition 2.4* (Gluing sprays).** *Let (D_0,D_1) be a Cartan pair of class C^ℓ ($\ell\geq 2$) in $\mathbb{C}$ (Definition 2.3). Set $D=D_0\cup D_1$, $D_{0,1}=D_0\cap D_1$. Let X be a complex space. Given an integer $r\in\{0,1,\dots,\ell\}$ and a spray $f\colon P_0\times\bar D_0\to X$ of class $\mathcal{A}^r(D_0)$ with the exceptional set σ such that $\sigma\cap\bar D_{0,1}=\varnothing$, there is an open set $P\Subset P_0\subset\mathbb{C}^m$ containing $0\in\mathbb{C}^m$ and satisfying the following.*

For every spray $f'\colon P_0\times\bar D_1\to X$ of class $\mathcal{A}^r(D_1)$, with the exceptional set σ' such that f' is sufficiently C^r close to f on $P_0\times\bar D_{0,1}$ and $\sigma'\cap\bar D_{0,1}=\varnothing$, there exists a spray $F\colon P\times\bar D\to X$ of class $\mathcal{A}^r(D)$, with the exceptional set $\sigma\cup\sigma'$, enjoying the following properties:

(i) *the restriction $F\colon P\times\bar D_0\to X$ is close to $f\colon P\times\bar D_0\to X$ in the C^r-topology (depending on the C^r-distance of f and f' on $P_0\times\bar D_{0,1}$),*
(ii) *the core map $F_0=F(0,\cdot)$ is homotopic to $f_0=f(0,\cdot)$ on $\bar D_0$, and F_0 is homotopic to $f'_0=f'(0,\cdot)$ on $\bar D_1$,*
(iii) *F_0 agrees with f_0 on σ, and it agrees with f'_0 on σ', and*
(iv) *$F(t,z)\in\{f'(s,z)\colon s\in P_0\}$ for each $t\in P$ and $z\in\bar D_1$.*

Here is a brief outline of the proof. The first step is to find a domain $P' \subset \mathbb{C}^m$ such that $0 \in P' \Subset P_0$, and a transition map between the sprays f and f', that is, a C^r map

$$\gamma\colon P' \times \bar{D}_{0,1} \to P_0 \times \bar{D}_{0,1}, \quad \gamma(t,z) = (c(t,z),z)$$

that is holomorphic in $P' \times D_{0,1}$ and is C^r close to the identity map (depending on the C^r-distance between f and f' over $P_0 \times \bar{D}_{0,1}$), such that

$$f = f' \circ \gamma \quad \text{on } P' \times \bar{D}_{0,1}.$$

This is an application of both the implicit function theorem and the fact that Cartan's Theorem B holds for holomorphic vector bundles on domains in $\mathbb{C}$ that are smooth of class C^r up to the boundary. The key step is to split the map γ in the form

$$\gamma = \beta \circ \alpha^{-1},$$

where $\alpha(t,z) = (a(t,z),z)$ and $\beta(t,z) = (b(t,z),z)$ are maps with similar properties over $P \times \bar{D}_0$ and $P \times \bar{D}_1$, respectively, for some slightly smaller parameter set $0 \in P \Subset P'$. This splitting is accomplished by nonlinear operators whose linearization involves a solution operator for the $\bar{\partial}$-equation with C^r estimates on $D = D_0 \cup D_1$. The final step is to observe that over $P \times \bar{D}_{0,1}$ we have

$$f = f' \circ \gamma = f' \circ \beta \circ \alpha^{-1} \Longrightarrow f \circ \alpha = f' \circ \beta.$$

Hence the two sides amalgamate into a spray F over $\bar{D}$, and it is easily verified that F satisfies Proposition 2.4.

3. A RIEMANN-HILBERT PROBLEM

In this section we explain how to find an approximate solution of a Riemann-Hilbert problem with the control of the average of a given function on a boundary arc. Results of this kind have been used by several authors; see, e.g., [Po1, Po2, BS, FG1, FG2].

Recall that $\mathbb{T} = b\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Given a measurable subset $I \subset \mathbb{T}$ and a measurable function $v : I \to \mathbb{R}$, $\int_I v(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$ will denote the integral over the set of points $t \in [0, 2\pi]$ for which $\mathrm{e}^{\mathrm{i}t} \in I$.

Lemma 3.1. *Let $f \in \mathcal{A}(\mathbb{D}, \mathbb{C}^n)$, and let $g\colon \mathbb{T} \times \bar{\mathbb{D}} \to \mathbb{C}^n$ be a continuous map such that for each $\zeta \in \mathbb{T}$ we have $g(\zeta, \cdot) \in \mathcal{A}(\mathbb{D}, X, f(\zeta))$. Given numbers $\varepsilon > 0$ and $0 < r < 1$, an arc $I \subset \mathbb{T}$, and a continuous function $u\colon \mathbb{C}^n \to \mathbb{R}$, there are a number $r' \in [r, 1)$ and a disc $h \in \mathcal{A}(\mathbb{D}, \mathbb{C}^n, f(0))$ satisfying*

$$\int_I u(h(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < \int_0^{2\pi}\int_I u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} + \varepsilon \tag{3.1}$$

and also the following properties:

(i) *for any* $\zeta \in \mathbb{T}$ *we have* $\mathrm{dist}(h(\zeta), g(\zeta, \mathbb{T})) < \varepsilon$,
(ii) *for any* $\zeta \in \mathbb{T}$ *and* $\rho \in [r', 1]$ *we have* $\mathrm{dist}(h(\rho\zeta), g(\zeta, \bar{\mathbb{D}})) < \varepsilon$,
(iii) *for any* $|\zeta| \le r'$ *we have* $|h(\zeta) - f(\zeta)| < \varepsilon$, *and*
(iv) *if* $g(\zeta, \cdot) = f(\zeta)$ *is the constant disc for all* $\zeta \in \mathbb{T} \setminus J$, *where* $J \subset \mathbb{T}$ *is an arc containing* $\bar{I}$, *then we can choose* h *such that* $|h - f| < \varepsilon$ *holds outside any given neighborhood of* J *in* $\bar{\mathbb{D}}$.

Proof. Write

$$g(\zeta, z) = f(\zeta) + \lambda(\zeta, z), \quad \zeta \in \mathbb{T},\ z \in \bar{\mathbb{D}},$$

where $\lambda(\zeta, z)$ is continuous on $(\zeta, z) \in \mathbb{T} \times \bar{\mathbb{D}}$, and for every fixed $\zeta \in \mathbb{T}$ the function $\bar{\mathbb{D}} \ni z \mapsto \lambda(\zeta, z)$ is holomorphic on $\mathbb{D}$ and satisfies $\lambda(\zeta, 0) = 0$. We can approximate λ uniformly on $\mathbb{T} \times \bar{\mathbb{D}}$ by Laurent polynomials of the form

$$\tilde{\lambda}(\zeta, z) = \frac{1}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^j = \frac{z}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^{j-1}$$

with polynomial coefficients $A_j(\zeta)$. Hence we can choose a map $\tilde{\lambda}$ as above and a number $r' \in [r, 1)$ such that

$$|\tilde{\lambda}(\rho e^{it}, z) - \lambda(e^{it}, z)| < \frac{\varepsilon}{2}, \quad t \in \mathbb{R},\ r' \le \rho \le 1,\ |z| \le 1, \tag{3.2}$$

and

$$|f(\rho e^{it}) - f(e^{it})| < \frac{\varepsilon}{2}, \quad t \in \mathbb{R},\ r' \le \rho \le 1. \tag{3.3}$$

Choose an integer $k > m$ and a number $c = e^{i\varphi} \in \mathbb{T}$, and set

$$h_k(\zeta, c) = f(\zeta) + \tilde{\lambda}(\zeta, c\zeta^k) = f(\zeta) + c\zeta^{k-m} \sum_{j=1}^{N} A_j(\zeta)(c\zeta^k)^{j-1}, \quad |\zeta| \le 1.$$

This is an analytic disc in $\mathbb{C}^n$ satisfying $h_k(0, c) = f(0)$; therefore it belongs to $\mathcal{A}(\mathbb{D}, \mathbb{C}^n, f(0))$. For $\zeta = e^{it} \in \mathbb{T}$ we have

$$h_k(e^{it}, c) = f(e^{it}) + \tilde{\lambda}(e^{it}, e^{i\varphi} e^{kit}) \approx g(e^{it}, e^{i(\varphi + kt)}), \tag{3.4}$$

and hence property (i) holds in view of (3.2). Similarly, if $r' \le \rho \le 1$, then

$$\begin{aligned} &|h_k(\rho e^{it}, c) - g(e^{it}, c\rho^k e^{ikt})| \\ &\qquad \le |\tilde{\lambda}(\rho e^{it}, c\rho^k e^{ikt}) - \lambda(e^{it}, c\rho^k e^{ikt})| + |f(\rho e^{it}) - f(e^{it})| < \varepsilon \end{aligned} \tag{3.5}$$

by (3.2) and (3.3), so (ii) holds as well.

If $k \to +\infty$, then $h_k(\zeta, \cdot\,) \to f(\zeta)$ uniformly on the set $\{|\zeta| \le r'\} \times \mathbb{T}$, so we get condition (iii) if k is chosen big enough. Property (iv) is a consequence of (ii) and (iii).

It remains to show that the inequality (3.1) can be achieved by a suitable choice of the number $c = \mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{T}$. We clearly have

$$\int_0^{2\pi} \int_I u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \int_I u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}(\theta+kt)}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi}.$$

By the mean value theorem there exists $\varphi \in [0, 2\pi)$ such that this equals

$$\int_I u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}(\varphi+kt)}))\frac{\mathrm{d}t}{2\pi}.$$

By (3.4) this number differs by at most ε from $\int_0^{2\pi} u(h_k(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\varphi}))\,\mathrm{d}t/(2\pi)$ if k is chosen big enough. This completes the proof. ◻

4. PROOF OF THEOREM 1.1

Step 1. We reduce to the case when X is a normal complex space.

Recall that a function on a complex space X is *weakly holomorphic* if it is holomorphic on the regular part X_{reg} and is locally bounded near each singular point. A reduced complex space X is said to be *normal* if every weakly holomorphic function is in fact holomorphic; it is *seminormal* if every continuous weakly holomorphic function on X is holomorphic. A holomorphic map $\pi\colon \tilde{X} \to X$ of complex spaces is called a *seminormalization* (or a *maximalization*) of X if $\tilde{X}$ is a seminormal complex space and π is a homeomorphism. We will use the following facts (see [Rem]):

- every reduced complex space admits a seminormalization,
- every locally irreducible seminormal complex space is normal, and
- seminormalization is a functor; in particular, the lift $\pi^{-1} \circ f$ of any holomorphic disc $f\colon \mathbb{D} \to X$ is a holomorphic disc in $\tilde{X}$.

This implies that if $\pi\colon \tilde{X} \to X$ is a seminormalization of X, then the composition $u \mapsto u \circ \pi$ induces an isomorphism from $\mathrm{Psh}(X)$ onto $\mathrm{Psh}(\tilde{X})$.

Assume now that Theorem 1.1 holds for normal complex spaces. Let X be a locally irreducible complex space and $u\colon X \to \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function. Let $\pi\colon \tilde{X} \to X$ be a seminormalization of X. The function $\tilde{u} = u \circ \pi$ is upper semicontinuous on $\tilde{X}$. Since $\tilde{X}$ is a normal complex space, the Poisson envelope $\tilde{v} = \hat{\tilde{u}}$ of $\tilde{u}$ is plurisubharmonic on $\tilde{X}$. Each holomorphic disc $f \in \mathcal{A}(\mathbb{D}, X)$ admits a holomorphic lifting $\tilde{f} \in \mathcal{A}(\mathbb{D}, \tilde{X})$ with $\pi \circ \tilde{f} = f$, and we clearly have $P_{\tilde{u}}(\tilde{f}) = P_u(f)$. This implies that $\tilde{v}(\pi^{-1}(x)) = \hat{u}(x)$ and hence $\hat{u}$ is plurisubharmonic.

Step 2. We reduce to the case when the function $u\colon X \to \mathbb{R}$ is continuous and bounded from below.

Since $u\colon X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous, there is a decreasing sequence of continuous functions $u_1 \geq u_2 \geq \cdots \geq u$ such that $u = \lim_{k\to\infty} u_k$ pointwise in X. Replacing u_k by $\max\{u_k, -k\}$, we may assume in addition that $u_k \geq -k$ on X; hence $\hat{u}_k \geq -k$ as well. Assuming that the result holds for each u_k, the Poisson envelopes $\hat{u}_1 \geq \hat{u}_2 \geq \cdots$ form a decreasing sequence of plurisubharmonic functions, and hence $v = \lim_{k\to\infty} \hat{u}_k$ is also plurisubharmonic or identically $-\infty$. Since $\hat{u}_k \leq u_k$, it follows that $v \leq u$. For any point $x \in X$ and disc $f \in \mathcal{A}(\mathbb{D}, X, x)$ we have $v(x) \leq P_v(f) \leq P_u(f) \leq P_{u_k}(f)$ by monotonicity. Taking the infimum over all such f shows that $v(x) \leq \hat{u}(x) \leq \hat{u}_k(x)$. Letting $k \to \infty$ gives $v(x) = \hat{u}(x)$, so $\hat{u}$ is plurisubharmonic or $-\infty$.

Step 3. We show that the Poisson envelope $v = \hat{u}$ given by (1.1) is upper semicontinuous on the regular locus X_{reg}. (We assume that X is a reduced complex space. The reductions in Step 1 and Step 2 will not be used here.)

Pick a point $x \in X_{\mathrm{reg}}$ and a number $\varepsilon > 0$. Assume first that $v(x) > -\infty$. By the definition of v, there exists a disc $f_0 \in \mathcal{A}(\mathbb{D}, X, x)$ such that $v(x) \leq P_u(f_0) < v(x) + \varepsilon$. By shrinking $\mathbb{D}$ slightly we may assume that $f_0(\mathbb{T}) \subset X_{\mathrm{reg}}$; hence the set $\sigma = \{\zeta \in \mathbb{D}\colon f_0(\zeta) \in X_{\mathrm{sing}}\}$ is finite and $0 \notin \sigma$. We embed f_0 as the central map $f_0 = f(0, \cdot)$ in a spray of holomorphic discs $f\colon P \times \bar{\mathbb{D}} \to X$ with the exceptional set σ, where P is an open set in $\mathbb{C}^m$ containing the origin. (See Definition 2.1 and Lemma 2.2; on $X = \mathbb{C}^n$ we can simply use the family of translates $f_y = f_0 + (y - x)$.) If $P' \Subset P$ is a small open neighborhood of $0 \in \mathbb{C}^m$, then $P_u(f(t, \cdot)) < P_u(f_0) + \varepsilon$ for each $t \in P'$, and hence

$$v(f(t,0)) \leq P_u(f(t,\cdot)) \leq P_u(f_0) + \varepsilon < v(x) + 2\varepsilon.$$

By the domination property, the set $\{f(t,0)\colon t \in P'\}$ fills a neighborhood of the point $x = f(0)$ in X, so we see that v is upper semicontinuous at x. A similar argument works at points where $v(x) = -\infty$.

Step 4. In this main step of the proof we assume that X is an irreducible normal complex space and that $u\colon X \to \mathbb{R}$ is a continuous function which is bounded from below (see Step 1 and Step 2). We shall prove that the Poisson envelope $v = \hat{u}$ given by (1.1) is plurisubharmonic on X_{reg}. (Plurisubharmonicity on X_{sing} will be proved in Step 5.)

We need to show that for every point $x \in X_{\mathrm{reg}}$ and for every analytic disc $f \in \mathcal{A}(\mathbb{D}, X, x)$ we have the submeanvalue property

$$v(x) = v(f(0)) \leq \int_0^{2\pi} v(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi}. \tag{4.1}$$

Since plurisubharmonicity is a local property, it is enough to consider small discs; hence we may assume that $f(\bar{\mathbb{D}}) \subset X_{\mathrm{reg}}$ and that f is holomorphic on a larger disc $r_0\mathbb{D}$ for some $r_0 > 1$.

Pick a number $\varepsilon > 0$. Fix a point $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$. By the definition of v, there exists an analytic disc $g_t = g(\mathrm{e}^{\mathrm{i}t}, \cdot\,) \in \mathcal{A}(\mathbb{D}, X, f(\mathrm{e}^{\mathrm{i}t}))$ such that

$$v(f(\mathrm{e}^{\mathrm{i}t})) \le \int_0^{2\pi} u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}\theta}{2\pi} < v(f(\mathrm{e}^{\mathrm{i}t})) + \varepsilon. \tag{4.2}$$

Since the set $\{\zeta \in \mathbb{D} \colon g_t(\zeta) \in X_{\mathrm{sing}}\}$ is discrete, we can replace g_t by the map $\zeta \mapsto g_t(r\zeta)$ for some $r < 1$ very close to 1 so that the new map still satisfies (4.2) and $g_t(\mathbb{T}) \subset X_{\mathrm{reg}}$. By Lemma 2.2 there is a domain $P \subset \mathbb{C}^m$ containing the origin and a holomorphic spray of discs $G \colon P \times \bar{\mathbb{D}} \to X$ with the central map $G(0, \cdot\,) = g_t$. (We use sprays of class $\mathcal{A}^0(\mathbb{D})$.) Since $G(0,0) = g_t(0) = f(\mathrm{e}^{\mathrm{i}t}) \in X_{\mathrm{reg}}$ and the spray G is dominating, the set $G(P,0) = \{G(w,0) \colon w \in P\}$ is a neighborhood of the point $f(\mathrm{e}^{\mathrm{i}t})$ in X. By the implicit mapping theorem, there are a disc $D \subset r_0\mathbb{D}$ centered at the point $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$ and a holomorphic map $\varphi \colon D \to P$ such that

$$\varphi(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{and} \quad G(\varphi(\zeta), 0) = f(\zeta), \quad \zeta \in D.$$

Consider the continuous map $g \colon D \times \bar{\mathbb{D}} \to X$ defined by

$$g(\zeta, z) = G(\varphi(\zeta), z), \quad \zeta \in D,\ z \in \bar{\mathbb{D}}.$$

Note that g is holomorphic on $D \times \mathbb{D}$, and

$$g(\mathrm{e}^{\mathrm{i}t}, \cdot\,) = G(0, \cdot\,) = g_t; \quad g(\zeta, 0) = f(\zeta),\ \zeta \in D.$$

Since $g(\zeta, \cdot\,)$ is uniformly close to g_t when ζ is close to $\mathrm{e}^{\mathrm{i}t}$, it follows from (4.2) that there is a small arc $I \Subset \mathbb{T} \cap D$ around the point $\mathrm{e}^{\mathrm{i}t}$ such that

$$\int_0^{2\pi} \int_I u(g(\mathrm{e}^{\mathrm{i}\eta}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}\eta}{2\pi}\frac{\mathrm{d}\theta}{2\pi} \le \int_I v(f(\mathrm{e}^{\mathrm{i}\eta}))\frac{\mathrm{d}\eta}{2\pi} + \frac{|I|\varepsilon}{2\pi}.$$

By repeating this construction at other points of $\mathbb{T}$ we find finitely many pairs of arcs $I_j \Subset I'_j \Subset \mathbb{T} \cap D_j$ ($j = 1, \dots, \ell$), where D_j is a disc contained in $r_0\mathbb{D}$, such that $\bar{I}'_j \cap \bar{I}'_k = \varnothing$ if $j \ne k$ and the set $E = \mathbb{T} \setminus \bigcup_{j=1}^m I_j$ has arbitrarily small measure $|E|$, and holomorphic families of discs $g_j(\zeta, z)$ for $\zeta \in D_j$ and $z \in \bar{\mathbb{D}}$ such that

$$\int_0^{2\pi} \int_{I_j} u(g_j(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} < \int_{I_j} v(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} + \frac{|I_j|\varepsilon}{2\pi}. \tag{4.3}$$

For each $j = 1, \dots, \ell$ we choose a smoothly bounded simply connected domain $\Delta_j \subset D_j \cap \mathbb{D}$ such that $\bar{\Delta}_j \subset D$, $\bar{\Delta}_j \cap \mathbb{T} = \overline{I'_j}$, and $\bar{\Delta}_j \cap \bar{\Delta}_k = \varnothing$ when $1 \le j \ne k \le \ell$. (The situation with $\ell = 3$ is illustrated in Figure 4.1. The reader

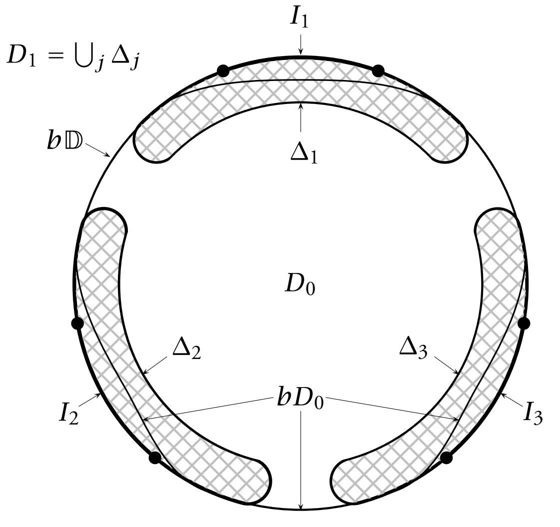

FIGURE 4.1. The Cartan pair (D_0, D_1)

should keep in mind that the gaps between the segments I_j have very small total length. The role of the sets $D_0, D_1 \subset \mathbb{D}$ with $D_0 \cup D_1 = \mathbb{D}$ is explained in the proof of Lemma 4.1 below.)

Let $\chi\colon \mathbb{C} \to [0,1]$ be a smooth function such that $\chi = 1$ on $\bigcup_{j=1}^{\ell} I_j \subset \mathbb{T}$ and $\chi = 0$ on a neighborhood of the set $\bar{\mathbb{D}} \setminus \bigcup_{j=1}^{\ell}(\Delta_j \cup I'_j)$. Consider the map $\xi\colon \bar{\mathbb{D}} \times \bar{\mathbb{D}} \to X$ defined by

$$\xi(\zeta, z) = \begin{cases} g_j(\zeta, \chi(\zeta)z), & \zeta \in \bar{\Delta}_j,\ z \in \bar{\mathbb{D}},\ j = 1, \dots, \ell; \\ f(\zeta), & \chi(\zeta) = 0,\ z \in \bar{\mathbb{D}}. \end{cases} \tag{4.4}$$

The latter condition holds in particular if $\zeta \in \mathbb{T} \setminus \bigcup_{j=1}^{m} I'_j$. Note that ξ is continuous and is holomorphic in the second variable. Then

$$\int_0^{2\pi}\int_0^{2\pi} u(\xi(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} < \int_0^{2\pi} v(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} + 2\varepsilon. \tag{4.5}$$

Indeed, the integral over $\bigcup_{j=1}^{\ell} I_j$ is estimated by adding up the inequalities (4.3), while the integral over the complementary set $E = \mathbb{T} \setminus \bigcup_{j=1}^{\ell} I_j$ gives at most ε if the measure $|E|$ is small enough. (When estimating the integral over E it is important to observe that $u \ge -M$ by the assumption, and u is bounded from above on each compact set. Since $-M \le v = \hat{u} \le u$, the same holds for v.)

To conclude the proof we apply the following lemma to the data that we have just constructed. We state it in a more general form since we shall need it again in Step 5 below.

Lemma 4.1. *Let X be a reduced complex space, and let $u\colon X \to \mathbb{R}$ be a continuous function on X. Assume that $f \in \mathcal{A}(\mathbb{D}, X)$ is a holomorphic disc such that $f(\mathbb{T}) \subset X_{\mathrm{reg}}$, and $\xi\colon \mathbb{T} \times \bar{\mathbb{D}} \to X$ is a continuous map such that $\xi(\zeta, \cdot) \in \mathcal{A}(\mathbb{D}, X, f(\zeta))$ for every $\zeta \in \mathbb{T}$. Given $\varepsilon > 0$, there exists an analytic disc $h \in \mathcal{A}(\mathbb{D}, X, f(0))$ such that*

$$\int_0^{2\pi} u(h(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < \int_0^{2\pi}\int_0^{2\pi} u(\xi(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} + \varepsilon. \tag{4.6}$$

Note that the estimate (4.6) is the same as (3.1) in Lemma 3.1, which pertains to the case $X = \mathbb{C}^n$, but we do not get (and do not need) the other approximation statements in that lemma.

Combining the inequalities (4.5) and (4.6), we obtain

$$v(x) \le \int_0^{2\pi} u(h(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < \int_0^{2\pi} v(f(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} + 3\varepsilon.$$

Since this holds for every $\varepsilon > 0$, the property (4.1) follows. This proves that v is plurisubharmonic on X_{reg} provided that Lemma 4.1 holds.

Proof of Lemma 4.1. In the case when X is a complex manifold, the first proof of Rosay [Ro1] uses a rather delicate construction of Stein neighborhoods; this approach was developed further by Lárusson and Sigurdsson [LS2]. Later Rosay [Ro2] gave another proof using an initial approximation by non-holomorphic discs with small $\bar{\partial}$-derivatives, approximating these by holomorphic discs, and finally patching the partial solutions together by solving a nonlinear Cousin-I problem. None of these methods seems to extend to complex spaces with singularities without major technical difficulties. On the other hand, the method that we use here works in essentially the same way as in the nonsingular case.

Since the function u is bounded on compacts, it is a trivial matter to reduce the proof to the special situation considered above so that the double integral in (4.5) changes by less than ε; in the sequel we consider this case. Let Δ_j be the discs chosen in the paragraph following (4.3) (see Figure 4.1). Fix an index $j \in \{1, \ldots, \ell\}$. We shall apply Lemma 3.1 over $\bar{\Delta}_j$ to find an analytic disc $f'_j\colon \bar{\Delta}_j \to X$ that approximates f uniformly as close as desired outside of a small neighborhood of the arc $\bar{I}_j$ in $\bar{\Delta}_j$ and satisfies the estimate

$$\int_{I_j} u(f'_j(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < \int_0^{2\pi}\int_{I_j} u(\xi(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} + \frac{|I_j|\varepsilon}{2\pi}. \tag{4.7}$$

Recall from (4.4) that $\xi(\zeta, z) = g_j(\zeta, \chi(\zeta)z)$ for $\zeta \in \bar{\Delta}_j$ and $z \in \bar{\mathbb{D}}$. Consider the function

$$u'_j(\zeta, z) = u(g_j(\zeta, z)), \quad \zeta \in \bar{\Delta}_j,\ z \in \bar{\mathbb{D}},$$

and the smooth family of analytic discs in $\mathbb{C}^2_{(\zeta,z)}$ given by

$$g'_j(\zeta,z) = (\zeta, \chi(\zeta)z), \quad \zeta \in b\Delta_j,\ z \in \bar{\mathbb{D}}.$$

Then

$$\xi = g_j \circ g'_j \quad \text{and} \quad u'_j = u \circ g_j \quad \text{on } b\Delta_j \times \bar{\mathbb{D}}. \tag{4.8}$$

Applying Lemma 3.1 with $\mathbb{D}$ replaced by Δ_j, u replaced by u'_j, and g replaced by g'_j furnishes an analytic disc $h'_j \in \mathcal{A}(\Delta_j, \mathbb{C}^2)$ which approximates the disc $\zeta \mapsto (\zeta, 0)$ outside of a small neighborhood of the arc $\bar{I}_j$ and such that

$$\int_{I_j} u'_j(h'_j(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} < \int_0^{2\pi}\int_{I_j} u'(g'_j(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} + \frac{|I_j|\varepsilon}{2\pi}. \tag{4.9}$$

Since $g_j \colon \bar{\Delta}_j \times \bar{\mathbb{D}} \to X$ is holomorphic in the interior $\Delta_j \times \mathbb{D}$, the map

$$f'_j := g_j \circ h'_j \colon \bar{\Delta}_j \to X \tag{4.10}$$

is an analytic disc in X such that $f'_j(\zeta) \approx g_j(\zeta, 0) = f(\zeta)$ for ζ outside a small neighborhood of $\bar{I}_j$ in $\bar{\Delta}_j$. From (4.8) and (4.10) it follows that

$$u \circ f'_j = u \circ g_j \circ h'_j = u'_j \circ h'_j, \quad u \circ \xi = u \circ g_j \circ g'_j = u'_j \circ g'_j$$

hold on $b\Delta_j \times \bar{\mathbb{D}}$. Hence the integrals in (4.7) equal the corresponding integrals in (4.9), and so the disc f'_j satisfies the desired properties.

If the approximation of f by f'_j is close enough for each $j = 1, \ldots, \ell$, we can use Proposition 2.4 to glue this collection of discs into a single analytic disc $h \colon \bar{\mathbb{D}} \to X$ which approximates f away from the union of arcs $\bigcup_{j=1}^{\ell} I_j$ and which approximates the disc f'_j over a neighborhood of $\bar{I}_j$ for each j. In particular, we can ensure that for each $j = 1, \ldots, \ell$ we have

$$\int_{I_j} u(h(\mathrm{e}^{\mathrm{i}t}))\frac{\mathrm{d}t}{2\pi} \approx \int_0^{2\pi}\int_{I_j} u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi}.$$

By adding up these terms and estimating the difference over the remaining set $E = \mathbb{T} \setminus \bigcup_j I_j$ with small length $|E|$, we obtain (4.6).

Let us explain the gluing. We apply Proposition 2.2 to embed f as the central map f_0 in a dominating spray of discs $f_p \in \mathcal{A}(\mathbb{D}, X)$ depending holomorphically on a parameter p in an open ball P in some $\mathbb{C}^N$. Next we apply the above approximation procedure simultaneously to all discs f_p in the spray, with a holomorphic dependence on the parameter. (The obvious details need not be repeated.) This gives for every j a holomorphic spray of discs $f'_{p,j} \in \mathcal{A}(\Delta_j, X)$ $(p \in P)$ approximating the spray f_p over the complement of a small neigborhood of $\bar{I}_j$ in $\bar{\Delta}_j$. If the approximations are close enough, we can glue these sprays into a spray of discs $F_p \in \mathcal{A}(\mathbb{D}, X)$ by using Proposition 2.4. (Here we allow the parameter set P to shrink.) For gluing we use a Cartan pair (D_0, D_1) with $D_0 \cup D_1 = \mathbb{D}$ obtained as follows (see Figure 4.1). We get D_0 by denting the boundary circle $\mathbb{T} = b\mathbb{D}$ slightly inward along each of the arcs $\bar{I}_j \subset \mathbb{T}$. The set D_1 equals $\bigcup_{j=1}^{\ell} \Delta_j$. Then the two sprays are close to each other over $\bar{D}_0 \cap \bar{D}_1$, so Proposition 2.4 applies.

The core disc $h = F_0$ obtained in this way is close to the initial disc f over the complement of a small neighborhood of $\bigcup_j \bar{I}_j$ in $\bar{\mathbb{D}}$, while over a neighborhood of $\bar{I}_j$ it is close to f'_j. If the approximations are close enough, then h also satisfies the estimate (4.6) in Lemma 4.1. ◻

Step 5. We now prove that $v = \hat{u}$ is plurisubharmonic on all of X. Let $w\colon X \to \mathbb{R} \cup \{-\infty\}$ be the upper regularization of $v|X_{\mathrm{reg}}$:

$$w(p) = \begin{cases} v(p), & p \in X_{\mathrm{reg}}; \\ \limsup_{q \in X_{\mathrm{reg}},\, q \to p} v(q), & p \in X_{\mathrm{sing}}. \end{cases}$$

It is easy to see that $w \le u$. Since X is normal, w is plurisubharmonic on X according to a result of Grauert and Remmert [GR1, Satz 4].

To complete the proof of the theorem we show that $v = w$ on X_{sing}. Fix a point $p \in X_{\mathrm{sing}}$ and a disc $f \in \mathcal{A}(\mathbb{D}, X, p)$. The fact that w is plurisubharmonic implies that $w(p) \le P_w(f)$. Since $w \le u$, we also have $P_w(f) \le P_u(f)$. Therefore $w(p) \le P_u(f)$ for every $f \in \mathcal{A}(\mathbb{D}, X, p)$, which implies that $w(p) \le v(p)$.

Suppose now that $w(p) < v(p)$; we shall reach a contradiction. Choose $\varepsilon > 0$ so that $w(p) + 3\varepsilon < v(p)$. Since w is upper semicontinuous, there is a neighborhood U of p such that

$$w(q) \le w(p) + \varepsilon \quad \text{for each } q \in U. \tag{4.11}$$

We can choose an analytic disc $f \in \mathcal{A}(\mathbb{D}, U, p)$ such that $f(\mathbb{T}) \subset U \cap X_{\mathrm{reg}}$. Since $w = v$ on X_{reg}, we have, using (4.11),

$$P_v(f) = P_w(f) \le \sup\{w \circ f(\zeta) \colon \zeta \in \mathbb{T}\} \le w(p) + \varepsilon. \tag{4.12}$$

For each point $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$ we choose a disc $g_t = g(\mathrm{e}^{\mathrm{i}t}, \cdot) \in \mathcal{A}(\mathbb{D}, X, f(\mathrm{e}^{\mathrm{i}t}))$ such that $P_u(g_t) < v(f(\mathrm{e}^{\mathrm{i}t})) + \varepsilon$. As in Step 3, we can deform g_t to a continuous family $g\colon \mathbb{T} \times \bar{\mathbb{D}} \to X$ of analytic discs such that

$$\left| \int_0^{2\pi} v(f(\mathrm{e}^{\mathrm{i}t})) \frac{\mathrm{d}t}{2\pi} - \int_0^{2\pi} \int_0^{2\pi} u(g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta})) \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\theta}{2\pi} \right| < \varepsilon. \tag{4.13}$$

Lemma 4.1 furnishes an analytic disc $h \in \mathcal{A}(\mathbb{D}, X, p)$ such that

$$P_u(h) < \int_0^{2\pi} \int_0^{2\pi} u(g(e^{it}, e^{i\theta})) \frac{dt}{2\pi} \frac{d\theta}{2\pi} + \varepsilon.$$

By (4.13) and (4.12) we get

$$P_u(h) \leq \int_0^{2\pi} v(f(e^{it})) \frac{dt}{2\pi} + 2\varepsilon \leq w(p) + 3\varepsilon < v(p),$$

which contradicts the definition of v. This concludes the proof.

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Added to page proofs: Since the submission of this paper, the authors have used the techniques developed in this paper to obtain similar results for some other classes of disc functionals; see [DF2].

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