Surjective holomorphic maps onto Oka manifolds

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Abstract Let X be a connected Oka manifold, and let S be a Stein manifold with $\dim S \geq \dim X$. We show that every continuous map $S \to X$ is homotopic to a surjective strongly dominating holomorphic map $S \to X$. We also find strongly dominating algebraic morphisms from the affine n-space onto any compact n-dimensional algebraically subelliptic manifold. Motivated by these results, we propose a new holomorphic flexibility property of complex manifolds, the *basic Oka property with surjectivity*, which could potentially provide another characterization of the class of Oka manifolds.

Keywords Stein manifold, Oka manifold, holomorphic map, algebraic map

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1. Introduction

A complex manifold X is said to be an *Oka manifold* if every holomorphic map $U \to X$ from an open convex set U in a complex Euclidean space $\mathbb{C}^N$ can be approximated uniformly on compacts in U by holomorphic maps $\mathbb{C}^N \to X$. This *convex approximation property* (CAP) of X, which was first introduced in [12], implies that maps from any Stein manifold S to X satisfy the parametric Oka principle with approximation and interpolation (see [13, Theorem 5.4.4]; it suffices to verify CAP for the integer $N = \dim S + \dim X$). In particular, every continuous map $f\colon S \to X$ from a Stein manifold S to an Oka manifold X is homotopic to a holomorphic map $F\colon S \to X$, and F can chosen to approximate f on a compact $\mathscr{O}(S)$-convex set $K \subset S$ provided that f is holomorphic on a neighborhood of K. For the theory of Oka manifolds, we refer to the monograph [13] and the surveys [14, 15, 22]; for the theory of Stein manifolds, see [18, 19].

In this note, we construct *surjective* holomorphic maps from Stein manifolds to Oka manifolds, and surjective algebraic morphisms of affine algebraic manifolds to certain compact algebraic manifolds. We say that a (necessarily surjective) holomorphic map $F\colon S \to X$ is *strongly dominating* if for every point $x \in X$ there exists a point $p \in S$ such that $F(p) = x$ and $dF_p\colon T_pS \to T_xX$ is surjective. Equivalently, $F(S \setminus \mathrm{br}_F) = X$ where $\mathrm{br}_F \subset S$ is the branch locus of F.

Theorem 1.1. *Let X be a connected Oka manifold. If S is a Stein manifold and* $\dim S \geq \dim X$ *then every continuous map $f\colon S \to X$ is homotopic to a strongly dominating (surjective) holomorphic map $F\colon S \to X$. In particular, there exists a strongly dominating holomorphic map $F\colon \mathbb{C}^n \to X$ for $n = \dim X$.*

Theorem 1.1 answers a question that arose in author's discussion with Jörg Winkelmann (see the Acknowledgement). The result also holds, with the same proof, if S is a reduced Stein space. A similar result in the algebraic category is given by Theorem 1.6.

A version of Theorem 1.1, with $X \subset \mathbb{C}^n$ a non-autonomous basin of a sequence of attracting automorphisms with uniform bounds, is due to Fornæss and Wold [9, Theorem 1.4]. With the exception of surjectivity, the results in the cited theorem had been know earlier for maps of Stein manifolds to Oka manifolds; see [13, Theorem 7.9.1 and Corollary 7.9.3, pp. 324-325] for the existence of embeddings, while the existence of maps with dense images follows immediately from the fact that Oka manifolds enjoy the Oka property with interpolation [13, Theorem 5.4.4].

According to the standard terminology, a holomorphic map $F\colon \mathbb{C}^n \to X$ is said to be *dominating* at the point $x_0 = F(0) \in X$ if the differential $dF_0\colon T_0\mathbb{C}^n \to T_{x_0}X$ is surjective; if such F exists then X is *dominable at* x_0. A complex manifold which is dominable at every point is called *strongly dominable*. Every Oka manifold is strongly dominable, but the converse is not known. For a discussion of this subject, see e.g. [16]. Theorem 1.1 furnishes a map $F : \mathbb{C}^n \to X$ such that the family of maps $\{F \circ \phi_a\}_{a\in\mathbb{C}^n}$, where $\phi_a\colon \mathbb{C}^n \to \mathbb{C}^n$ is the translation $z \mapsto z + a$, dominates at every point of X.

On the other hand, we do not know whether every Oka manifold X is the image of a *locally biholomorphic* map $\mathbb{C}^n \to X$ with $n = \dim X$. A closely related problem is to decide whether locally biholomorphic self-maps of $\mathbb{C}^n$ for $n > 1$ satisfy the Runge approximation theorem; see [13, Problem 8.11.3 and Theorem 8.12.4].

Theorem 1.1 is proved in Section 3. The proof is based on an approximation result for holomorphic maps from Stein manifolds to Oka manifolds which we formulate in Section 2 (see Theorem 2.1). The approximation takes place on a locally finite sequence of compact sets in a Stein manifold S which are separated by the level sets of a strongly plurisubharmonic exhaustion function and satisfy certain holomorphic convexity conditions. Although Theorem 2.1 follows easily from the proof of the Oka principle with approximation (see [13, Chapter 5]), this formulation is useful in certain situations like the one considered here, and hence we feel it worthwhile to record it.

Theorem 1.1 is motivated in part by results to the effect that certain complex manifolds S are *universal sources*, in the sense that they admit a surjective holomorphic map $S \to X$ onto every complex manifold of the same dimension. This holds for a polydisk and a ball in $\mathbb{C}^n$ (see Fornæss and Stout [7, 8]; in this case, the map can be chosen locally biholomorphic and finitely sheeted), and also for any bounded domain with $\mathscr{C}^2$ boundary in $\mathbb{C}^n$ (see Løw [24]). Further results, with emphasis on the case $X = \mathbb{C}^n$, were obtained by Chen and Wang [4]. In these results, the source manifold is Kobayashi hyperbolic. This condition cannot be substantially weakened since a holomorphic map is distance decreasing with respect to the Kobayashi pseudometrics on the respective manifolds. In particular, a manifold with vanishing Kobayashi pseudometric (such as $\mathbb{C}^n$) does not admit any nonconstant holomorphic map to a hyperbolic manifold. Furthermore, the existence of a nondegenerate holomorphic map $\mathbb{C}^n \to X$ to a connected compact complex manifold X of dimension n implies that X is not of general type (see Kodaira [21] and Kobayashi and Ochiai [20]). By an extension of the Kobayashi-Ochiai argument, Campana proved that such X is *special* [2, Corollary 8.11]. Special manifolds are important in Campana's structure theory of compact Kähler manifolds. Recently, Diverio and Trapani [5] and Wu and Yau [28, 29] proved that a compact connected complex manifold X, which admits a Kähler metric whose holomorphic sectional curvature is everywhere nonpositive and is strictly negative at least at one point, has positive canonical bundle K_X. (See also Tosatti and Yang [26] and Nomura [25].) Hence, such X is projective and of general type, and therefore it does not admit any nondegenerate holomorphic map $\mathbb{C}^n \to X$ with $n = \dim X$.

These observations justify the hypothesis in Theorem 1.1 that X be an Oka manifold.

Let us recall a related but weaker holomorphic flexibility property introduced by Gromov [17]. A complex manifold X is said to enjoy the *basic Oka property*, BOP, if every continuous map $S \to X$ from a Stein manifold S is homotopic to a holomorphic map. The only difference with respect to the class of Oka manifolds is that BOP does not include any approximation or interpolation conditions. Thus, every Oka manifold satisfies BOP, but the converse fails e.g. for contractible hyperbolic manifolds (such as bounded convex domains in $\mathbb{C}^n$). The basic Oka property property was studied by Winkelmann [27] for maps between Riemann surfaces, and by Campana and Winkelmann [3] for more general complex manifolds. (Their use of the term *homotopy principle* is equivalent to BOP.) In particular, they proved in [3, Main Theorem] that a projective manifold satisfying BOP is special in the sense of [2]. (The converse is an open problem.) We thus have

$$\mathrm{Oka} \Longrightarrow \mathrm{BOP} \Longrightarrow \mathrm{special},$$

where the second implication holds for compact projective manifolds (and is expected to be true for all compact Kähler manifolds).

Concerning the relationship between Oka manifolds and manifolds with BOP, one has the feeling that these two classes are essentially the same after eliminating the obvious counterexamples provided by contractible hyperbolic manifolds; the latter may be used as building blocks in manifolds with BOP, but not in Oka manifolds. With this in mind, we propose the following new Oka property.

Definition 1.2. A connected complex manifold X satisfies the *basic Oka property with surjectivity*, abbreviated BOPS, if every continuous map $f : S \to X$ from a Stein manifold S with $\dim S \geq \dim X$ is homotopic to a *surjective* holomorphic map $F : S \to X$.

Theorem 1.1 says that Oka $\Rightarrow$ BOPS. Applying the BOPS axiom to a constant map $\mathbb{C}^n \to x_0 \in X$ gives the following observation.

Proposition 1.3. *A connected complex manifold X satisfying* BOPS *admits a surjective holomorphic map $\mathbb{C}^n \to X$ with $n = \dim X$. In particular, the Kobayashi pseudometric of a complex manifold satisfying* BOPS *vanishes identically.*

Since the BOPS axiom eliminates the obvious counterexamples to the (false) implication BOP $\Rightarrow$ Oka, the following seems a reasonable question.

Problem 1.4. (a) Assuming that a complex manifold X satisfies BOPS, does it follow that X is an Oka manifold? That is, do we have the implication $\mathrm{BOPS} \Rightarrow \mathrm{Oka}$?

(b) Do the properties BOP and BOPS coincide in the class of compact (or compact Kähler, or compact projective) manifolds?

Let us mention another question related to Theorem 1.1. Let $\mathbb{B}^n$ denote the open ball in $\mathbb{C}^n$. It is an open problem whether $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is an Oka manifold when $n > 1$.

Problem 1.5. Let $n > 1$. Does there exist a surjective holomorphic map $\mathbb{C}^n \to \mathbb{C}^n \setminus \overline{\mathbb{B}}^n$?

In this connection, we mention that Dixon and Esterle (see [6, Theorem 8.13, p. 182]) constructed for every $\epsilon > 0$ a finitely sheeted holomorphic map $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ whose image avoids the closed unit ball $\overline{\mathbb{B}}^2$ but contains the complement of the ball of radius $1 + \epsilon$: $\mathbb{C}^2 \setminus (1+\epsilon)\overline{\mathbb{B}}^2 \subset f(\mathbb{C}^2) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}^2$.

Theorem 1.1 shows that a negative answer to Problem 1.5 would imply that $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ fails to be Oka. Since $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is a union of Fatou-Bieberbach domains (obtained for example as attracting basins of holomorphic automorphisms of $\mathbb{C}^n$ which map the ball $\mathbb{B}^n$ into itself), this would provide an example of a strongly dominable manifold which is not Oka.

The above example is also connected to the open problem whether every Oka manifold is elliptic or subelliptic, the latter being the main known geometric conditions implying all versions of the Oka property (see Gromov [17], Forstnerič [10], and [13, Definition 5.5.11 (d) and Corollary 5.5.12]). The following implications hold for any complex manifold:

$$\text{homogeneous} \Longrightarrow \text{elliptic} \Longrightarrow \text{subelliptic} \Longrightarrow \text{Oka} \Longrightarrow \text{strongly dominable}.$$

It was shown by Andrist et al [1] that $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is not subelliptic when $n \geq 3$. Since $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is strongly dominable, at least one of the two right-most implications cannot be reversed. Therefore, the question whether $\mathbb{C}^3 \setminus \overline{\mathbb{B}}^3$ is an Oka manifold is of particular interest.

It is natural to look for an analogue of Theorem 1.1 in the algebraic category. At this time, we do not have a good notion of an algebraic Oka manifold. However, a useful geometric condition on an algebraic manifold X, which gives the approximation of certain holomorphic maps $S \to X$ from affine algebraic manifolds S by algebraic morphisms $S \to X$, is *algebraic subellipticity*; see [13, Definition 5.5.11 (e)] or Section 4 below. (We emphasize that all algebraic maps in this paper are understood to be morphisms, i.e., without singularities.) In Section 4 we prove the following result in this direction.

Theorem 1.6. *Assume that X is a compact algebraically subelliptic manifold and S is an affine algebraic manifold such that $\dim S \geq \dim X$. Then, every algebraic map $S \to X$ is homotopic (through algebraic maps) to a surjective strongly dominating algebraic map $S \to X$. In particular, X admits a surjective strongly dominating algebraic map $F\colon \mathbb{C}^n \to X$ with $n = \dim X$.*

The proof of Theorem 1.6 is based on Theorem 4.1 which is taken from [11]. It says in particular that, given an affine algebraic manifold S and an algebraically subelliptic manifold X, a holomorphic map $S \to X$ that is homotopic to an algebraic map through a family of holomorphic maps can be approximated by algebraic maps $S \to X$.

Example 1.7. Let X be an algebraic manifold of dimension n which is covered by Zariski open sets that are biregularly isomorphic to $\mathbb{C}^n$. Such manifolds are said to be of Class $\mathcal{A}_0$ (see [13, Definition 6.4.5]). Then X is algebraically subelliptic (see [13, Proposition 6.4.6]). Furthermore, the total space Y of any blow-up $Y \to X$ along a closed algebraic submanifold of X is also algebraically subelliptic according to Lárusson and Truong [23, Corollary 2]. If X (and hence Y) is compact, then Theorem 1.6 furnishes a strongly dominating morphisms $\mathbb{C}^n \to Y$. This holds for example if Y is obtained by blowing up a projective space or a Grassmanian along a compact submanifold.

It was shown by Lárusson and Truong [23, Proposition 6] that every algebraically subelliptic manifold (not necessarily compact) is strongly algebraically dominable. We do not know whether the analogue of Theorem 1.6 holds if X is a *noncompact* algebraically subelliptic manifold.

2. Approximation of maps from a Stein manifold to an Oka manifold on a sequence of Stein compacts

We denote by $\mathscr{O}(S)$ the algebra of all holomorphic functions on a complex manifold S, endowed with the compact-open topology. Recall that a compact set K in S is said to be $\mathscr{O}(S)$*-convex* if $K = \widehat{K}_{\mathscr{O}(S)}$, where the holomorphic hull of K is defined by

$$\widehat{K}_{\mathscr{O}(S)} = \big\{p \in S : |f(p)| \le \sup_K |f| \ \ \forall f \in \mathscr{O}(S)\big\}.$$

In this section, we prove the following approximation result. In the next section, we will apply it to prove Theorem 1.1.

Theorem 2.1. *Let S be a reduced Stein space and $(K_j)_{j=1}^{\infty}$ be a sequence of compact pairwise disjoint subsets of S satisfying the following properties:*

(a) *Every compact set in S intersects at most finitely many of the sets K_j.*
(b) *The union $\cup_{j=1}^{k} K_j$ is $\mathscr{O}(S)$-convex for each $k \in \mathbb{N}$.*
(c) *Set $K = \cup_{j=1}^{\infty} K_j$. There exist a strongly plurisubharmonic exhaustion function $\rho\colon S \to \mathbb{R}_+ = [0, +\infty)$ and an increasing sequence $0 < a_1 < a_2 < \dots$ with $\lim_{j\to\infty} a_j = +\infty$ such that for every $j \in \mathbb{N}$ we have $K \cap \{\rho = a_j\} = \emptyset$ and*

$$\text{the compact set } M_j := \{\rho \le a_j\} \cup (K \cap \{\rho \le a_{j+1}\}) \text{ is } \mathscr{O}(S)\text{-convex.} \tag{2.1}$$

Let X be an Oka manifold, and let $f\colon S \to X$ be a continuous map which is holomorphic on a neighborhood of the set $K = \cup_{j=1}^{\infty} K_j$. Let dist *be a distance function on X inducing the manifold topology. Given a sequence $\epsilon_j > 0$ $(j \in \mathbb{N})$, there exists a holomorphic map $F\colon S \to X$, homotopic to f by a family of maps $F_t\colon S \to X$ $(t \in [0,1])$ that are holomorphic on a neighborhood of K, such that*

$$\sup_{p \in K_j} \operatorname{dist}(f(p), F(p)) < \epsilon_j \quad \text{for all } j = 1, 2, \dots. \tag{2.2}$$

Furthermore, given a discrete sequence of points $(p_j)_{j\in\mathbb{N}} \subset K$ and integers $k_j \in \mathbb{N}$, we can choose F to agree with f to order k_j at p_j.

Proof. We may assume that dist is a complete metric on X and that $\sum_j \epsilon_j < \infty$. Let $(a_j)_{j\in\mathbb{N}}$ be the sequence of real numbers in condition (c). Set

$$S_j := \{p \in S : \rho(p) \le a_j\}, \quad A_j := \{p \in S : a_j \le \rho(p) \le a_{j+1}\}, \quad j \in \mathbb{N}.$$

Note that S_j is compact $\mathscr{O}(S)$-convex, and we have

$$S_{j+1} = S_j \cup A_j \ \text{ and } \ M_j = S_j \cup (K \cap A_j) \ \text{ for every } j = 1, 2, \dots.$$

(Recall that $K = \cup_{j=1}^{\infty} K_j$.) For consistency of notation we also set

$$S_0 = \emptyset, \quad M_0 := K \cap S_1, \quad F_0 = f.$$

By hypothesis (c), we have that $K \cap bS_j = \emptyset$ for all $j \in \mathbb{N}$. Furthermore, condition (a) in the theorem implies that each set S_j contains at most finitely many of the sets K_i. Set

$$\eta_j := \min\{\epsilon_i : K_i \subset S_j\} > 0, \quad j = 1, 2, \dots. \tag{2.3}$$

To prove the theorem, we shall construct sequences of continuous maps $F_j\colon S \to X$, homotopies $F_{j,t}\colon S \to X$ $(t \in [0,1])$, and numbers $b_j, c_j > 0$ satisfying the following conditions for every $j \in \mathbb{N}$:

(i$_j$) $a_j < b_j < c_j < a_{j+1}$ and $K \cap A_j \subset \{c_j < \rho < a_{j+1}\}$.

(ii$_j$) F_j is holomorphic on $\{\rho < b_j\}$ and $F_j = F_{j-1}$ on $\{\rho \ge c_j\}$.

(iii$_j$) $\mathrm{dist}(F_j(p), F_{j-1}(p)) < 2^{-j}\eta_j$ for every $p \in M_{j-1}$.

(iv$_j$) $F_{j,0} = F_{j-1}$ and $F_{j,1} = F_j$.

(v$_j$) For every $t \in [0,1]$ the map $F_{j,t}$ is holomorphic on a neighborhood of M_{j-1} and $F_{j,t} = F_{j-1}$ holds on $\{\rho \ge c_j\}$.

(vi$_j$) $\mathrm{dist}(F_{j,t}(p), F_{j-1}(p)) < 2^{-j}\eta_j$ for every $p \in M_{j-1}$ and $t \in [0,1]$.

We could also add a suitable condition on F_j to ensure jet interpolation along a discrete sequence $(p_j) \subset K$ (see the last sentence in the theorem). Since this interpolation is a trivial addition in what follows, we shall delete it to simplify the exposition.

A sequence of maps and homotopies satisfying these properties can be constructed recursively by using [13, Theorem 5.4.4] at every step; we offer some details.

Assume that maps $F_0, F_1, \ldots, F_j$ and homotopies $F_{1,t}, \ldots, F_{j,t}$ with these properties have been found for some $j \in \mathbb{N}$. (Recall that $F_0 = f$.) In view of property (ii$_j$) the map F_j is holomorphic on the set $\{\rho < b_j\}$, and we have $F_j = F_{j-1} = \cdots = F_0$ on $\{\rho \ge c_j\}$. Since $K \cap A_j \subset \{c_j < \rho < a_{j+1}\}$ by property (i$_j$), it follows that F_j is holomorphic on a neighborhood of the set M_j (2.1). Since M_j is $\mathscr{O}(S)$-convex, we can apply [13, Theorem 5.4.4] fo find a number $c_{j+1} > a_{j+1}$ close to a_{j+1}, a holomorphic map $F_{j+1} \colon \{\rho < c_{j+1}\} \to X$ satisfying property (iii$_{j+1}$), and a homotopy of maps $F_{j+1,t} \colon \{\rho < c_j\} \to X$ ($t \in [0,1]$) satisfying properties (iv$_{j+1}$) and (vi$_{j+1}$). It remains to extend this homotopy to all of S such that condition (v$_{j+1}$) holds as well. This is accomplished by using a cut-off function in the parameter of the homotopy. Explicitly, pick a number b_{j+1} such that $a_{j+1} < b_{j+1} < c_{j+1}$, and let $\chi \colon S \to [0,1]$ be a continuous function which equals 1 on the set $\{\rho \le b_{j+1}\}$ and has support contained in $\{\rho < c_{j+1}\}$. The homotopy of continuous maps

$$(p,t) \longmapsto F_{j+1,\chi(p)t}(p) \in X, \quad p \in S,\ t \in [0,1]$$

then agrees with the homotopy $F_{j+1,t}$ on the set $\{\rho \le b_{j+1}\}$ (since $\chi = 1$ there), and it agrees with the map F_j (and hence with $F_0 = f$) on $\{\rho \ge c_{j+1}\}$ since χ vanishes there. This established the condition (v$_{j+1}$) and completes the induction step.

In view of (iii$_j$) and the definition of the numbers η_j (2.3), the sequence $F_j \colon S \to X$ converges uniformly on compacts in S to a holomorphic map $F = \lim_{j\to\infty} F_j \colon S \to X$ satisfying the estimates (2.2). Furthermore, conditions (iv$_{j+1}$)–(vi$_{j+1}$) imply that the sequence of homotopies $F_{j,t} \colon S \to X$ ($j \in \mathbb{N}$) can be assembled into a homotopy $F_t \colon S \to X$ ($t \in [0,1]$) connecting $F_0 = f$ to the final holomorphic map $F_1 = F$ such that F_t is holomorphic on a neighborhood of the set K for every $t \in [0,1]$ and every map F_t in the homotopy satisfies the estimates (2.2). This assembling is accomplished by writing $[0,1) = \cup_{j=1}^{\infty} I_j$, where $I_j = [1-2^{-j+1}, 1-2^{-j}]$, and placing the homotopy $(F_{j,t})_{t\in[0,1]}$ onto the subinterval $I_j \subset [0,1]$ by suitably reparametrizing the t-variable. □

3. Construction of surjective holomorphic maps to Oka manifolds

Proof of Theorem 1.1. Let X be a complex manifold of dimension n. Choose a countable family of compact sets $L'_j \subset L_j \subset X$ ($j \in \mathbb{N}$) satisfying the following conditions:

(i) $L'_j \subset \mathring{L}_j$ for every $j \in \mathbb{N}$.

(ii) $\cup_{j=1}^\infty L'_j = X$.

(iii) For every $j\in\mathbb{N}$ there are an open set $V_j\subset X$ containing L_j and a biholomorphic map $\psi_j\colon V_j\to\psi_j(V_j)\subset\mathbb{C}^n$ such that $\psi_j(L_j)=\overline{\mathbb{B}}^n$ is the closed unit ball in $\mathbb{C}^n$.

A compact set $L_j\subset X$ satisfying condition (iii) will be called a (closed) *ball* in X. If the manifold X is compact, then we can cover it by a finite family of such balls.

Let S be a Stein manifold of dimension $m=\dim S\ge n$. Choose a smooth strongly plurisubharmonic exhaustion function $\rho\colon S\to\mathbb{R}_+=[0,+\infty)$. Pick an increasing sequence of real numbers $a_j>0$ with $\lim_{j\to\infty}a_j=+\infty$. For each $j\in\mathbb{N}$ we choose a small $\mathscr{O}(S)$-convex ball K_j in S such that

$$K_j\subset\{p\in S\colon a_j<\rho(p)<a_{j+1}\} \tag{3.1}$$

and

$$\text{the compact set } M_j:=K_j\cup\{\rho\le a_j\} \text{ is } \mathscr{O}(S)\text{-convex.} \tag{3.2}$$

The last condition can be achieved by taking the balls K_j small enough; here is an explanation. By the assumption, there are an open set $U_j\subset S$ containing K_j and a biholomorphic coordinate map $\phi_j\colon U_j\to\phi_j(U_j)\subset\mathbb{C}^m$ such that $\phi_j(K_j)=\overline{\mathbb{B}}^m\subset\mathbb{C}^m$. In view of (3.1) we may assume that $\overline{U}_j\cap\{\rho\le a_j\}=\emptyset$. Let $p_j:=\phi_j^{-1}(0)\in K_j$ be the center of K_j. The compact set $\{\rho\le a_j\}\cup\{p_j\}$ is clearly $\mathscr{O}(S)$-convex, and hence it has a basis of compact $\mathscr{O}(S)$-convex neighborhoods. In particular, there is a compact neighborhood $T\subset U_j$ of the point p_j such that $T\cup\{\rho\le a_j\}$ is $\mathscr{O}(S)$-convex. Choose a number $0<r_j<1$ small enough such that $r_j\overline{\mathbb{B}}^m\subset\phi_j(T)$. The ball $K'_j:=\phi_j^{-1}(r_j\overline{\mathbb{B}}^m)$ is then contained in T and is $\mathscr{O}(T)$-convex. Hence, the set $K'_j\cup\{\rho\le a_j\}$ is $\mathscr{O}(S)$-convex. Replacing K_j by K'_j and rescaling the coordinate map ϕ_j accordingly so that it takes this set onto $\overline{\mathbb{B}}^m$, condition (3.2) is satisfied.

Denote by $\pi\colon\mathbb{C}^m\to\mathbb{C}^n$ the coordinate projection $(z_1,\ldots,z_m)\mapsto(z_1,\ldots,z_n)$. (Recall that $m\ge n$.) Then $\pi(\overline{\mathbb{B}}^m)=\overline{\mathbb{B}}^n$. Let $U_j\supset K_j$ and $\phi_j\colon U_j\to\phi_j(U_j)\subset\mathbb{C}^m$ be as above. There is an open neighborhood $U'_j\subset S$ of K_j, with $U'_j\subset U_j$, such that the map

$$f_j=\psi^{-1}\circ\pi\circ\phi_j\colon U'_j\to V_j\subset X$$

is a well defined holomorphic submersion satisfying $f_j(K_j)=L_j$ for every $j\in\mathbb{N}$.

Since the set L'_j is contained in the interior of L_j and f_j is a submersion, there is a compact set K'_j contained in the interior of K_j such that $L'_j\subset f_j(\mathring{K}'_j)$. By Rouché's theorem we can choose $\epsilon_j>0$ small enough such that for every holomorphic map $F\colon K_j\to X$ defined on a neighborhood of K_j we have that

$$\sup_{p\in K_j}\operatorname{dist}(f_j(p),F(p))<\epsilon_j \implies L'_j\subset F(K'_j). \tag{3.3}$$

Let $f\colon S\to X$ be a continuous map. By a homotopic deformation of f, supported on a contractible neighborhood of the ball $K_j\subset S$ for each j, we can arrange that $f=f_j$ on a neighborhood of K_j for each $j\in\mathbb{N}$. The homotopy is kept fixed outside a somewhat bigger neighborhood of each K_j in S, and these neighborhoods are chosen to have pairwise disjoint closures. We denote the new map by the same letter f.

Theorem 2.1, applied to the map f and the sequences K_j and ϵ_j, furnishes a holomorphic map $F\colon S \to X$ that is homotopic to f and satisfies the estimate

$$\sup_{p\in K_j} \mathrm{dist}(f(p), F(p)) < \epsilon_j, \quad j = 1, 2, \ldots$$

(see (2.2)). By the choice of ϵ_j (3.3) it follows that $L'_j \subset F(K'_j)$ for each $j \in \mathbb{N}$, and hence

$$F(S) = \cup_{j=1}^{\infty} F(K'_j) = \cup_{j=1}^{\infty} L'_j = X.$$

Furthermore, if the numbers $\epsilon_j > 0$ are chosen small enough, then F has maximal rank equal to $\dim X$ at every point of K'_j (since this holds for the map f on the bigger set K_j), and hence F is strongly dominating. □

Remark 3.1. The same proof applies if S is a reduced Stein space with $\dim S \geq \dim X$. In this case, we just pick the balls K_j (3.1) in the regular locus of S.

4. Surjective algebraic maps to compact algebraically subelliptic manifolds

In this section we prove Theorem 1.6. We begin by recalling the relevant notions.

An algebraic manifold X is said to be *algebraically subelliptic* if it admits a finite family of algebraic sprays $s_j\colon E_j \to X$ ($j = 1, \ldots, k$), defined on total spaces E_j of algebraic vector bundles $\pi_j\colon E_j \to X$, which is *dominating* in the sense that for each point $x \in X$ the vector subspaces $(ds_j)_{0_x}(E_{j,x}) \subset T_xX$ span the tangent space T_xX:

$$(ds_1)_{0_x}(E_{1,x}) + \cdots + (ds_k)_{0_x}(E_{k,x}) = T_xX \quad \forall x \in X.$$

See [11, Definition 2.1] or [13, Definition 5.5.11 (e)] for the details. Here, X could be a projective (or quasi-projective) algebraic manifold, although the same theory applies to more general algebraic manifolds. By an *algebraic map*, we always mean an algebraic morphism without singularities.

The following result is [11, Theorem 3.1]; see also [13, Theorem 7.10.1]. As pointed out there, this is a version of the *h-Runge approximation theorem* in the algebraic category. For the analytic case of this result, see Gromov [17] and [13, Theorem 6.6.1].

Theorem 4.1. *Assume that S is an affine algebraic manifold and X is an algebraically subelliptic manifold. Given an algebraic map $f\colon S \to X$, a compact $\mathscr{O}(S)$-convex subset K of S, an open set $U \subset S$ containing K, and a homotopy $f_t\colon U \to X$ of holomorphic maps ($t \in [0, 1]$) with $f_0 = f|_U$, there exists for every $\epsilon > 0$ an algebraic map $F\colon S \times \mathbb{C} \to X$ such that*

$$F(\cdot\,, 0) = f \quad \textit{and} \quad \sup_{p\in K,\, t\in[0,1]} \mathrm{dist}\,(F(p, t), f_t(p)) < \epsilon.$$

Proof of Theorem 1.6. The proof uses Theorem 4.1 and is similar to that of Theorem 1.1. The main difference is that the initial map $f\colon S \to X$ must be algebraic. For the sake of simplicity, we present the details only in the special case when $S = \mathbb{C}^n$ with $n = \dim X$.

Fix a point $x_0 \in X$ and let $f\colon \mathbb{C}^n \to X$ be the constant map $f(z) = x_0 \in X$.

Since X is compact, there is finite family of pairs of compact sets $L'_j \subset L_j \subset X$ ($j = 1, \ldots, \ell$) satisfying properties (i)–(iii) stated at the beginning of proof of Theorem 1.1 (see Section 3). In particular, each set L_j is a ball in a suitable local coordinate, and we have that $\cup_{j=1}^{\ell} L'_j = X$.

Let $n = \dim X$. Choose pairwise disjoint closed balls $K_1, \ldots, K_\ell$ in $\mathbb{C}^n$ whose union $K := \cup_{j=1}^{\ell} K_j$ is polynomially convex. Let $p_j \in K_j$ denote the center of K_j. For each $j = 1, \ldots, \ell$ there are an open ball $U_j \subset \mathbb{C}^n$ containing K_j and a biholomorphic map $g_j \colon U_j \to g_j(U_j) \subset X$ such that $g_j(K_j) = L_j$. We may assume that the sets $U_1, \ldots, U_\ell$ are pairwise disjoint. By using a contraction of K_j and L_j to their respective centers, and after shrinking the neighborhoods $U_j \supset K_j$ if necessary, we can find homotopies of holomorphic maps $f_{j,t} \colon U_j \to X$ $(t \in [0,1],\ j = 1, \ldots, \ell)$ such that

$$f_{j,0} = f|_{U_j} \quad \text{and} \quad f_{j,1} = g_j \quad \text{for all } j = 1, \ldots, \ell.$$

Set $U = \cup_{j=1}^{\ell} U_j$ and denote by $f_t \colon U \to X$ the holomorphic map whose restriction to U_j agrees with $f_{j,t}$ for each $j = 1, \ldots, \ell$. Then $f_0 = f|_U$ is the constant map $U \to x_0$.

Applying Theorem 4.1 to the source manifold $S = \mathbb{C}^n$, the constant (algebraic) map $f \colon S \to x_0 \in X$, and the homotopy $\{f_t\}_{t \in [0,1]}$ furnishes an algebraic map $F \colon \mathbb{C}^n \to X$ whose restriction to K_j approximates the map g_j for each $j = 1, \ldots, \ell$. Assuming that the approximation is close enough, we see as in the proof of Theorem 1.1 that $F(\mathbb{C}^n) = X$ and that F can be chosen to be strongly dominating. □

Remark 4.2. A major source of examples of algebraically elliptic manifolds are the *algebraically flexible* manifods; see e.g. [22, Definition 12]. An algebraic manifold X is said to be algebraically flexible of it admits finitely many algebraic vector fields $V_1, \ldots, V_N$ with complete algebraic flows $\phi_{j,t}$ $(t \in \mathbb{C},\ j = 1, \ldots, N)$, such that the vectors $V_1(x), \ldots, V_N(x)$ span the tangent space $T_x X$ at every point $x \in X$. Note that every $(\phi_{j,t})_{t \in \mathbb{C}}$ is a unipotent 1-parameter group of algebraic automorphisms of X. The composition of the flows $\phi_{1,t_1} \circ \cdots \circ \phi_{N,t_N}$ is a dominating algebraic spray $X \times \mathbb{C}^N \to X$, and hence such X is algebraically elliptic.

For a survey of this subject, we refer to Kutzschebauch's paper [22].

Remark 4.3. The argument in the proof of Theorem 1.6 also applies in the holomorphic case and gives a simple proof of Theorem 1.1 when the manifold X is compact and the initial map $f \colon S \to X$ is assumed to be holomorphic. I wish to thank Tuyen Truong for this observation.

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