A properly embedded holomorphic disc in the ball with finite area and dense boundary curve

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Abstract In this paper we construct a properly embedded holomorphic disc in the unit ball $\mathbb{B}^2$ of $\mathbb{C}^2$ having a surprising combination of properties: on the one hand, it has finite area and hence is the zero set of a bounded holomorphic function on $\mathbb{B}^2$; on the other hand, its boundary curve is everywhere dense in the sphere $b\mathbb{B}^2$.

Keywords holomorphic disc, conformal map, Legendrian curve

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1. Introduction and main results

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc in the complex plane $\mathbb{C}$ and by $\mathbb{B}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{j=1}^n |z_j|^2 < 1\}$ the open unit ball in the Euclidean space $\mathbb{C}^n$ for any $n \in \mathbb{N}$.

In this paper we prove the following result which answers a question posed by Filippo Bracci (private communication, June 2017).

Theorem 1.1. *For every $n > 1$ and $\epsilon > 0$ there exists a proper holomorphic embedding $F\colon \mathbb{D} \hookrightarrow \mathbb{B}^n$ which extends to a smooth injective immersion $F\colon \overline{\mathbb{D}} \setminus \{\pm 1\} \to \overline{\mathbb{B}}^n$ such that $\mathrm{Area}(F(\mathbb{D})) < \epsilon$ and the boundary $F(b\mathbb{D} \setminus \{\pm 1\})$ is everywhere dense in $b\mathbb{B}^n$.*

The last statement is clearly equivalent to $\overline{F(\mathbb{D})} = F(\mathbb{D}) \cup b\mathbb{B}^n$. Our construction actually gives an injective holomorphic immersion F of an open neighborhood $U \subset \mathbb{C}$ of $\overline{\mathbb{D}} \setminus \{\pm 1\}$ into $\mathbb{C}^n$ which is transverse to the sphere $b\mathbb{B}^n$ and satisfies $F(U) \cap \mathbb{B}^n = F(\mathbb{D})$. It is clearly impossible for F to be smooth on all of $\overline{\mathbb{D}}$ since in this case $F(b\mathbb{D})$ cannot be dense in $b\mathbb{B}^n$. In the actual construction, we work on thin simply connected and smoothly bounded strips $\Omega \subset \mathbb{C}$ around the real axis $\mathbb{R}$ such that $F(\Omega)$ is a properly embedded holomorphic disc in $\mathbb{B}^n$, and the main action (i.e., creating a dense image curve $F(b\Omega) \subset b\mathbb{B}^n$) takes place at infinity. By a minor modification of our proof it is possible to find a disc of this kind such that the map F extends holomorphically to a neighborhood of $\overline{\mathbb{D}} \setminus \{1\}$. (Indeed, it suffices to replace the use of Lemma 4.3 by Lemma 4.4.)

The result seems especially interesting in dimension $n = 2$, for two reasons. One is that self-intersections of a complex curve in $\mathbb{C}^2$ are stable under deformations, and hence it is a rather delicate task to find *embedded* complex curves satisfying additional conditions such as these. Another is that an embedded holomorphic disc of finite area in the ball $\mathbb{B}^2$ is the zero set of a bounded holomorphic function on $\mathbb{B}^2$ according to a theorem of Berndtsson [3, Theorem 1.1], so we obtain the following corollary to Theorem 1.1.

Corollary 1.2. *There is a bounded holomorphic function on $\mathbb{B}^2$ whose zero set is a smooth curve of finite area, biholomorphic to the disc, and its boundary cluster set equals $b\mathbb{B}^2$.*

A related result was proved by Globevnik and Stout in 1989 [10, Theorem VI.1]: every strongly pseudoconvex domain $D \subset \mathbb{C}^n$ $(n \geq 2)$ with real analytic boundary contains a proper holomorphic disc $F\colon \mathbb{D} \to D$ of arbitrarily small area such that $\overline{F(\mathbb{D})} = F(\mathbb{D}) \cup \overline{\omega}$, where ω is a given nonempty connected subset of bD. (It is easy to achieve the latter improvement also in our result.) The main new point here is that we find properly *embedded* holomorphic discs with these properties, even in the lowest dimensional case $n = 2$. Our construction is considerably more involved than the one in [10].

Our result contributes to the understanding of the H^∞-hull of noncompact subsets of the ball, showing that the H^∞-hull of an embedded complex submanifold may coincide with its polynomial hull even if the submanifold has big boundary cluster set. As Bracci pointed out, it would be interesting to know whether there exists a holomorphic fibration $\mathbb{B}^2 \to \mathbb{D}$ with all fibres being discs of the type in Theorem 1.1; we do not know the answer. If such a fibration does not exist, this would lead to an analytic proof of the theorem of Koziarz and Mok [15] that there is no fibration of the ball over the disc which is invariant under the action of a co-compact group of automorphisms of the ball.

Our construction is based on a result from contact geometry which we also prove in the paper. Let $n \geq 2$. The sphere $b\mathbb{B}^n = S^{2n-1}$ carries the contact structure ξ given by the distribution of complex tangent lines. Removing a point from $b\mathbb{B}^n$ we obtain the Euclidean space $(\mathbb{R}^{2n-1}_{(x,y,z)}, \xi_0)$ with its standard contact structure $\xi_0 = \ker(dz + xdy)$, where $x, y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$ and $xdy = \sum_{j=1}^{n-1} x_j dy_j$; see Sec. 2. A smooth curve $f\colon \mathbb{R} \to b\mathbb{B}^n$ is said to be *complex tangential*, or *ξ-Legendrian*, if $\dot f(t) \in \xi_{f(t)}$ holds for every $t \in \mathbb{R}$.

The following result is proved in Sec. 2 as an application of Theorem 2.1. We write $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}_- = \{t \in \mathbb{R} : t \leq 0\}$.

Proposition 1.3. *For every $n \in \mathbb{N}$ there is an injective real analytic complex tangential immersion $f\colon \mathbb{R} \hookrightarrow b\mathbb{B}^n$ such that the curves $\Lambda_\pm = f(\mathbb{R}_\pm)$ are everywhere dense in $b\mathbb{B}^n$.*

A suitably chosen (thin) complexification of the embedded Legendrian curve $\Lambda = f(\mathbb{R}) \subset b\mathbb{B}^n$ is an embedded complex disc $\Sigma_0 \subset \mathbb{C}^n$ of arbitrarily small area such that $\Sigma_0 \cap \overline{\mathbb{B}}^n = \Lambda$ (see Lemma 5.1). By pulling Σ_0 slightly inside the ball along Λ by a suitably chosen holomorphic multiplier, where the amount of pulling diminishes sufficiently fast as we approach either of the two ends of Λ so that the boundary of Σ_0 remains in the complement of the closed ball, we obtain a properly embedded holomorphic disc Σ in $\mathbb{B}^n$ of arbitrarily small area whose boundary approximates Λ as closely as desired in the fine $\mathscr{C}^0$ topology, and hence they are dense in the sphere $b\mathbb{B}^n$. Since Λ is everywhere dense in $b\mathbb{B}^n$, it is a rather subtle task to obtain injectivity of the limit disc. The principal difficulty is that injectivity is not an open condition among immersions of noncompact manifolds in any fine topology. (On the other hand, immersions form an open set in the fine $\mathscr{C}^1$ topology; see e.g. [17, Sec. 2.15].) We find a disc $F(\mathbb{D}) \subset \mathbb{B}^n$ satisfying Theorem 1.1 as a limit of an inductively constructed sequence of properly embedded holomorphic discs $F_k\colon \mathbb{D} \hookrightarrow \mathbb{B}^n$, where each F_k is holomorphic on a neighborhood of the closed disc $\overline{\mathbb{D}}$. In the induction step we are given a properly embedded complex disc $\Sigma_k = F_k(\mathbb{D}) \subset \mathbb{B}^n$ with smooth boundary $b\Sigma_k = F_k(b\mathbb{D}) \subset b\mathbb{B}^n$ which intersects the Legendrian curve $\Lambda = f(\mathbb{R}) \subset b\mathbb{B}^n$ transversely at a pair of points $p_k^\pm$. (There may be other intersection points. The disc Σ_k is actually the intersection of a somewhat bigger embedded holomorphic disc in $\mathbb{C}^n$ with the ball $\mathbb{B}^n$.) Let E_k^+ (resp. E_k^-) be a compact arc in Λ with an endpoint p_k^+ (resp. p_-^+). The first step is to find a small perturbation of $\overline{\Sigma}_k$, fixing the points $p_k^\pm$, such that the boundary of the new disc intersects the arcs $E_k^\pm$ only at the points $p_k^\pm$ (see Lemma 3.1). The next disc Σ_{k+1} is then obtained by stretching Σ_k along the arcs $E_k^\pm$ to the other endpoints $q_k^\pm$ of $E_k^\pm$ so that the stretched out part lies in a thin tube around $E_k^+ \cup E_k^-$; see Lemma 4.3. The sequence of embedded discs obtained in this way converges in the weak $\mathscr{C}^1$ topology and also in the fine $\mathscr{C}^0$ topology on $\overline{\mathbb{D}} \setminus \{\pm 1\}$ to an embedded disc satisfying the conclusion of Theorem 1.1. The details are given in Sec. 5.

2. Densely embedded real analytic Legendrian curves

Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Let (x, y, z) be the Euclidean coordinates on $\mathbb{R}^{2n+1}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and $z \in \mathbb{R}$. The standard contact structure $\xi_0 = \ker \alpha_0$ on $\mathbb{R}^{2n+1}$ is given by the 1-form

$$\alpha_0 = dz + \sum_{i=1}^{n} x_i dy_i. \tag{2.1}$$

If M is a smooth manifold then a smooth immersion $f\colon M \to (\mathbb{R}^{2n+1}, \xi_0 = \ker \alpha_0)$ is said to be *isotropic* if $f^*\alpha_0 = 0$; an isotropic immersion is *Legendrian* if M has the maximal possible dimension n. (See the monographs by Cieliebak and Eliashberg [6] and Geiges [9] for background on contact geometry.) In this paper we only consider maps from $\mathbb{R}$ and call them Legendrian irrespectively of the dimension of the target.

Let $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. A neighborhood of a $\mathscr{C}^k$ map $f_0\colon \mathbb{R} \to \mathbb{R}^n$ in the *fine $\mathscr{C}^k$ topology* on the space $\mathscr{C}^k(\mathbb{R}, \mathbb{R}^n)$ is of the form

$$\left\{ f \in \mathscr{C}^k(\mathbb{R}, \mathbb{R}^n) : |f^{(j)}(t) - f_0^{(j)}(t)| < \epsilon(t) \ \ \forall t \in \mathbb{R} \ \forall j = 0, \ldots, k \right\}$$

where $\epsilon : \mathbb{R} \to (0, +\infty)$ is a positive continuous function, $f^{(j)}(t)$ denotes the derivative of order j of f at the point $t \in \mathbb{R}$, and $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^n$. (See Whitney [20] or Golubitsky and Guillemin [11] for more information.)

In this section, we prove the following result.

Theorem 2.1. *Let $n \in \mathbb{N}$. Every continuous map $f_0\colon \mathbb{R} \to \mathbb{R}^{2n+1}$ can be approximated in the fine $\mathscr{C}^0$ topology by real analytic injective Legendrian immersions $f\colon \mathbb{R} \hookrightarrow (\mathbb{R}^{2n+1}, \xi_0)$.*

The following is an immediate corollary to Theorem 2.1.

Corollary 2.2. *For every nonempty open connected subset ω of $\mathbb{R}^{2n+1}$ there exists a real analytic injective ξ_0-Legendrian immersion $f\colon \mathbb{R} \hookrightarrow \omega$ such that the cluster set of each of the sets $f(\mathbb{R}_+)$ and $f(\mathbb{R}_-)$ equals $\overline{\omega}$.*

The conclusion of the corollary means that for every point $p \in \overline{\omega}$ there exists a sequence $t_1 < t_2 < \ldots$ with $\lim_{j\to\infty} t_j = +\infty$ and $\lim_{j\to\infty} f(t_j) = p$, as well a sequence t_j decreasing to $-\infty$ with the same property.

Proof of Theorem 2.1. Let (x, y, z) be coordinates on $\mathbb{R}^{2n+1}$ as above. We consider $\mathbb{R}^{2n+1}$ as the standard real subspace of $\mathbb{C}^{2n+1}$ and use the same letters to denote complex coordinates on $\mathbb{C}^{2n+1}$. The standard contact form α_0 on $\mathbb{R}^{2n+1}$ (2.1) then extends to a holomorphic contact form on $\mathbb{C}^{2n+1}$, and a holomorphic map $f\colon D \to \mathbb{C}^{2n+1}$ from a domain $D \subset \mathbb{C}$ will be called Legendrian if $f^*\alpha_0 = 0$. Note that every real analytic Legendrian map $f\colon I \to \mathbb{R}^{2n+1}$ from a segment $I \subset \mathbb{R}$ extends by complexification to a holomorphic Legendrian map from a neighborhood of I in $\mathbb{C}$. Furthermore, if a domain $D \subset \mathbb{C}$ is invariant with respect to the conjugation $\tau(z) = \bar z$ and $f\colon D \to \mathbb{C}^{2n+1}$ is a holomorphic Legendrian map, then the map $F\colon D \to \mathbb{C}^{2n+1}$ defined by

$$F(z) = \frac{1}{2}\left(f(z) + \overline{f(\bar z)}\right), \quad z \in D \tag{2.2}$$

is a holomorphic Legendrian map satisfying $F(z) = \overline{F(\bar z)}$; in particular, $F\colon D \cap \mathbb{R} \to \mathbb{R}^{2n+1}$ is a real analytic Legendrian map.

Recall that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\epsilon : \mathbb{R} \to (0, +\infty)$ be a positive continuous function. Pick $\eta_0 > 0$. We shall construct a sequence of holomorphic Legendrian maps $f_j\colon j\overline{\mathbb{D}} \to \mathbb{C}^{2n+1}$ and a decreasing sequence of positive number $\eta_j > 0$ such that the following conditions hold for every $j \in \mathbb{N}$, where the condition (d_1) is vacuous:

(a_j) $f_j(z) = \overline{f_j(\bar z)}$ for $z \in j\overline{\mathbb{D}}$; in particular, f_j is real-valued on $[-j, j] = j\overline{\mathbb{D}} \cap \mathbb{R}$,

(b_j) $f_j\colon [-j, j] \hookrightarrow \mathbb{R}^{2n+1}$ is a Legendrian embedding,

(c_j) $|f_j(t) - f_0(t)| < \epsilon(t)$ for $t \in [-j, j]$,

(d_j) $||f_j - f_{j-1}||_{\mathscr{C}^1((j-1)\overline{\mathbb{D}})} < \eta_{j-1}$, and

(e_j) $0 < \eta_j < \eta_{j-1}/2$ and every $\mathscr{C}^1$ map $g\colon [-j, j] \to \mathbb{R}^{2n+1}$ with $||g - f_j||_{\mathscr{C}^1([-j,j])} < 2\eta_j$ is an embedding.

It is immediate that the sequence f_j converges uniformly on compacts in $\mathbb{C}$ to a holomorphic Legendrian map $f = \lim_{j\to\infty} f_j\colon \mathbb{C} \to \mathbb{C}^{2n+1}$ whose restriction to $\mathbb{R}$ is a real analytic Legendrian embedding $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^{2n+1}$ satisfying $|f(t) - f_0(t)| \le \epsilon(t)$ for all $t \in \mathbb{R}$. This will prove Theorem 2.1.

We begin by explaining the base of the induction $(j = 1)$. Set

$$\epsilon_1 = \min\{\epsilon(t) : -1 \le t \le 1\} > 0.$$

We can find a smooth Legendrian embedding $g\colon [-1, 1] \hookrightarrow \mathbb{R}^{2n+1}$ such that

$$|g(t) - f_0(t)| < \epsilon_1/2 \quad \text{for all } \ t \in [-1, 1]. \tag{2.3}$$

(See e.g. Geiges [9, Theorem 3.3.1, p. 101] or Gromov [13].) Let us recall this elementary argument for the case $n = 1$, i.e., in $\mathbb{R}^3$. From the contact equation $dz + xdy = 0$ we see that the third component z of a Legendrian curve $g(t) = (x(t), y(t), z(t))$ for $t \in [-1, 1]$ is uniquely determined by the formula

$$z(t) = z(0) - \int_0^t x(s)\dot y(s)ds, \quad t \in [-1, 1]. \tag{2.4}$$

Hence, a loop γ in the Lagrangian (x, y)-plane adds a displacement for the amount $-\int_\gamma xdy$ to the z-variable. By Stokes's theorem, this equals the negative of the signed area of the region enclosed by γ. Hence, it suffices to approximate the (x, y)-projection of the given continuous arc $f_0\colon [-1, 1] \to \mathbb{R}^3$ by a smooth immersed arc containing small loops whose signed area creates a suitable displacement in the z-direction, thereby uniformly approximating f_0 by a smooth Legendrian arc $g\colon [-1, 1] \to \mathbb{R}^3$. Furthermore, by a general position argument (see e.g. [11]) we can approximate its Lagrange projection $g_L = (x, y) : [-1, 1] \to \mathbb{R}^2$ in $\mathscr{C}^2([-1, 1])$ by a smooth immersion with only simple (transverse) double points. Note that $g(t_1) = g(t_2)$ for some pair of numbers $t_1 \ne t_2$ if and only if $g_L(t_1) = g_L(t_2)$ and $\int_{t_1}^{t_2} x(s)\dot y(s)ds = 0$. Since g_L has at most finitely many double point loops, we can arrange by a generic $\mathscr{C}^2$-small perturbation of g_L away from its double points that the signed area enclosed by any of its double point loops is nonzero, thereby ensuring that the new map g is a Legendrian embedding and the estimate (2.3) still holds. Furthermore, we see from (2.4) that $\mathscr{C}^2$ approximation of the Lagrange projection g_L gives $\mathscr{C}^1$ approximation of the last component z. A similar argument applies in any dimension.

Pick a number δ with $0 < \delta < \epsilon_1/2$ and apply Weierstrass's theorem to find a polynomial Legendrian map $f_1 \colon \mathbb{C} \to \mathbb{C}^{2n+1}$ such that

$$\|f_1 - \tilde{g}\|_{\mathscr{C}^1([-1,1])} < \delta < \epsilon_1/2. \tag{2.5}$$

(It suffices to apply Weierstrass's theorem to the Lagrange projection and obtain the last component by integration (2.4) as in the previous step.) If $\delta > 0$ is chosen small enough, then $f_1 \colon [-1,1] \to \mathbb{C}^{2n+1}$ is an embedding whose imaginary component is small in $\mathscr{C}^1([-1,1])$. Hence, replacing f_1 by its symmetrization (2.2) we obtain a holomorphic Legendrian map $\mathbb{C} \to \mathbb{C}^{2n+1}$ which is real-valued on $\mathbb{R}$ and whose restriction to $[-1,1]$ is a Legendrian embedding $f_1 \colon [-1,1] \hookrightarrow \mathbb{R}^{2n+1}$ satisfying (2.5). Thus, conditions (a_1) and (b_1) hold, and the inequalities (2.3) and (2.5) yield (c_1). Condition (d_1) is vacuous. Pick a constant $\eta_1 > 0$ such that condition (e_1) holds.

This provides the base of the induction, and we proceed to the induction step.

Assume that for some $j \in \mathbb{N}$ we have found maps $f_1, \ldots, f_j$ and numbers $\eta_1, \ldots, \eta_j$ satisfying conditions (a_k)–(e_k) for $k = 1, \ldots, j$. Thus, there is $\delta > 0$ such that f_j is a holomorphic Legendrian map on the disc $(j+\delta)\mathbb{D}$ and $f_j \colon [-1-\delta, 1+\delta] \hookrightarrow \mathbb{R}^{2n+1}$ is an embedding. Set $E_j = j\overline{\mathbb{D}} \cup [-j-1, j+1] \subset \mathbb{C}$. Decreasing $\delta > 0$ if necessary, we can apply the same arguments as in the initial step in order to extend f_j to a map $f_j \colon (j+\delta)\mathbb{D} \cup [-j-1, j+1] \to \mathbb{C}^{2n+1}$ such that $f_j \colon [-j-1, j+1] \to \mathbb{R}^{2n+1}$ is a smooth Legendrian embedding satisfying

$$|f_j(t) - f_0(t)| < \epsilon(t), \quad t \in [-j-1, j+1].$$

Mergelyan's theorem for generalized Legendrian maps (see [2, Lemma 4.3]) gives an entire Legendrian map $f_{j+1} \colon \mathbb{C} \to \mathbb{C}^{2n+1}$ approximating f_j in $\mathscr{C}^1(E_j)$. If the approximation is close enough and we replace f_{j+1} by its symmetrization (2.2), then f_{j+1} satisfies conditions (a_{j+1})–(d_{j+1}). Finally, pick $\eta_{j+1} > 0$ satisfying condition (e_{j+1}) and the induction may proceed. This completes the proof of Theorem 2.1. $\square$

Proof of Proposition 1.3. For simplicity of notation we give the details in the case $n = 2$; the same proof applies in general. Thus, let ξ denote the contact structure on the sphere $b\mathbb{B}^2$ whose fibre ξ_z at any point $z = (z_1, z_2) \in b\mathbb{B}^2$ consists of all real vectors $v \in T_z b\mathbb{B}^2$ such that $Jv \in T_z b\mathbb{B}^2$. (Here, $J \colon T\mathbb{C}^2 \to T\mathbb{C}^2$ denotes the almost complex structure operator associated to the standard complex structure on $\mathbb{C}^2$.) Recall (see e.g. Rudin [19, Sec. 2.3.3]) that $b\mathbb{B}^2 \setminus \{pt\}$ is biholomorphically equivalent to the Heisenberg sphere

$$\Sigma = \{(x + \mathrm{i}y, u + \mathrm{i}v) \in \mathbb{C}^2 : v = x^2 + y^2\} \cong \mathbb{R}^3.$$

Thus, $\rho = -v + x^2 + y^2$ is a defining function for Σ and $(x, y, u) \in \mathbb{R}^3$ are global coordinates on Σ. Let $\phi \colon \mathbb{R}^3 \to \Sigma \subset \mathbb{C}^2$ be the map $\phi(x, y, u) = (x + \mathrm{i}y, u + \mathrm{i}(x^2 + y^2))$. A tangent vector $V = \xi_1 \partial_x + \xi_2 \partial_y + \eta \partial_u \in T_{(x,y,u)}\mathbb{R}^3$ is mapped by ϕ to the vector

$$\phi_* V = \widetilde{V} = \xi_1 \partial_x + \xi_2 \partial_y + \eta \partial_u + (2x\xi_1 + 2y\xi_2)\partial_v \in T_{\phi(x,y,u)}\Sigma.$$

The vector

$$J\widetilde{V} = -\xi_2 \partial_x + \xi_1 \partial_y - (2x\xi_1 + 2y\xi_2)\partial_u + \eta \partial_v$$

lies in $T_{\phi(x,y,u)}\Sigma$ if and only if

$$0 = \langle d\rho, J\widetilde{V}\rangle = \langle -dv + 2xdx + 2ydy, J\widetilde{V}\rangle = -\eta - 2x\xi_2 + 2y\xi_1.$$

This shows that the contact structure ξ is given in the coordinates (x, y, u) by the 1-form

$$\alpha = du + 2(xdy - ydx).$$

Introducing the new coordinate $\tilde{u}$ by $u = 4\tilde{u} + 2xy$, we have

$$\alpha = 4d\tilde{u} + 2d(xy) + 2(xdy - ydx) = 4(d\tilde{u} + xdy).$$

This shows that the contact structure ξ on $b\mathbb{B}^2 \setminus \{pt\}$ is contactomorphic to the standard structure (2.1) on $\mathbb{R}^3$. Hence, Proposition 1.3 follows from Corollary 2.2. $\square$

Remark 2.3. It is likely that Theorem 2.1 and Corollary 2.2 hold for every real analytic contact manifold (X, ξ). In this direction, Globevnik and Stout [10, Theorem V.1] constructed a not necessarily injective real analytic Legendrian immersion $f : \mathbb{R} \hookrightarrow (X, \xi)$ with dense image in any real analytic contact manifold. Results on approximation of smoothly embedded compact isotropic submanifolds by real analytic ones can be found in the monograph by Cieliebak and Eliashberg [6, Sec. 6.7]; however, their arguments do not seem to extend (at least not directly) to noncompact isotropic submanifolds. $\square$

3. A general position result

In this section we prove a general position result (see Lemma 3.1) which is used in the proof of Theorem 1.1. For simplicity of notation we focus on the case $n = 2$, although the proof carries over without any changes to the higher dimensional case $n > 2$.

Recall that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{T} = b\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. For any $k \in \mathbb{Z}_+ \cup \{+\infty\}$ and domain $D \subset \mathbb{C}$ with smooth boundary we denote by $\mathscr{A}^k(D)$ the space of functions $h \colon \overline{D} \to \mathbb{C}$ of class $\mathscr{C}^k(\overline{D})$ which are holomorphic in D, and we write $\mathscr{A}^0(D) = \mathscr{A}(D)$. We also introduce the function space

$$\mathscr{H} = \{h = u + \mathrm{i}v \in \mathscr{A}^\infty(\mathbb{D}) : h(\bar{z}) = \overline{h(z)},\ \ u|_{\mathbb{T}} = 0 \text{ near } \pm 1\}. \tag{3.1}$$

Note that for every $h = u + \mathrm{i}v \in \mathscr{H}$ we have $h(\pm 1) = 0$, $u(\bar{z}) = u(z)$ and $v(\bar{z}) = -v(z)$ for all $z \in \overline{\mathbb{D}}$; in particular, $v(x) = 0$ for all $x \in [-1, 1]$. Note that $\mathscr{H}$ is a nonclosed real vector subspace of $\mathscr{A}^\infty(\mathbb{D})$. Every function in $\mathscr{H}$ is uniquely determined by a smooth real function $u \in \mathscr{C}^\infty(\mathbb{T})$ supported away from the points ± 1 and satisfying $u(e^{\mathrm{i}t}) = u(e^{-\mathrm{i}t})$ for all $t \in \mathbb{R}$. Indeed, if $u \colon \overline{\mathbb{D}} \to \mathbb{R}$ is the harmonic extension of $u \colon \mathbb{T} \to \mathbb{R}$ and v is the harmonic conjugate of u determined by $v(0) = 0$, then the function $h = u + \mathrm{i}v$ belongs to $\mathscr{H}$ and is given by the classical integral formula

$$h(z) = T[u](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z}\, u(e^{\mathrm{i}\theta})\, d\theta, \quad z \in \mathbb{D}. \tag{3.2}$$

The real part of the integral operator on the right hand side above is the Poisson integral, while the imaginary part is the Hilbert (conjugate function) transform. We shall write

$$\mathscr{H}^\pm = \{h = u + \mathrm{i}v \in \mathscr{H} : \pm u \geq 0\}, \quad \mathscr{H}^\pm_* = \mathscr{H}^\pm \setminus \{0\}. \tag{3.3}$$

Note that the sets $\mathscr{H}^\pm$ and $\mathscr{H}^\pm_*$ are cones, i.e., closed under addition and multiplication by nonnegative (resp. positive) real numbers, and we have $\mathscr{H}^+ \cap \mathscr{H}^- = \{0\}$. Furthermore,

$$h = u + \mathrm{i}v \in \mathscr{H}^+_* \implies u > 0 \text{ on } \mathbb{D},\ \frac{\partial u}{\partial x}(-1) > 0,\ \frac{\partial u}{\partial x}(1) < 0 \tag{3.4}$$

where the last two inequalities follow from the Hopf lemma (since $u|_{\mathbb{T}} = 0$ near ± 1).

Let $\sigma\colon \mathbb{C}^2_* := \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ denote the projection onto the Riemann sphere whose fibres are complex lines through the origin.

Lemma 3.1. *Let $f = (f_1, f_2) : \overline{\mathbb{D}} \to \mathbb{C}^2_*$ be a map of class $\mathscr{A}^\infty(\mathbb{D})$ such that*

(a) $|f|^2 := |f_1|^2 + |f_2|^2 \le 1$ *on* $(-1, 1) = \mathbb{D} \cap \mathbb{R}$,
(b) $|f| > 1$ *on* $\mathbb{T} \setminus \{\pm 1\}$, *and*
(c) $\sigma \circ f\colon \overline{\mathbb{D}} \to \mathbb{CP}^1$ *is an immersion.*

Assume that $E \subset b\mathbb{B}^2$ is a compact smoothly embedded curve such that $f(\pm 1) \notin E$. Given a number $\eta \in (0, 1)$, there is a function $h \in \mathscr{H}^+_*$ arbitrarily close to 0 in $\mathscr{C}^1(\overline{\mathbb{D}})$ such that the immersion

$$f_h := e^{-h} f : \overline{\mathbb{D}} \to \mathbb{C}^2_* \tag{3.5}$$

satisfies the following conditions:

(1) $|f_h| < 1$ *on* $(-1, 1)$ *and* $|f_h| > (1 - \eta)|f| + \eta > 1$ *on* $\mathbb{T} \setminus \{\pm 1\}$,
(2) f_h *is transverse to* $b\mathbb{B}^2$, *and*
(3) $f_h(\overline{\mathbb{D}}) \cap E = \varnothing$.

Remark 3.2. Note that $\sigma \circ f_h = \sigma \circ f\colon \overline{\mathbb{D}} \to \mathbb{CP}^1$ is an immersion by the assumption, and hence f_h is an immersion. If f_h satisfies the conclusion of the lemma, then the set

$$C = \{z \in \overline{\mathbb{D}} : f_h(z) \in b\mathbb{B}^2\} \subset \mathbb{D} \cup \{\pm 1\}$$

is a smooth, closed, not necessarily connected curve containing the points ± 1, and its image $f_h(C) = f_h(\overline{\mathbb{D}}) \cap b\mathbb{B}^2$ is a smooth curve disjoint from E. Hence, C and $f_h(C)$ are finite unions of pairwise disjoint smooth Jordan curves. Each connected component of $f_h(C)$ bounds a connected component of the complex curve $f_h(\mathbb{D}) \cap \mathbb{B}^2$ (a properly immersed complex disc in $\mathbb{B}^2$). Since $|f_h| < 1$ on $(-1, 1)$, there is a component Ω of $\mathbb{D} \setminus C$ containing $(-1, 1)$, and $b\Omega \subset \mathbb{D} \cup \{\pm 1\}$ is a closed Jordan curve containing the points ± 1. This component Ω will be of main interest in the proof of Theorem 1.1. $\square$

Proof. Given $h = u + \mathrm{i}v \in \mathscr{H}$, we define the functions $\rho = \rho_0$ and ρ_h by

$$\rho = \log|f|\colon \overline{\mathbb{D}} \to \mathbb{R}, \quad \rho_h := \log|e^{-h} f| = -u + \rho : \overline{\mathbb{D}} \to \mathbb{R}. \tag{3.6}$$

Conditions (a) and (b) on f imply that

$$\rho \le 0 \text{ on } [-1, 1] = \overline{\mathbb{D}} \cap \mathbb{R}, \quad \rho(\pm 1) = 0, \quad \rho > 0 \text{ on } \mathbb{T} \setminus \{\pm 1\}. \tag{3.7}$$

It is obvious that for any function $h \in \mathscr{H}^+_*$ with sufficiently small $\mathscr{C}^0(\overline{\mathbb{D}})$ norm and such that the support of $u|_{\mathbb{T}} = \Re h|_{\mathbb{T}}$ avoids a certain fixed neighborhood of the points ± 1, the map f_h given by (3.5) satisfies condition (1) of the lemma. (Recall that $\Re h|_{\mathbb{T}}$ vanishes near the points ± 1 by the definition of the space $\mathscr{H}$.) In particular, for every fixed $h \in \mathscr{H}^+_*$ this holds for the map f_{th} for all small enough $t > 0$.

Note that the map f_h (3.5) intersects the sphere $b\mathbb{B}^2$ transversely if and only if 0 is a regular value of the function ρ_h (3.6). From (3.7) it follows that $\frac{\partial \rho}{\partial x}(-1) \le 0$ and $\frac{\partial \rho}{\partial x}(1) \ge 0$. Together with (3.4) we see that for every $h \in \mathscr{H}^+_*$ we have

$$\frac{\partial \rho_h}{\partial x}(-1) = \frac{\partial \rho}{\partial x}(-1) - \frac{\partial u}{\partial x}(-1) < 0, \quad \frac{\partial \rho_h}{\partial x}(1) > 0.$$

Replacing f by $e^{-h} f$ for some such h close to 0, we may assume that $\rho = \log|f|$ satisfies these conditions. Hence, there are discs $U^\pm \subset \mathbb{C}$ around the points ± 1, respectively, such that $d\rho \neq 0$ on $\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)$. Since this set is compact, it follows that for all $h \in \mathscr{H}$ with sufficiently small $\mathscr{C}^1(\overline{\mathbb{D}})$ norm we have that

$$d\rho_h \neq 0 \text{ on } \overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-). \tag{3.8}$$

Furthermore, since the curve E does not contain the points $f(\pm 1)$, we may choose the discs $U^\pm$ small enough such that

$$f_h\big(\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)\big) \cap E = \varnothing \tag{3.9}$$

holds for all $h \in \mathscr{H}$ sufficiently close to 0 in $\mathscr{C}^1(\overline{\mathbb{D}})$.

Recall that $\rho = \log|f| < 0$ on $(-1,1) = \mathbb{D} \cap \mathbb{R}$. Hence, there is an open set $U_0 \Subset \mathbb{D}$ containing the compact interval $(-1,1) \setminus (U^+ \cup U^-) \subset \mathbb{R}$ such that $\rho \leq -c < 0$ on $\overline{U}_0$ for some constant $c > 0$. Since $\rho > 0$ on $\mathbb{T} \setminus \{\pm 1\}$, a similar argument gives an open set $U_1 \Subset \mathbb{C}$ containing the compact set $\mathbb{T} \setminus (U^+ \cup U^-)$ (the union of two closed circular arcs) such that $\rho \geq c' > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$ for some $c' > 0$. It follows that for all $h \in \mathscr{H}$ sufficiently close to 0 in $\mathscr{C}^1(\overline{\mathbb{D}})$ we have that $\rho_h < 0$ on $\overline{U}_0$, $\rho_h > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$, and hence

$$f_h\big(\overline{\mathbb{D}} \cap (\overline{U}_0 \cup \overline{U}_1)\big) \cap b\mathbb{B}^2 = \varnothing. \tag{3.10}$$

For such h it follows in view of (3.8) that

$$\{z \in \overline{\mathbb{D}} : f_h(z) \in b\mathbb{B}^2,\ d\rho_h(z) = 0\} \subset K := \overline{\mathbb{D}} \setminus (U_0 \cup U_1 \cup U^+ \cup U^-). \tag{3.11}$$

Note that the set on the left hand side above is precisely the set of points in $\overline{\mathbb{D}}$ at which the map f fails to be transverse to the sphere $b\mathbb{B}^2$. The set K is compact and contained in $\mathbb{D} \setminus (-1,1)$. Pick $h = u + \mathrm{i}v \in \mathscr{H}_*^+$ and consider the family of functions $\rho_{th} = -tu + \rho$ for $t \in \mathbb{R}$. Since $\partial\rho_{th}/\partial t = -u < 0$ on $\mathbb{D}$, transversality theorem (see Abraham [1]) implies that for a generic choice of t, 0 is a regular value of the function $\rho_{th}|_{\mathbb{D}}$. By choosing $t > 0$ small enough and taking into account also (3.8) and (3.10), we infer that the map $f_{th} = e^{-th} f : \overline{\mathbb{D}} \to \mathbb{C}^2_*$ is transverse to $b\mathbb{B}^2$ (hence condition (2) holds) and it also satisfies condition (1). Replacing f by f_{th}, we may assume that f satisfies conditions (1) and (2).

It remains to achieve also condition (3) in the lemma. From (3.9) and (3.10) it follows that $\{z \in \overline{\mathbb{D}} : f_h(z) \in E\}$ is contained in the compact set $K \subset \mathbb{D}$ defined in (3.11). To conclude the proof, it suffices to find finitely many functions $h_1, \ldots, h_N \in \mathscr{H}_*^+$ such that, writing $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$, the family of maps

$$f_t(z) = \exp\Big(-\sum_{j=1}^N t_j h_j(z)\Big) f(z), \quad z \in \overline{\mathbb{D}} \tag{3.12}$$

satisfies $f_t(\overline{\mathbb{D}}) \cap E = \varnothing$ for a generic choice of $t \in \mathbb{R}^N$ near 0. By choosing $t = (t_1, \ldots, t_N) \in \mathbb{R}^N$ close enough to 0 and such that $t_j > 0$ for all $j = 1, \ldots, N$, the map f_t will also satisfy conditions (1) and (2) in the lemma, thereby completing the proof.

Claim: For every point $z \in \mathbb{D} \setminus (-1,1)$ there exists functions $h_1, h_2 \in \mathscr{H}^+$ such that the values $h_1(z),\ h_2(z) \in \mathbb{C}$ are $\mathbb{R}$-linearly independent.

Proof of the claim. Let δ_θ denote the probability measure on $\mathbb{T}$ representing the Dirac mass of the point $e^{\mathrm{i}\theta} \in \mathbb{T}$. Choose a sequence of smooth nonnegative even functions $u_j : \mathbb{T} \to \mathbb{R}_+$ supported near $\pm\mathrm{i}$ such that $u_j(e^{\mathrm{i}\theta}) = u_j(e^{-\mathrm{i}\theta})$ for all $\theta \in \mathbb{R}$ and

$$\lim_{j\to\infty} \frac{1}{2\pi} u_j d\theta = \delta_{\pi/2} + \delta_{-\pi/2}$$

as measures. From (3.2) we have that

$$\lim_{j\to\infty} T[u_j](z) = T[\delta_{\pi/2}+\delta_{-\pi/2}](z) = \frac{i+z}{i-z} + \frac{-i+z}{-i-z} = 2\frac{1-|z|^4-2\mathrm{i}\Im(z^2)}{|1+z^2|^2}. \tag{3.13}$$

The imaginary part of this expression vanishes precisely when $\Im(z^2)=0$ which is the union of the two coordinate axes. Since u_j is an even function, we see that $h_j = T[u_j]$ is real on the segment $J=\{\mathrm{i}y : y\in(-1,1)\} = \mathbb{D}\cap \mathrm{i}\mathbb{R}$, and it is nonvanishing at any given point $z_0=\mathrm{i}y_0\in J\backslash\{0\}$ for all big $j\in\mathbb{N}$ as follows from (3.13). Let $\phi_a(z)=(z-a)/(1-az)$ for $a\in(-1,1)$; this is a holomorphic automorphism of the disc which maps the interval $[-1,1]$ to itself, and if $a\neq 0$ then $\phi_a(J)\cap J=\varnothing$. Note that $\phi_a(\bar z)=\overline{\phi_a(z)}$ and $\phi_a(\pm 1)=\pm 1$; hence, the precomposition $h\mapsto h\circ\phi_a$ preserves the class $\mathscr{H}_*^+$. Choosing $a\in(-1,1)\backslash\{0\}$ we have that $\Im(h_j\circ\phi_a)(z_0)=\Im h_j(\phi_a(z_0))\neq 0$ for $j\in\mathbb{N}$ big enough as is seen from (3.13) and the fact that the point $\phi_a(z_0)\in\mathbb{D}$ does not lie in the union of the coordinate axis. This gives two functions in $\mathscr{H}_*^+$, namely h_j and $h_j\circ\phi_a$, whose values at the given point $z_0=\mathrm{i}y_0\in J\setminus\{0\}$ are $\mathbb{R}$-linearly independent. This establishes the claim for points in $J\setminus\{0\}$, and for other points in $\mathbb{D}\setminus(-1,1)$ we get the same conclusion by precomposing with an automorphisms ϕ_a, $a\in(-1,1)$. □

We continue with the proof of the lemma. Since the set K (3.11) is compact and contained in $\mathbb{D}\setminus(-1,1)$, the above claim yields functions $h_1,\ldots,h_N\in\mathscr{H}_*^+$ such that for every point $z\in K$ the vectors $h_j(z)\in\mathbb{C}$ $(j=1,\ldots,N)$ span $\mathbb{C}$ over $\mathbb{R}$. Consider the corresponding family of maps $f_t\colon\overline{\mathbb{D}}\to\mathbb{C}^2$ given by (3.12). Note that

$$\left.\frac{\partial f_t(z)}{\partial t_j}\right|_{t=0} = -h_j(z)f(z), \quad j=1,\ldots,l.$$

It follows that the map $\overline{\mathbb{D}}\times\mathbb{R}^N\ni(z,t)\mapsto f_t(z)\in\mathbb{C}^2$ is a submersion over the set K at $t=0$, and hence for all $t\in\mathbb{R}^N$ near 0. Indeed, we have $\sigma\circ f_t=\sigma\circ f\colon\overline{\mathbb{D}}\to\mathbb{CP}^1$ which is an immersion (and hence a submersion) by the assumption on f, while for each fixed $z\in K$ the partial differential $\partial_t f_t(z)|_{t=0}$ is surjective onto the radial direction $\mathbb{C}f(z)=\ker d\sigma_{f(z)}$ by the choice of the functions $h_1,\ldots,h_N$. Transversality theorem [1] implies that for a generic choice of $t\in\mathbb{R}^N$ near 0 the map $f_t|_K\colon K\to\mathbb{C}^2$ is transverse to the curve E, and hence it misses E by dimension reasons. Taking into account also (3.9) and (3.10) it follows that for any such t we have $f_t(\overline{\mathbb{D}})\cap E=\varnothing$. This completes the proof. □

4. A lemma on conformal mappings

The main results of this section are Lemmas 4.3 and 4.4 on the behaviour of biholomorphic maps from planar domains onto domains with exposed boundary points. These more precise versions of [8, Lemma 2.1] are used in the proof of Theorem 1.1.

Let $z=x+\mathrm{i}y$ denote the coordinate on $\mathbb{C}$. Consider the antiholomorphic involutions

$$\tau_x(x+\mathrm{i}y)=-x+\mathrm{i}y,\quad \tau_y(x+\mathrm{i}y)=x-\mathrm{i}y. \tag{4.1}$$

Note that $\tau_x\circ\tau_y=\tau_y\circ\tau_x$ is the reflection $z\mapsto -z$ across the origin $0\in\mathbb{C}$. The involutions τ_x,τ_y generate an abelian group

$$\Gamma=\langle\tau_x,\tau_y\rangle\cong\mathbb{Z}_2^2 \tag{4.2}$$

of order 4. A set $D\subset\mathbb{C}$ is said to be Γ-*invariant* if $\gamma(D)=D$ holds for all $\gamma\in\Gamma$. A map $\phi\colon D\to\mathbb{C}$ defined on a Γ-invariant set is said to be Γ-*equivariant* if

$$\phi=\gamma\circ\phi\circ\gamma \quad \text{holds for all } \gamma\in\Gamma.$$

Since every $\gamma \in \Gamma$ is an involution, this condition is equivalent to $\gamma \circ \phi = \phi \circ \gamma$. Note that a Γ-equivariant map $\phi\colon D \to \mathbb{C}$ takes $\mathbb{R} \cap D$ into $\mathbb{R}$ and $\mathrm{i}\mathbb{R} \cap D$ into $\mathrm{i}\mathbb{R} \cap D'$; in particular, $\phi(0) = 0$. For every map ϕ from a Γ-invariant domain, the map $\tilde{\phi} = \frac{1}{4}\sum_{\gamma\in\Gamma} \gamma \circ \phi \circ \gamma$ is Γ-equivariant.

Definition 4.1. A nonempty connected domain $D \subset \mathbb{C}$ is *special* if it is bounded with $\mathscr{C}^\infty$ smooth boundary, simply connected, and Γ-invariant.

It is easily seen that a special domain D intersects the real line in an interval $(-a, a)$ for some $a > 0$, and at the points $\pm a$ the boundary bD is tangent to the vertical line $x = \pm a$. This interval $(-a, a)$ will be called the *base* of D. The analogous observation holds for the intersection of D with the imaginary axis. Recall that a biholomorphism between a pair of bounded planar domains with smooth boundaries extends to a smooth diffeomorphism between their closures in view of the theorems by Carathéodory [5] and Kellogg [14]. We record the following observation.

Lemma 4.2. *Assume that D is a special domain (Def. 4.1) and $\phi\colon D \to D'$ is a biholomorphic map onto a bounded domain $D' = \phi(D)$ with smooth boundary satisfying*

$$\phi(0) = 0 \ \text{ and } \ \phi'(0) > 0. \tag{4.3}$$

Then ϕ is Γ-equivariant if and only if the domain D' is special. In particular, a special domain $D \subset \mathbb{C}$ with the base $(-a, a)$ admits a Γ-equivariant biholomorphism $\phi\colon \mathbb{D} \to D$ satisfying (4.3) *and $\phi(\pm 1) = \pm a$.*

Proof. Assume that D' is special. For every $\gamma \in \Gamma$ the map $\gamma \circ \phi \circ \gamma\colon D \to D'$ is then a well defined biholomorphism satisfying the normalization (4.3), so it equals ϕ. This shows that ϕ is Γ-equivariant. The converse is obvious. □

The following is the main result of this section.

Lemma 4.3. *Assume that $D \subset \mathbb{C}$ is a special domain with the base $(-a, a)$. Fix a number $b > a$ and set $I^+ = [a, b]$, $I^- = [-b, -a]$, and $I = I^+ \cup I^-$. Given an open neighborhood $V \subset \mathbb{C}$ of I, a number $\epsilon > 0$, and an integer $k \in \mathbb{Z}_+$ there exists a Γ-equivariant biholomorphism $\phi\colon D \to \phi(D) = D'$ onto a special domain D' with the base $(-b, b)$ satisfying the following conditions:*

(a) *$\phi(0) = 0$, $\phi'(0) > 0$, $\phi(\pm a) = \pm b$,*
(b) *$D \subset D' \subset D \cup V$ (hence $D \setminus V = D' \setminus V$), and*
(c) *$\|\phi - \mathrm{Id}\|_{\mathscr{C}^k(\overline{D}\setminus V)} < \epsilon$.*

Proof. We shall follow the proof of [8, Lemma 2.1] with certain refinements.

By using a biholomorphism $\mathbb{D} \to D$ furnished by Lemma 4.2 (which extends to a $\mathscr{C}^\infty$ diffeomorphism $\overline{\mathbb{D}} \to \overline{D}$), we see that it suffices to prove the result when $D = \mathbb{D}$ and hence $a = 1$. Choose a smaller open neighborhood $V_0 \subset \mathbb{C}$ of $I = I^+ \cup I^-$ such that $\overline{V}_0 \subset V$. Pick a number $\epsilon_0 \in (0, \epsilon)$; it precise value will be specified later. Fix a pair of small discs $U_0^+ \Subset U_1^+$ centered at the point $1 \in \mathbb{C}$, let $U_0^- \Subset U_1^-$ be the corresponding discs centered at the point -1 given by $U_j^- = \tau_x(U_j^+)$, and set $U_j = U_j^+ \cup U_j^-$ for $j = 0, 1$. (Here, τ_x and τ_y are the involutions (4.1).) We choose these discs small enough such that

$$U_1 \subset V_0 \Subset V. \tag{4.4}$$

Decreasing the number $\epsilon_0 > 0$ if necessary we may assume that

$$\mathrm{dist}(U_0, \mathbb{C} \setminus U_1) > \frac{\epsilon_0}{2}. \tag{4.5}$$

Fix an integer $n \in \mathbb{N}$ with $n > 1/(b-1)$ and let

$$I_n = [1, 1+1/n] \cup [-1-1/n, -1].$$

Recall that $I = [1, b] \cup [-b, -1]$. Choose a smooth $\mathbb{C}$-valued map θ_n on a neighborhood of the compact set

$$K_n = \big(1 + \frac{1}{2n}\big)\overline{\mathbb{D}} \cup I_n \tag{4.6}$$

which equals the identity on a neighborhood of the closed disc $(1+\frac{1}{2n})\overline{\mathbb{D}}$, maps the interval $[1, 1+\frac{1}{n}] \subset \mathbb{R}$ diffeomorphically onto the interval $[1, b] \subset \mathbb{R}$, and satisfies

$$\tau_x \circ \theta_n \circ \tau_x = \theta_n.$$

In particular, we have that

$$\theta_n(1 + 1/n) = b, \quad \theta_n(-1-1/n) = -b. \tag{4.7}$$

By Mergelyan's theorem [16] we can approximate θ_n as closely as desired in $\mathscr{C}^1(K_n)$ a polynomial map $\vartheta_n \colon \mathbb{C} \to \mathbb{C}$. (Mergelyan's theorem provides uniform approximation, but we can apply his result to the derivative of θ_n and integrate back in order to get $\mathscr{C}^1$ approximation.) Furthermore, we can achieve that ϑ_n satisfies the interpolation conditions (4.7) and also $\vartheta_n(\pm 1) = \pm 1$. Finally, replacing ϑ_n by $\frac{1}{4}\sum_{\gamma\in\Gamma} \gamma \circ \vartheta_n \circ \gamma$ we ensure that ϑ_n is Γ-equivariant. Assuming that ϑ_n is sufficiently close to θ_n in $\mathscr{C}^1(K_n)$, it follows that ϑ_n is biholomorphic in an open neighborhood of K_n (4.6). Since ϑ_n is Γ-equivariant, it maps the interval $[1, 1+1/n]$ diffeomorphically onto $[1, b]$ and maps $[-1-1/n, -1]$ diffeomorphically onto $[-b, -1]$. Furthermore, we may assume that

$$|\vartheta_n(z) - z| < \frac{\epsilon_0}{2} \quad \text{for all } \ z \in \big(1 + \frac{1}{2n}\big)\overline{\mathbb{D}} \ \text{ and } n > \frac{1}{b-1}, \tag{4.8}$$

and that the Γ-invariant domain

$$\Theta_n := \vartheta_n^{-1}(\mathbb{D}) \Subset \big(1 + \frac{1}{4n}\big)\mathbb{D} \tag{4.9}$$

is an arbitrarily small smooth perturbation of the disc $\mathbb{D}$, with $\pm 1 \in b\Theta_n$.

Pick an open neighborhood $W_n^+ \subset \mathbb{C}$ of the interval $[1, 1+1/n] \subset \mathbb{R}$ and set $W_n = W_n^+ \cup \tau_x(W_n^+)$. By choosing n big and W_n^+ small, we may assume that

$$W_n \subset U_0 \quad \text{and} \quad \vartheta_n(W_n) \subset V_0. \tag{4.10}$$

Let

$$\Omega_n = \Theta_n \cup R_n \tag{4.11}$$

be a special domain with the base $(-1-1/n, 1+1/n)$, obtained by attaching to Θ_n a thin Γ-invariant strip R_n around the arcs $[1, 1+1/n) \cup (-1-1/n, -1]$ such that $R_n \subset W_n$. Together with the second inclusion in (4.10) we get

$$\vartheta_n(R_n) \subset V_0. \tag{4.12}$$

The (unique) biholomorphic map

$$\psi_n \colon \mathbb{D} \to \Omega_n, \quad \psi_n(0) = 0, \ \ \psi_n'(0) > 0 \tag{4.13}$$

is Γ-equivariant and satisfies $\psi_n(\pm 1) = \pm(1 + 1/n)$ by Lemma 4.2. As $n \to \infty$, the domains $\Theta_n \subset \Omega_n$ converge to the disc $\mathbb{D}$ in the sense of Carathéodory (the *kernel* *convergence*, see [18, Theorem 1.8]), and their closures $\overline{\Theta}_n \subset \overline{\Omega}_n$ also converge to the closed disc $\overline{\mathbb{D}}$. It follows that the sequence of conformal diffeomorphisms $\psi_n \colon \overline{\mathbb{D}} \to \overline{\Omega}_n$ converges to the identity map uniformly on $\overline{\mathbb{D}}$ by Rado's theorem (see Pommerenke [18, Corollary 2.4, p. 22] or Goluzin [12, Theorem 2, p. 59]). In particular, we have that

$$|\psi_n(z) - z| < \frac{\epsilon_0}{2}, \quad z \in \overline{\mathbb{D}} \tag{4.14}$$

for all big enough $n \in \mathbb{N}$. We claim that, for such n, the Γ-invariant domain

$$D' = \vartheta_n(\Omega_n) \tag{4.15}$$

and the Γ-equivariant biholomorphic map

$$\phi = \vartheta_n \circ \psi_n \colon \mathbb{D} \to D' \tag{4.16}$$

satisfy the conclusion of the lemma provided that the number $\epsilon_0 > 0$ is chosen small enough. Indeed, condition (a) holds by the construction. We now verify condition (b). Firstly, by (4.9), (4.11), and (4.13) we have that

$$\mathbb{D} = \vartheta_n(\Theta_n) \subset \vartheta_n(\Omega_n) = \vartheta_n(\psi_n(\mathbb{D})) = \phi(\mathbb{D}) = D'.$$

Assume now that $w \in D' \setminus V_0$. By (4.15) we have $w = \vartheta_n(\zeta)$ for some $\zeta \in \Omega_n = \Theta_n \cup R_n$. Since $\vartheta_n(R_n) \subset V_0$ by (4.12) while $w \notin V_0$, we have $\zeta \in \Theta_n$. By (4.9) it follows that $w = \vartheta_n(\zeta) \in \mathbb{D}$. This shows that

$$D' \setminus V_0 = \mathbb{D} \setminus V_0 \tag{4.17}$$

and hence establishes condition (b) with V_0 in place of V (and hence also for V).

Finally we verify condition (c). Assume that $z \in \mathbb{D} \setminus U_1$. Conditions (4.5) and (4.14) imply $\psi_n(z) \in \Omega_n \setminus U_0$. Since $\Omega_n = \Theta_n \cup R_n$ by (4.11) and $R_n \subset W_n \subset U_0$ by (4.10), we infer in view of (4.9) that $\Omega_n \setminus U_0 \subset \Theta_n \subset \big(1 + \frac{1}{4n}\big)\mathbb{D}$ and hence

$$\psi_n(z) \in \big(1 + \frac{1}{4n}\big)\overline{\mathbb{D}} \quad \text{for all} \ \ z \in \overline{\mathbb{D}} \setminus U_1.$$

By (4.8), (4.14), and (4.16) we conclude that

$$|\phi(z) - z| \le |\vartheta_n(\psi_n(z)) - \psi_n(z)| + |\psi_n(z) - z| < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0, \quad z \in \overline{\mathbb{D}} \setminus U_1. \tag{4.18}$$

Since $V_0 \subset V$ and $\mathbb{D} \setminus V_0 \subset \mathbb{D} \setminus U_1$ by (4.4), this establishes condition (c) for $k = 0$.

To complete the proof, we will show that the $\mathscr{C}^k$ estimates of $\phi - \mathrm{Id}$ on $\overline{\mathbb{D}} \setminus V$ follow from the uniform estimate (4.18) on $\mathbb{D} \setminus V_0$ in view of the Cauchy estimates and the reflection principle. Pick a compact arc $J \subset b\mathbb{D} \setminus \overline{V}_0$ such that $J \setminus V$ lies in the relative interior of J. Choose an open neighborhood $E \subset \mathbb{C}$ of J which is invariant with respect to the antiholomorphic reflection $\tau(z) = 1/\bar{z}$ around the circle $b\mathbb{D}$ and such that $\overline{E} \cap \overline{V}_0 = \varnothing$. By decreasing $\epsilon_0 > 0$ if necessary we may assume that $\epsilon_0 < \mathrm{dist}(\overline{E}, \overline{V}_0)$. It follows from (4.18) that for every $z \in E \cap \overline{\mathbb{D}}$ we have $|\phi(z) - z| < \epsilon_0$ and hence $\phi(z) \in \overline{\mathbb{D}} \setminus V_0$. In view of (4.17) it follows in particular that

$$\phi(E \cap b\mathbb{D}) \subset b\mathbb{D} \setminus V_0. \tag{4.19}$$

We extend ϕ to $\overline{\mathbb{D}} \cup E$ by setting

$$\phi(w) = \tau \circ \phi \circ \tau(w), \quad w \in E \setminus \mathbb{D}.$$

Since τ fixes the circle bD pointwise, the extended map agrees with ϕ on $E \cap b\mathbb{D}$ in view of (4.19). Since $\tau(w) \in E \cap \overline{\mathbb{D}}$ and hence $\phi \circ \tau(w) \in \overline{\mathbb{D}} \setminus V_0$ by what was said above, the extended map $\phi \colon \overline{\mathbb{D}} \cup E \to \mathbb{C}$ satisfies the estimate $|\phi(z) - z| < C\epsilon_0$ for all $z \in (\overline{\mathbb{D}} \setminus V_0) \cup E$, where the constant $C > 1$ depends only on the distortion caused by the reflection τ on

E. Since the compact set $\overline{\mathbb{D}} \setminus V$ is contained in the open set $\Omega = (\mathbb{D} \setminus \overline{V}_0) \cup E$, the Cauchy estimates give $\|\phi - \mathrm{Id}\|_{\mathscr{C}^k(\overline{\mathbb{D}} \setminus V)} \leq C' \|\phi - \mathrm{Id}\|_{\mathscr{C}^0(\Omega)} < C' C \epsilon_0$ for some constant C' depending only on k and $\mathrm{dist}(\overline{\mathbb{D}} \setminus V, \mathbb{C} \setminus \Omega)$. Choosing $\epsilon_0 > 0$ small enough (which amounts to choosing the index $n \in \mathbb{N}$ in the definition of the domain D' (4.15) and the map $\phi\colon \mathbb{D} \to D'$ (4.16) big enough), this is $< \epsilon$. The proof is complete. $\square$

The same proof gives the following lemma which we record for future applications.

Lemma 4.4. *Let $b > 1$. Given an open neighborhood $V \subset \mathbb{C}$ of the interval $I = [1, b] \subset \mathbb{R}$ and numbers $\epsilon > 0$ and $k \in \mathbb{Z}_+$, there exists a biholomorphism $\phi\colon \mathbb{D} \to \phi(\mathbb{D}) = D'$ onto a smoothly bounded domain D' satisfying the following conditions:*

(a) $\phi(0) = 0$, $\phi'(0) > 0$, $\phi(1) = b$, $\phi(x - \mathrm{i}y) = \overline{\phi(x + \mathrm{i}y)}$,
(b) $\mathbb{D} \subset D' \subset \mathbb{D} \cup V$, *and*
(c) $\|\phi - \mathrm{Id}\|_{\mathscr{C}^k(\overline{D} \setminus V)} < \epsilon$.

Remark 4.5. Condition (a) implies that $D' \supset \mathbb{D} \cup [1, b)$. We believe that the approximation condition (c) is an automatic consequence of thinness of the attached strip $D' \setminus \mathbb{D}$.

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. For simplicity of notation we focus on the case $n = 2$ which is of main interest, although the proof holds for any $n \geq 2$.

Let $z = x + \mathrm{i}y$ denote the coordinate on $\mathbb{C}$. Let Γ denote the group (4.2). Given a positive continuous even function $g > 0$ on $\mathbb{R}$, we let $S_g \subset \mathbb{C}$ denote the Γ-invariant strip

$$S_g = \{x + \mathrm{i}y : x \in \mathbb{R},\ \ |y| < g(x)\}. \tag{5.1}$$

Let $f = (f_1, f_2)\colon \mathbb{R} \hookrightarrow b\mathbb{B}^2$ be a real analytic complex tangential embedding with dense image, furnished by Proposition 1.3. By complexification, f extends to a holomorphic immersion $f : \overline{S}_{g_0} \hookrightarrow \mathbb{C}^2$ for some g_0 as above. Fix g_0 and write $S = S_{g_0}$. Since the function $|f|^2 = |f_1|^2 + |f_2|^2$ is strongly subharmonic on S and constantly equal to 1 on $\mathbb{R}$, we have $|f(x + \mathrm{i}y)| \geq 1 + c(x)|y|^2$ for a positive smooth function $c\colon \mathbb{R} \to (0, \infty)$ (see e.g. [4] for the details). Hence, if the strip S is chosen thin enough then

$$1 \leq |f|^2 = |f_1|^2 + |f_2|^2 < 2 \ \text{ on } \overline{S}, \tag{5.2}$$

and $|f| = 1$ holds precisely on $\mathbb{R}$. This means that the immersed complex curve $f(\overline{S}) \subset \mathbb{C}^2$ touches the sphere $b\mathbb{B}^2$ tangentially along $f(\mathbb{R})$ and satisfies

$$f(\overline{S} \setminus \mathbb{R}) \subset \sqrt{2}\mathbb{B}^2 \setminus \overline{\mathbb{B}}^2. \tag{5.3}$$

Lemma 5.1. *Given $\epsilon > 0$, there is a strip S_g of the form* (5.1)*, with $0 < g(x) < g_0(x)$ for all $x \in \mathbb{R}$, such that $f\colon \overline{S}_g \to \mathbb{C}^2$ is and injective immersion and*

$$\mathrm{Area}(f(S_g)) = \int_{S_g} |f'|^2 dxdy < \epsilon.$$

Proof. Consider a double sequence $b_j > 0$ ($j \in \mathbb{Z}$) such that

$$0 < b_j < \min\{g_0(x) : j - 1 \leq x \leq j\}, \quad j \in \mathbb{Z}.$$

It follows that the rectangle

$$P_j = \{z = x + \mathrm{i}y \in \mathbb{C} : j - 1 \leq x \leq j,\ |y| < b_j\} \Subset S$$

is compactly contained in S. We claim that the sequence b_j can be chosen such that f is injective on the union $\bigcup_{j\in\mathbb{Z}} P_j$. Indeed, assume that the numbers b_j for $j = 0, \pm 1, \ldots, \pm k$ have already been chosen such that f is an injective immersion on the set $Q_k = \bigcup_{|j|\le k} P_j$. In view of (5.3) if follows that f is an injective immersion on $Q_k \cup \mathbb{R}$; hence it is an injective immersion in an open neighborhood of the compact set $Q_k \cup [-k-1, k+1]$. Therefore we can choose the constants $b_{k+1} > 0$ and $b_{-k-1} > 0$ small enough such that f is also an injective immersion on Q_{k+1}, and hence the induction may proceed. Finally, choosing a positive even continuous function $g > 0$ on $\mathbb{R}$ satisfying

$$\max\{g(x) : j-1 \le x \le j\} < b_j, \quad j \in \mathbb{Z},$$

it follows that $S_g \subset \bigcup_{j\in\mathbb{Z}} P_j = \bigcup_{k\in\mathbb{N}} Q_k$ and hence f is injective on S_g. By decreasing g if necessary we can clearly achieve that the area of the disc $f(S_g)$ is as small as desired. □

Replacing the function g_0 by g and the initial strip $S = S_{g_0}$ by the strip S_g furnished by Lemma 5.1, we shall assume that

$$f\colon S \hookrightarrow \mathbb{C}^2 \text{ is an injective holomorphic immersion.} \tag{5.4}$$

Recall that $\sigma\colon \mathbb{C}^2_* \to \mathbb{CP}^1$ denotes the canonical projection onto the Riemann sphere. At each point $z = (z_1, z_2) \in b\mathbb{B}^2$ the complex line $\xi_z \subset T_z\mathbb{C}^2$ tangent to $b\mathbb{B}^2$ is transverse to the line $\mathbb{C}z = \sigma^{-1}(\sigma(z)) \cup \{0\}$, and hence $d\sigma_z\colon \xi_z \to T_{\sigma(z)}\mathbb{CP}^1$ is an isomorphism. Since the immersed curve $f\colon \mathbb{R} \to b\mathbb{B}^2$ satisfies $\dot f(t) \in \xi_{f(t)}$ for every $t \in \mathbb{R}$, it follows that $\sigma \circ f\colon \mathbb{R} \to \mathbb{CP}^1$ is an immersion. Hence, if the strip S is chosen thin enough then

$$\sigma \circ f : S \to \mathbb{CP}^1 \ \text{ is an immersion.} \tag{5.5}$$

We shall frequently use the following observation.

Lemma 5.2. *Let D be a relatively compact domain in $\mathbb{C}$ and $F\colon \overline{D} \to \mathbb{C}^2_*$ be an injective immersion of class $\mathscr{A}^1(D)$ such that $\sigma \circ F\colon \overline{D} \to \mathbb{CP}^1$ is an immersion. If $h \in H^\infty(D)$ is sufficiently small in the sup-norm, then $e^{-h}F\colon D \to \mathbb{C}^2_*$ is an injective immersion.*

Proof. Let $c = \sup\{|F(z)| : z \in \overline{D}\} > 0$. Consider the set

$$\Delta = \big\{(z,w) \in \overline{D} \times \overline{D} : \sigma \circ F(z) = \sigma \circ F(w)\big\} = \Delta_0 \cup \Delta'$$

where $\Delta_0 = \{(z,z) : z \in \overline{D}\}$ and $\Delta' = \Delta \setminus \Delta_0$. Since $\sigma \circ F\colon \overline{D} \to \mathbb{CP}^1$ is an immersion, it is locally an embedding, and hence the set Δ' is compact. Since $F\colon \overline{D} \to \mathbb{C}^2_*$ is injective, it follows that

$$\delta := \inf\big\{|F(z) - F(w)| : (z,w) \in \Delta'\big\} > 0.$$

Choose $\mu > 0$ such that $|e^\zeta - 1| < \delta/3c$ when $|\zeta| < \mu$. Assuming that $|h(z)| < \mu$ for all $z \in D$ and taking into account that $|F| < c$ on $\overline{D}$, we have for every $(z,w) \in \Delta'$ that

$$\begin{aligned} \big|e^{-h(z)}F(z) - e^{-h(w)}F(w)\big| \ &\ge\ |F(z) - F(w)| - \\ &\quad -\big|e^{-h(z)}F(z) - F(z)\big| - \big|e^{-h(w)}F(w) - F(w)\big| \\ &\ge\ \delta - c\delta/3c - c\delta/3c = \delta/3 > 0. \end{aligned}$$

This shows that the map $e^{-h}F\colon D \to \mathbb{C}^2$ is injective. Since $\sigma \circ (e^{-h}F) = \sigma \circ F$ is an immersion by the assumption, $e^{-h}F$ is also an immersion. □

We shall find an embedded disc in $\mathbb{B}^2$ satisfying Theorem 1.1 by pulling the embedded strip $f(S) \subset \mathbb{C}^2_*$ slightly into the ball along the curve $f(\mathbb{R}) \subset b\mathbb{B}^2$, where the amount of pulling decreases fast enough as we go to infinity inside the strip. When doing so, we shall pay special attention to ensure injectivity; this is a fairly delicate task since the curve $f(\mathbb{R}) = f(S) \cap b\mathbb{B}^2$ is dense in $b\mathbb{B}^2$. To this end, we shall be considering smoothly bounded, simply connected, Γ-invariant domains $D \subset \mathbb{C}$ satisfying

$$\mathbb{R} \subset D \subset \overline{D} \subset S. \tag{5.6}$$

Assume that $h = u + \mathrm{i}v \in \mathscr{A}^1(D)$ satisfies

$$u > 0 \text{ on } \mathbb{R} \quad \text{and} \quad e^{2u} < |f|^2 \text{ on } bD. \tag{5.7}$$

Consider the map F of class $\mathscr{A}^1(D)$ defined by

$$F = (F_1, F_2) = e^{-h}(f_1, f_2) : \overline{D} \to \mathbb{C}^2_*. \tag{5.8}$$

On the real axis we have that $|F|^2 = e^{-2u}|f|^2 < |f|^2 = 1$, while on the boundary bD we have that $|F|^2 = e^{-2u}|f|^2 > 1$ in view of (5.7). This means that

$$F(\mathbb{R}) \subset \mathbb{B}^2 \quad \text{and} \quad F(bD) \cap \overline{\mathbb{B}}^2 = \varnothing. \tag{5.9}$$

Assume in addition that F is transverse to the sphere $b\mathbb{B}^2$. Let

$$\Omega \subset \{z \in D : F(z) \in \mathbb{B}^2\} \tag{5.10}$$

denote the connected component of the set on the right hand side containing $\mathbb{R}$. It follows that $\overline{\Omega} \subset D$ and $F|_\Omega \colon \Omega \to \mathbb{B}^2$ is a proper holomorphic map extending smoothly to $\overline{\Omega}$ and mapping $b\Omega$ to the sphere $b\mathbb{B}^2$. Clearly, Ω is Runge in $\mathbb{C}$ and hence conformally equivalent to the disc; indeed, there is a biholomorphism $\mathbb{D} \to \Omega$ extending smoothly to $\overline{\mathbb{D}} \setminus \{\pm 1\}$. The proof of Theorem 1.1 is concluded by the following lemma.

Lemma 5.3. *Given $\epsilon > 0$, there exist a smoothly bounded, simply connected, Γ-invariant domain $D \subset \mathbb{C}$ satisfying* (5.6) *and a function $h = u + \mathrm{i}v \in \mathscr{A}^1(D)$ satisfying* (5.7) *such that the map $F = e^{-h} f \colon \overline{D} \to \mathbb{C}^2_*$ is an injective immersion transverse to $b\mathbb{B}^2$ satisfying*

- (α) $\mathrm{Area}(F(\Omega)) < \epsilon$, *where Ω is defined by* (5.10)*, and*
- (β) *the curve $F(b\Omega) \subset b\mathbb{B}^2$ is everywhere dense in the sphere $b\mathbb{B}^2$.*

Proof. We begin by explaining the scheme of proof.

We shall construct an increasing sequence of special domains $D_1 \subset D_2 \subset D_3 \subset \cdots \subset S$ (see Def. 4.1) whose union $D = \bigcup_{j=1}^\infty D_j$ is a simply connected, smoothly bounded, Γ-invariant domain satisfying (5.6). The first domain D_1 is a round disc centered at 0; by rescaling the coordinate on $\mathbb{C}$ we may assume that $D_1 = \mathbb{D}$ is the unit disc. For every $n \in \mathbb{N}$ we let $D_{n+1} = D_n \cup S_n$ be a special domain with the base $(-n-1, n+1) \subset \mathbb{R}$, furnished by Lemma 4.3. (Recall that S_n is a thin strip around the interval $(-n-1, n+1)$.) For each $n \geq 2$ let ψ_n be the biholomorphism

$$\psi_n \colon D_n \to D_{n-1}, \quad \psi_n(0) = 0, \;\; \psi_n'(0) > 0.$$

By Lemma 4.2, ψ_n extends to a smooth Γ-equivariant diffeomorphism $\psi_n \colon \overline{D}_n \to \overline{D}_{n-1}$ satisfying $\psi_n([-n, n]) = [-n+1, n-1]$ and $\psi_n(\pm n) = \pm(n-1)$. Set

$$\Psi_1 = \mathrm{Id}|_{\overline{D}_1}, \qquad \Psi_n = \psi_2 \circ \cdots \psi_n \colon \overline{D}_n \to \overline{D}_1 \quad \forall n = 2, 3, \ldots. \tag{5.11}$$

At the same time, we shall find a sequence of multipliers $h_n \in \mathscr{A}^\infty(D_n)$ $(n \in \mathbb{N})$ of the form $h_n = \tilde h_n \circ \Psi_n$, with $\tilde h_n \in \mathscr{H}^+_*$ (see (3.3)), such that the map

$$F_n = e^{-h_n} f \colon \overline{D}_n \hookrightarrow \mathbb{C}^2_*$$

is an embedding of class $\mathscr{A}^\infty(D_n)$ that is transverse to $b\mathbb{B}^2$, and F_{n+1} approximates F_n as closely as desired in $\mathscr{C}^0(\overline{D}_n)$ and in $\mathscr{C}^1(\overline{D}_n \setminus U_n)$, where $U_n = U_n^+ \cup U_n^-$ is a small neighborhood of the points $\pm n$ for every $n \in \mathbb{N}$. In the induction step, we shall use Lemma 3.1 in order to find a small perturbation of F_n such that $F_n(\overline{D}_n)$ intersects the pair of arcs $E_n^+ = f([n, n+1]) \subset b\mathbb{B}^2$ and $E_n^- = f([-n-1, -n]) \subset b\mathbb{B}^2$ only at the points $f(\pm n)$. This will allow us to construct the next map $F_{n+1} = e^{-h_{n+1}} f \colon \overline{D}_{n+1} \hookrightarrow \mathbb{C}^2_*$ which is an embedding mapping the strip $\overline{D_{n+1} \setminus D_n}$ into a small neighborhood of the arcs $E_n^+ \cup E_n^-$. The sequence h_n will be chosen such that it converges to a function $h = u + \mathrm{i}v = \tilde h \circ \Psi \in \mathscr{A}^1(D)$ satisfying (5.7), with $\tilde h = \lim_{n\to\infty} \tilde h_n \in \mathscr{A}^1(D_1)$. Furthermore, we will ensure that the limit map $F = \lim_{n\to\infty} F_n = e^{-h} f \colon \overline{D} \to \mathbb{C}^2_*$ (which satisfies (5.9) in view of (5.7)) is an injective immersion that is transverse to the sphere $b\mathbb{B}^2$. The domain Ω (5.10) will then satisfy the conclusion of the lemma, and $F(\Omega) \subset \mathbb{B}^2$ will be a properly embedded holomorphic disc satisfying Theorem 1.1.

We now turn to the details. Recall that $|f|^2 > 1$ on $S \setminus \mathbb{R}$. We begin by choosing a function $h_1 = u_1 + \mathrm{i}v_1 \in \mathscr{H}^+_*$ on $\overline{D}_1 = \overline{\mathbb{D}}$, close to 0 in $\mathscr{C}^1(\overline{D}_1)$, such that

$$e^{2u_1} < \frac{1}{2}\left(|f|^2 + 1\right) \quad \text{on } bD_1 \setminus \{\pm 1\} \tag{5.12}$$

and the map $F_1 = e^{-h_1} f \colon \overline{D}_1 \to \mathbb{C}^2_*$ of class $\mathscr{A}^\infty(D_1)$ is an embedding (see Lemma 5.2) which is transverse to $b\mathbb{B}^2$ (see Lemma 3.1). From (5.12) we infer that $|F_1| = e^{-u_1}|f| > 1$ on $bD_1 \setminus \{\pm 1\}$. Recall that $|F_1| < 1$ on $(-1,1)$ since $u_1 > 0$ on D_1 and $|f| = 1$ on $\mathbb{R}$. Let

$$C_1 = \{z \in \overline{D}_1 : F_1(z) \in b\mathbb{B}^2\}, \quad \Gamma_1 = F_1(C_1) = F_1(\overline{D}_1) \cap b\mathbb{B}^2.$$

We have that $C_1 \subset D_1 \cup \{\pm 1\}$, each of the sets C_1 and Γ_1 is a union of finitely many smooth closed Jordan curves, and Γ_1 bounds the embedded complex curve $F_1(D_1) \cap \mathbb{B}^2$ (see Remark 3.2). By [7] the curve Γ_1 is transverse to the distribution $\xi \subset T(b\mathbb{B}^2)$ of complex tangent planes, and hence the ξ-Legendrian embedding $f \colon \mathbb{R} \hookrightarrow b\mathbb{B}^2$ is not tangent to Γ_1 at the points $f(\pm 1) \in \Gamma_1$. Thus, there is a number $0 < \delta < 1$ such that

$$f\big([1-\delta, 1+\delta] \cup [-1-\delta, -1+\delta]\big) \cap \Gamma_1 = \{f(1), f(-1)\}.$$

Applying Lemma 3.1 with the smooth compact curve

$$E = f([-2, -1-\delta] \cup [1+\delta, 2]) \subset b\mathbb{B}^2 \setminus \Gamma_1$$

we can approximate $h_1 \in \mathscr{H}^+_*$ as closely as desired in the $\mathscr{C}^1(\overline{D}_1)$ norm by a function $\tilde h_1 \in \mathscr{H}^+_*$ such that, after redefining the map F_1 by setting $F_1 = e^{-\tilde h_1} f$ and also redefining the curves C_1 and Γ_1 accordingly, the above conditions still hold and in addition we have

$$f([1,2] \cup [-2,-1]) \cap \Gamma_1 = f([1,2] \cup [-2,-1]) \cap F_1(\overline{D}_1) = \{f(1), f(-1)\}. \tag{5.13}$$

Write $\tilde h_1 = \tilde u_1 + \mathrm{i}\tilde v_1$. This completes the initial step.

We now explain how to obtain the next embedding $F_2 \colon \overline{D}_2 \hookrightarrow \mathbb{C}^2_*$. This is the first step of the induction, and all subsequent steps will be of the same kind.

Choose a compact set $M \subset S$ containing $\overline{D}_1 \cup [-2, +2]$ in the interior. Pick a small number $\mu = \mu_1 > 0$. By (the proof of) Lemma 5.2 we may decrease $\mu > 0$ if necessary such that for any domain $D' \subset M$ and function $h \in \mathscr{A}(D')$ satisfying $|h| < \mu$ on $\overline{D'}$ the map $e^{-h}f\colon \overline{D'} \to \mathbb{C}^2_*$ is injective. Since $\tilde u_1$ vanishes on $bD_1 = \mathbb{T}$ near ± 1 by the definition of the class $\mathscr{H}$, there are small discs $U_1^\pm$ around the points ± 1 such that

$$\tilde u_1 \text{ vanishes on } bD_1 \cap U_1^\pm. \tag{5.14}$$

Furthermore, since $\tilde h_1(\pm 1) = 0$, we can shrink the discs $U_1^\pm$ if necessary to get

$$|\tilde h_1| < \mu \ \text{ on } \ \overline{D}_1 \cap (\overline{U}_1^+ \cup \overline{U}_1^-)). \tag{5.15}$$

Choose a pair of smaller open discs $W_1^\pm \Subset V_1^\pm \Subset U_1^\pm$ around the points ± 1. Lemma 4.3 furnishes a special domain D_2 with the base $(-2,+2)$ satisfying $D_1 \subset D_2 \subset M$ and a Γ-equivariant conformal diffeomorphism $\psi_2 = \Psi_2\colon \overline{D}_2 \to \overline{D}_1$ with $\psi_2(0) = 0$ and $\psi_2'(0) > 0$. (By the construction, D_2 is the union of D_1 and an arbitrarily thin Γ-invariant strip S_2 around the interval $(-2,2) \subset \mathbb{R}$, with $\pm 2 \in bD_2$.) We may choose D_2 such that the attaching set $\overline{D_2 \cap bD_1}$ is contained in $W_1^+ \cup W_1^-$. Consider the pair of compact sets

$$K = \overline{D}_1 \setminus (W_1^+ \cup W_1^-), \quad L = \overline{D_2 \setminus D_1} \cup (\overline{D}_1 \cap (\overline{V}_1^+ \cup \overline{V}_1^-)). \tag{5.16}$$

Note that

$$K \cup L = \overline{D}_2, \quad K \cap L = \overline{D}_2 \cap \big((\overline{V}_1^+ \setminus W_1^+) \cup (\overline{V}_1^- \setminus W_1^-)\big).$$

By Lemma 4.3, the domain D_2 can be chosen such that Ψ_2 is as close as desired to the identity map in $\mathscr{C}^1(K)$ and

$$\Psi_2(L) \subset U_1^+ \cup U_1^-. \tag{5.17}$$

Set

$$h_2 = u_2 + \mathrm{i} v_2 := \tilde h_1 \circ \Psi_2 \in \mathscr{A}^\infty(D_2). \tag{5.18}$$

Note that $u_2 > 0$ on D_2, u_2 vanishes on $bD_2 \setminus \overline{D}_1$ by (5.14) and (5.17), $h_2(\pm 2) = 0$, and

$$|h_2| < \mu \quad \text{on } \overline{D_2 \setminus D_1} \tag{5.19}$$

which follows from (5.15), (5.16), and (5.17). Assuming that the approximations are close enough, we see from (5.12) that

$$e^{2u_2} < \frac{1}{2}\big(|f|^2 + 1\big) \ \text{ on } bD_2 \setminus \{\pm 1\}. \tag{5.20}$$

We claim that the immersion

$$F_2 = e^{-h_2} f\colon \overline{D}_2 \to \mathbb{C}^2_*$$

of class $\mathscr{A}^\infty(D_2)$ is injective provided that the approximations are close enough. Indeed, F_2 is injective on L by the choice of the constant $\mu > 0$, the estimate (5.15), the inclusion (5.17), and the definition (5.18) of h_2. Assuming as we may that Ψ_2 is close enough to the identity on K (see Lemma 4.3), the function $h_2|_K$ is so close to $\tilde h_1|_K$ that F_2 is injective on K in view of Lemma 5.2. To obtain injectivity of F_2 on $\overline{D}_2$, it remains to see that

$$F_2(\overline{L \setminus K}) \cap F_2(\overline{K \setminus L}) = \varnothing.$$

Note that F_2 maps $\overline{L \setminus K}$ into a small neighborhood of the two arcs $f([1,2] \cup [-2,-1])$. Since these arcs intersects $F_1(\overline{D}_1)$ only at the points $F_1(\pm 1) = f(\pm 1) \in \Gamma_1$ (cf. (5.13)) and F_2 can be chosen as close as desired to F_1 on the set K which does not contain the points ± 1, the claim follows.

By a slight adjustment of $\tilde h_1$ (and hence of h_2, see (5.18)), keeping the above conditions, we may assume that the embedding $F_2\colon \overline{D}_2 \hookrightarrow \mathbb{C}^2$ is transverse to $b\mathbb{B}^2$ (see Lemma 3.1). Each of the sets $C_2 = \{z \in \overline{D}_2 : F_2(z) \in b\mathbb{B}^2\}$ and $\Gamma_2 = F_2(C_2) = F_2(\overline{D}_2) \cap b\mathbb{B}^2$ is then a finite unions of smooth Jordan curves. By another application of Lemma 3.1 we can find a $\mathscr{C}^1(\overline D_1)$-small deformation $\tilde h_2 \in \mathscr{H}_*^+$ of $\tilde h_1$ such that, redefining h_2 by setting $h_2 = \tilde h_2 \circ \Psi_2 \in \mathscr{A}^\infty(D_2)$ and adjusting the map F_2 accordingly, the above conditions remain valid and in addition we have that

$$f([-3,-2]\cup[2,3]) \cap F_2(\overline D_2) = \{f(2), f(-2)\}.$$

This complete the first step of the induction, and we are now ready to apply the same arguments to the map F_2 and the domain D_2 to find the next embedding $F_3\colon \overline D_3 \hookrightarrow \mathbb{C}^2_*$.

Clearly this construction can be continued inductively. It yields

- an increasing sequence of special domains $D_1 \subset D_2 \subset D_3 \subset \cdots$ such that D_n has the base $(-n,n)$, $D_{n+1} = D_n \cup S_n$ where S_n is a thin strip around the interval $(-n-1, n+1)$, and the union $\overline D = \bigcup_{n=1}^\infty \overline D_n \subset S$ is a smoothly bounded, simply connected, Γ-invariant domain,
- a sequence $U_n = U_n^- \cup U_n^+$ of small pairwise disjoint neighborhood of the points $\{-n,n\}$ such that $\overline D_{n-1} \cap \overline U_n = \varnothing$ for every $n = 2,3,\ldots$;
- a sequence of Γ-equivariant diffeomorphisms $\Psi_n = \psi_2 \circ \cdots \psi_n\colon \overline D_n \to \overline D_1$ of class $\mathscr{A}^\infty(D_n)$, where $\Psi_1 = \mathrm{Id}|_{\overline D_1}$ and $\psi_n\colon \overline D_n \to \overline D_{n-1}$ for $n \ge 2$ (see (5.11)),
- a sequence of multipliers

$$h_n = \tilde h_n \circ \Psi_n \in \mathscr{A}^\infty(D_n) \quad \text{with } \tilde h_n \in \mathscr{H}_*^+, \tag{5.21}$$

- a sequence of embeddings $F_n = e^{-h_n} f\colon \overline D_n \hookrightarrow \mathbb{C}^2_*$ of class $\mathscr{A}^\infty(D_n)$,

such that the following conditions hold for every $n \in \mathbb{N}$:

(a$_n$) the conformal diffeomorphism $\psi_{n+1}\colon \overline D_{n+1} \to \overline D_n$ is arbitrarily close to the identity in $\mathscr{C}^1(\overline D_n \setminus U_n)$ and satisfies $\psi_{n+1}(\overline{D_{n+1}\setminus D_n}) \subset U_n$;

(b$_n$) the function $h_n = u_n + \mathrm{i}v_n$ (5.21) satisfies $u_n > 0$ on D_n and $e^{2u_n} < \frac12(|f|^2+1)$ on $bD_n \setminus \{\pm n\}$ (see (5.20));

(c$_n$) $\tilde h_{n+1}$ approximates $\tilde h_n$ as closely as desired in $\mathscr{C}^1(\overline D_1)$;

(d$_n$) h_{n+1} approximates h_n as closely as desired in $\mathscr{C}^0(\overline D_n)$ and in $\mathscr{C}^1(\overline D_n \setminus U_n)$,

(e$_n$) $|h_{n+1}|$ is as small as desired uniformly on $\overline{D_{n+1}\setminus D_n}$ (see (5.19));

(f$_n$) the map F_{n+1} approximates F_n as closely as desired uniformly on $\overline D_n$ and in $\mathscr{C}^1(\overline D_n \setminus U_n)$, and F_{n+1} is as close as desired to f uniformly on $\overline{D_{n+1}\setminus D_n}$.

Note that (f$_n$) is a consequence of (a$_n$), (c$_n$), (d$_n$), and (e$_n$).

Assuming that these approximations are close enough at every step, we can draw the following conclusions. The sequence $\Psi_n\colon \overline D_n \to \overline D_1$ converges to the Γ-equivariant biholomorphic map $\Psi\colon D = \bigcup_{n=1}^\infty D_n \to \mathbb{D}$ with $\Psi(0) = 0$ and $\Psi'(0) > 0$. Note that $\Psi(\mathbb{R}) = (-1,1)$, and Ψ extends to a $\mathscr{C}^\infty$ diffeomorphism $\overline D \to \overline{\mathbb{D}} \setminus \{\pm 1\}$. Secondly, the sequence $h_n \in \mathscr{A}^\infty(D_n)$ converges in the weak $\mathscr{C}^1(\overline D)$ topology (i.e., in the $\mathscr{C}^1$ topology on every compact subset of $\overline D$) to a function

$$h = u + \mathrm{i}v = \tilde h \circ \Psi \in \mathscr{A}^1(D) \quad \text{where } \tilde h = \lim_{n\to\infty} \tilde h_n \in \mathscr{A}^1(D_1) \tag{5.22}$$

(the second limit $\tilde h$ exists in the $\mathscr{C}^1(\overline D_1)$ topology in view of condition (c$_n$)) satisfying

$$u > 0 \text{ on } D, \quad e^{2u} \le \frac12\left(|f|^2+1\right) < |f|^2 \text{ on } bD. \tag{5.23}$$

Thirdly, the sequence of embeddings $F_n = e^{-h_n} f\colon \overline{D}_n \hookrightarrow \mathbb{C}^2_*$ converges in the weak $\mathscr{C}^1(\overline{D})$ topology to the map $F = e^{-h} f\colon \overline{D} \to \mathbb{C}^2$ of class $\mathscr{A}^1(D)$. Since $\sigma \circ F = \sigma \circ f\colon \overline{D} \to \mathbb{CP}^1$ is an immersion, F is an immersion on $\overline{D}$. Lemma 5.2 shows that F is injective (hence an embedding) on every domain $\overline{D}_n$, and therefore also on $\overline{D}$, provided that the approximation of F_n by F_{n+1} is close enough in $\mathscr{C}^0(\overline{D}_n)$ for every $n \in \mathbb{N}$ (see (f_n)). Note that conditions (5.7) holds in view of (5.23), and hence F satisfies condition (5.9). It follows that the simply connected domain Ω (5.10) (with $\mathbb{R} \subset \Omega \subset \overline{\Omega} \subset D$) is well defined, and the restricted map $F|_\Omega\colon \Omega \hookrightarrow \mathbb{B}^2$ is a properly embedded holomorphic disc. Since every map $F_n\colon \overline{D}_n \to \mathbb{C}^2$ is transverse to $b\mathbb{B}^2$, the same is true for $F\colon \overline{D} \to \mathbb{C}^2$ provided the approximation is close enough at every step.

It remains to show conditions (α) and (β) in the lemma.

Given a sequence $\epsilon_1 > \epsilon_2 > \cdots > 0$ with $\lim_{n\to\infty} \epsilon_n = 0$, conditions ($\mathrm{d}_n$) and ($\mathrm{e}_n$) show that the sequence h_n (5.21) can be chosen such that its limit h (5.22) satisfies

$$|h| < \epsilon_n \quad \text{on} \ \ \overline{D_{n+1} \setminus D_n} \ \ \text{for all} \ \ n \in \mathbb{N}.$$

This implies that $F(b\Omega) \subset b\mathbb{B}^2$ consists of a pair of curves in the sphere which are as close as desired to the curve $f(\mathbb{R}) \subset b\mathbb{B}^2$ in the fine $\mathscr{C}^0$ topology. Since $f(\mathbb{R})$ is dense in $b\mathbb{B}^2$, the same is true for $F(b\Omega)$ provided the approximation is close enough, so condition (β) holds. (This argument is similar to the one in [10, proof of Theorem VI.1].)

It remains to estimate the area of the disc $F(\Omega) \subset \mathbb{B}^2$. Set

$$c = \max\left\{|\tilde{h}'(z)| : z \in \overline{D}_1\right\} \tag{5.24}$$

where $\tilde{h} \in \mathscr{A}^1(D_1)$ is as in (5.22). Note that $c > 0$ can be made as small as desired by choosing all terms of the sequence $\tilde{h}_n \in \mathscr{H}^+_*$ small enough in the $\mathscr{C}^1(\overline{D}_1)$ norm. Recall that $h = \tilde{h} \circ \Psi$ (see (5.22)). We have $h'(z) = \tilde{h}'(\Psi(z))\Psi'(z)$ and hence $|h'(z)|^2 \le c^2|\Psi'(z)|^2$ for $z \in D$. Differentiation of $F = e^{-h} f$ gives $F' = -e^{-h}h'f + e^{-h}f'$. Note that $u > 0$ and hence $e^{-u} < 1$ on D. Recall also that $|f|^2 \le 2$ on S (see (5.2)). Using the inequality $|a+b|^2 \le 2(|a|^2 + |b|^2)$, which holds for any $a, b \in \mathbb{C}^n$, we thus obtain

$$\begin{aligned} \mathrm{Area}(F(\Omega)) &= \int_\Omega |F'|^2 dxdy \le 2\int_\Omega e^{-2u}|h'|^2|f|^2 dxdy + 2\int_\Omega e^{-2u}|f'|^2 dxdy \\ &\le 4c^2 \int_\Omega |\Psi'(z)|^2 dxdy + 2\int_\Omega |f'|^2 dxdy \\ &= 4c^2 \mathrm{Area}(\Psi(\Omega)) + 2\mathrm{Area}(f(\Omega)). \end{aligned}$$

Since $\Psi(\Omega) \subset \Psi(D) = \mathbb{D}$, we have $\mathrm{Area}(\Psi(\Omega)) \le \pi$, and hence the first term is bounded by $\epsilon/2$ if $c > 0$ is small enough. Lemma 5.1 shows that the second term can be chosen $< \epsilon/2$. This completes the proof of Lemma 5.3, and hence of Theorem 1.1. □

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