Complete densely embedded complex lines in $\mathbb{C}^2$

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Abstract In this paper we construct a complete injective holomorphic immersion $\mathbb{C} \to \mathbb{C}^2$ whose image is dense in $\mathbb{C}^2$. The analogous result is obtained for any closed complex submanifold $X \subset \mathbb{C}^n$ for $n > 1$ in place of $\mathbb{C} \subset \mathbb{C}^2$. We also show that, if X intersects the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ and K is a connected compact subset of $X \cap \mathbb{B}^n$, then there is a Runge domain $\Omega \subset X$ containing K which admits a complete injective holomorphic immersion $\Omega \to \mathbb{B}^n$ whose image is dense in $\mathbb{B}^n$.

Keywords complete complex submanifold, holomorphic embedding

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1. Introduction

A smooth immersion $f\colon X \to \mathbb{R}^n$ from a smooth manifold X to the Euclidean space $\mathbb{R}^n$ is said to be *complete* if the image of every divergent path in X has infinite length in $\mathbb{R}^n$; equivalently, if the metric $f^*(ds^2)$ on X induced by the Euclidean metric ds^2 on $\mathbb{R}^n$ is complete. An injective immersion will be called an *embedding*. If X is an open Riemann surface, $n \geq 3$, and $f\colon X \to \mathbb{R}^n$ is a conformal immersion, then it parametrizes a minimal surface in $\mathbb{R}^n$ if and only if it is a harmonic map.

A seminal result of Colding and Minicozzi [9, Corollary 0.13] states that a complete embedded minimal surface of finite topology in $\mathbb{R}^3$ is necessarily proper in $\mathbb{R}^3$; this was extended to surfaces of finite genus and countably many ends by Meeks, Pérez, and Ros [18]. This is no longer true for complex curves in $\mathbb{C}^2$ (a special case of minimal surfaces in $\mathbb{R}^4$). Indeed, there exist complete embedded complex curves in $\mathbb{C}^2$ with arbitrary topology which are bounded and hence non-proper (see [3]; the case of finite topology was previously shown in [4]). Furthermore, every relatively compact domain in $\mathbb{C}$ admits a complete non-proper holomorphic embedding into $\mathbb{C}^2$ (see [2, Corollary 4.7]). Since all examples in the cited sources are normalized by open Riemann surfaces of *hyperbolic* type (i.e., carrying non-constant negative subharmonic functions; see e. g. [10, p. 179]), one is led to wonder whether hyperbolicity plays a role in this context. The purpose of this note is to show that it actually does not. The following is our first main result.

Theorem 1.1. *Given a closed complex submanifold X of $\mathbb{C}^n$ for some $n > 1$, there exists a complete holomorphic embedding $f\colon X \to \mathbb{C}^n$ such that $f(X)$ contains any given countable subset of $\mathbb{C}^n$. In particular, $f(X)$ can be made dense in $\mathbb{C}^n$.*

By *dense* we shall always mean *everywhere dense*. Note that if $f(X)$ is dense in $\mathbb{C}^n$ then $f\colon X \to \mathbb{C}^n$ is non-proper. Taking $n = 2$ and $X = \mathbb{C}$ gives the following corollary.

Corollary 1.2. *There is a complete embedded complex line $\mathbb{C} \to \mathbb{C}^2$ with a dense image.*

Corollary 1.2 also holds if $\mathbb{C}$ is replaced by any open Riemann surface admitting a proper holomorphic embedding into $\mathbb{C}^2$. There are many open *parabolic* (i.e., non-hyperbolic) Riemann surfaces enjoying this condition; it is however not known whether all open Riemann surfaces do. For a survey of this classical embedding problem we refer to Sections 8.9 and 8.10 in [12] and the paper [15]. Without taking care of injectivity, every open Riemann surface admits complete dense holomorphic immersions into $\mathbb{C}^n$ for any $n \geq 2$ and complete dense conformal minimal immersions into $\mathbb{R}^n$ for $n \geq 3$ (see [1]).

These results provide additional evidence that there is much more room for conformal minimal surfaces (even those given by holomorphic maps) in $\mathbb{R}^4 = \mathbb{C}^2$ than in $\mathbb{R}^3$. We point out that it is quite easy to find injective holomorphic immersions $\mathbb{C} \to \mathbb{C}^2$ which are neither complete nor proper. For example, if $a > 0$ is irrational then the map $\mathbb{C} \ni z \mapsto (e^z, e^{az}) \in \mathbb{C}^2$ is an injective immersion, but the image of the negative real axis is a curve of finite length in $\mathbb{C}^2$ terminating at the origin. On the other hand, it is an open problem whether a conformal minimal embedding $\mathbb{C} \to \mathbb{R}^3$ is necessarily proper; see [11, Conjecture 1.2].

To prove Theorem 1.1, we use an idea from the recent paper by Alarcón, Globevnik, and López [4]. The construction relies on two ingredients. First, in any spherical shell in $\mathbb{C}^n$ one can find a compact polynomially convex set L, consisting of finitely many pairwise disjoint balls contained in affine real hyperplanes, such that any curve traversing this shell and avoiding L has length bigger than a prescribed constant. For a suitable choice of L with this property it is then possible to find a holomorphic automorphism of $\mathbb{C}^n$ which pushes a given complex submanifold $X \subset \mathbb{C}^n$ off L. The construction of such an automorphism uses the main result of the *Andersén-Lempert theory*. In [4] this construction was used to show that every closed complex submanifold $X \subset \mathbb{C}^n$ contains a bounded Runge domain Ω admitting a proper complete holomorphic embedding into the unit ball of $\mathbb{C}^n$; furthermore, Ω can be chosen to contain any given compact subset of X. Clearly, such Ω carries nonconstant negative plurisubharmonic functions and is Kobayashi hyperbolic, so in general one cannot map all of X into the ball. We choose instead a sequence of automorphisms which converges uniformly on compacts in X to a complete holomorphic embedding $X \hookrightarrow \mathbb{C}^n$ whose image contains a prescribed countable set of points in $\mathbb{C}^n$.

It is natural to ask whether the analogue of Theorem 1.1 holds for more general target manifolds in place of $\mathbb{C}^n$. Since our proof relies on the Andersén-Lempert theory which holds on any Stein manifold Y enjoying Varolin's *density property* (the latter meaning that every holomorphic vector field on Y can be approximated uniformly on compacts by Lie combinations of $\mathbb{C}$-complete holomorphic vector fields; see Varolin [19] or [12, Sec. 4.10]), the following is a reasonable conjecture.

Conjecture 1.3. Assume that Y is a Stein manifold with the density property. Choose a complete Riemannian metric $\mathfrak{g}$ on Y.

(a) If $\dim Y \geq 3$ then there exists a $\mathfrak{g}$-complete holomorphic embedding $\mathbb{C} \to Y$ with a dense image.
(b) More generally, if X is a Stein manifold, $\dim X < \dim Y$, and there is a proper holomorphic embedding $X \hookrightarrow Y$, then there exists a $\mathfrak{g}$-complete injective holomorphic immersion $X \to Y$ with a dense image.

It was recently shown in [7] that, if X and Y are as in assertion (b) above and satisfy $2\dim X + 1 \leq \dim Y$, then there exists a proper (hence complete) holomorphic embedding

$X \hookrightarrow Y$. Thus, for such dimensions we are just asking whether *proper* can be replaced by *dense*, keeping completeness.

It is known that for any $n > 1$ the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ contains complete properly embedded complex hypersurfaces (see [6, 16, 4] and the references therein); this settles in an optimal way a problem posed by Yang in 1977 about the existence of complete bounded complex submanifolds of $\mathbb{C}^n$ (see [20, 21]). Moreover, given a discrete subset $\Lambda \subset \mathbb{B}^2$ there are complete properly embedded complex curves in $\mathbb{B}^2$ containing Λ (see [17] for discs and [3] for examples with arbitrary topology). It remained and open problem whether $\mathbb{B}^n$ also admits complete densely embedded complex submanifolds. Our second main result gives an affirmative answer to this question.

Theorem 1.4. *Let X be a closed complex submanifold of $\mathbb{C}^n$ for some $n > 1$ such that $X \cap \mathbb{B}^n \neq \varnothing$. Given a connected compact subset $K \subset X \cap \mathbb{B}^n$, there are a pseudoconvex Runge domain $\Omega \subset X$ containing K and a complete holomorphic embedding $f \colon \Omega \to \mathbb{B}^n$ whose image $f(\Omega)$ contains any given countable subset of $\mathbb{B}^n$. In particular, $f(\Omega)$ can be made dense in $\mathbb{B}^n$.*

As above, if $f(\Omega) \subset \mathbb{B}^n$ is dense then the map $f \colon \Omega \to \mathbb{B}^n$ is non-proper. Taking $n = 2$ and $X = \mathbb{D} := \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ we obtain the following corollary.

Corollary 1.5. *There is a complete embedded complex disc $\mathbb{D} \to \mathbb{B}^2$ with a dense image.*

More generally, it follows from Theorem 1.4 that in $\mathbb{B}^2$ there are complete embedded complex curves with arbitrary finite topology and containing any given countable subset. (See Corollary 3.1.) Without taking care of injectivity, given an arbitrary domain (i.e., a connected open subset) D in $\mathbb{C}^n$ ($n \geq 2$), on each open connected orientable smooth surface there is a complex structure such that the resulting open Riemann surface admits complete dense holomorphic immersions into D; moreover, every bordered Riemann surface carries a complete holomorphic immersion into D with dense image (see [1]). The analogous results for conformal minimal immersions into any domain in $\mathbb{R}^n$ ($n \geq 3$) also hold (see [1]).

The proof of Theorem 1.4 uses arguments similar to those in the proof of Theorem 1.1, but with an additional ingredient to keep the image of the embedding f inside the ball.

Notation. Given a closed complex submanifold X of $\mathbb{C}^n$ ($n > 1$), a compact set $K \subset X$, and a map $f = (f_1, \ldots, f_n) \colon X \to \mathbb{C}^n$, we write $\|f\|_K = \sup\{|f(x)| : x \in K\}$ where $|f|^2 = \sum_{j=1}^n |f_j|^2$. Denote by ds^2 the Euclidean metric on $\mathbb{C}^n$. Given an immersion $f \colon X \to \mathbb{C}^n$, we denote by $\mathrm{dist}_f(x, y)$ the distance between points $x, y \in X$ in the metric $f^*(ds^2)$ on X. If $K \subset L$ are compact subsets of X, we set

$$\mathrm{dist}_f(K, X \setminus L) = \inf\{\mathrm{dist}_f(x, y) : x \in K,\ y \in X \setminus L\}. \tag{1.1}$$

2. Proof of Theorem 1.1

Let X be a closed complex submanifold of $\mathbb{C}^n$ for some $n > 1$ and let $A = \{a_j\}_{j \in \mathbb{N}}$ be any countable subset of $\mathbb{C}^n$. Pick a compact $\mathscr{O}(X)$-convex set $K_0 \subset X$ and a number $0 < \epsilon_0 < 1$. Let f_0 denote the inclusion map $X \hookrightarrow \mathbb{C}^n$. In order to prove Theorem 1.1, we shall inductively construct the following:

(a) an exhaustion of X by an increasing sequence of compact $\mathscr{O}(X)$-convex sets

$$K_1 \subset K_2' \subset K_2 \subset K_3' \subset K_3 \subset \cdots \subset \bigcup_{i=1}^{\infty} K_i = X$$

such that $K_{i-1} \subset \mathring{K}_i'$ and $K_i' \subset \mathring{K}_i$ hold for all $i \in \mathbb{N}$,

(b) a sequence of proper holomorphic embeddings $f_i \colon X \hookrightarrow \mathbb{C}^n$ $(i \in \mathbb{N})$,

(c) a discrete sequence of points $\{b_i\}_{i\in\mathbb{N}} \subset X$ with $b_i \in K_i$ for every $i \in \mathbb{N}$, and

(d) a decreasing sequence of numbers $\epsilon_i > 0$,

such that the following conditions hold for every $i \in \mathbb{N}$:

(i) $\|f_i - f_{i-1}\|_{K_{i-1}} < \epsilon_{i-1}$,

(ii) $a_j = f_i(b_j) \in f_i(K_i)$ for $j = 1, \ldots, i$ and $f_i(b_j) = f_{i-1}(b_j)$ for $j = 1, \ldots, i-1$,

(iii) $\mathrm{dist}_{f_i}(K_{i-1}, X \setminus K_i') > 1/\epsilon_{i-1}$ (see (1.1)),

(iv) $0 < \epsilon_i < \epsilon_{i-1}/2$,

(v) if $g \colon X \to \mathbb{C}^n$ is a holomorphic map such that $\|g - f_i\|_{K_i} < 2\epsilon_i$, then g is an injective immersion on K_{i-1} and $\mathrm{dist}_g(K_{i-1}, X \setminus K_i) > 1/(2\epsilon_{i-1})$.

Assume for a moment that sequences with these properties exist. Conditions (a) and (iv) ensure that the sequence f_i converges uniformly on compacts in X to a holomorphic map $f = \lim_{i\to\infty} f_i \colon X \to \mathbb{C}^n$. By (i) and (iv) we have for every $i \in \mathbb{N}$ that

$$\|f - f_i\|_{K_i} \le \sum_{k=i}^{\infty} \|f_{k+1} - f_k\|_{K_i} < \sum_{k=i}^{\infty} \epsilon_k < 2\epsilon_i.$$

Hence condition (v) implies that f is an injective immersion on K_{i-1} and

$$\mathrm{dist}_f(K_{i-1}, X \setminus K_i) > 1/(2\epsilon_{i-1}).$$

Since this holds for every $i \in \mathbb{N}$ and $\sum_i 1/\epsilon_i = +\infty$, it follows that $f \colon X \to \mathbb{C}^n$ is a complete injective immersion. Finally, condition (ii) implies that $f(X)$ contains the set $A = \{a_j\}_{j\in\mathbb{N}}$. This completes the proof.

Let us now explain the induction. We shall frequently use the well known fact that if $g \colon X \hookrightarrow \mathbb{C}^n$ is a proper holomorphic embedding and $K \subset X$ is a compact $\mathscr{O}(X)$-convex set, then the set $g(K) \subset \mathbb{C}^n$ is polynomially convex.

Assume that for some $i \in \mathbb{N}$ we have found maps f_j, sets $K_j' \subset K_j$ and numbers ϵ_j satisfying the stated conditions for $j = 0, \ldots, i-1$. The next map f_i will be of the form $f_i = \Phi \circ f_{i-1}$ for some holomorphic automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$ which will be found in two steps,

$$\Phi = \phi \circ \theta \quad \text{with} \quad \phi, \theta \in \mathrm{Aut}(\mathbb{C}^n).$$

Let $\mathbb{B} = \mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$. Choose a number $r > 0$ such that

$$f_{i-1}(K_{i-1}) \subset r\mathbb{B},$$

and then pick numbers $R > r'$ with $r' > r$. In the open spherical shell $S = R\mathbb{B} \setminus r'\overline{\mathbb{B}}$ we choose a labyrinth $L = \bigcup_{k=1}^{\infty} L_k$ of the type constructed in [4, Theorem 2.5], i.e., every set L_k is a ball in an affine real hyperplane $\Lambda_k \subset \mathbb{C}^n$ such that these balls are pairwise disjoint, the set $\widetilde{L}_k = \bigcup_{j=1}^{k} L_j$ is contained in an open half-space determined by Λ_{k+1} for every $k \in \mathbb{N}$, and any path $\lambda \colon [0,1) \to R\mathbb{B} \setminus L$ with $\lambda(0) \in r'\overline{\mathbb{B}}$ and $\lim_{t\to 1} |\lambda(t)| = R$ has infinite Euclidean length. (Alternatively, we may use a labyrinth of the type constructed by

Globevnik in [16, Corollary 2.2].) It follows that $\widetilde{L}_k \cap r'\overline{\mathbb{B}} = \varnothing$ and $\widetilde{L}_k \cup r'\overline{\mathbb{B}}$ is polynomially convex for every $k \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$ big enough such that every path $\lambda\colon [0,1] \to \mathbb{C}^n \setminus \widetilde{L}_{k_0}$ with $\lambda(0) \in r'\overline{\mathbb{B}}$ and $\lambda(1) \in \mathbb{C}^n \setminus R\,\mathbb{B}$ has length bigger than $1/\epsilon_{i-1}$. Choose a holomorphic automorphism $\theta \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying the following conditions:

(I) $|\theta(f_{i-1}(x)) - f_{i-1}(x)| < \min\{\epsilon_{i-1}/2, r' - r\}$ for all $x \in K_{i-1}$,

(II) $\theta(a_j) = a_j$ for $j = 1, \ldots, i-1$ (note that $a_j = f_{i-1}(b_j) \in f_{i-1}(K_{i-1})$ for $j = 1, \ldots, i-1$),

(III) $a_i \notin \theta(f_{i-1}(X))$, and

(IV) $\theta(f_{i-1}(X)) \cap \widetilde{L}_{k_0} = \varnothing$.

Such θ is found by an application of the Andersén-Lempert theory as explained in [4, Proofs of Lemma 3.1 and Theorem 1.6], using the fact that the set $f_{i-1}(K_{i-1}) \cup \widetilde{L}_{k_0}$ is polynomially convex (since $f_{i-1}(K_{i-1}) \subset r\overline{\mathbb{B}}$ and $r\overline{\mathbb{B}} \cup \widetilde{L}_{k_0}$ is polynomially convex). The explicit result used in their proof is [14, Theorem 2.1] which is also available in [12, Theorem 4.12.1].

Consider the proper holomorphic embedding $g_i = \theta \circ f_{i-1}\colon X \hookrightarrow \mathbb{C}^n$. The compact set

$$K_i' = \{x \in X : |g_i(x)| \le R + 1\}$$

is $\mathscr{O}(X)$-convex and contains K_{i-1} in its interior. By condition (I) we have $g_i(K_{i-1}) \subset r'\mathbb{B}$, and hence condition (IV) and the choice of k_0 imply

$$\mathrm{dist}_{g_i}(K_{i-1}, X \setminus K_i') > 1/\epsilon_{i-1}.$$

Choose a point $b_i \in X \setminus K_i'$. The set $K_i' \cup \{b_i\}$ is then $\mathscr{O}(X)$-convex, and hence its image $g_i(K_i') \cup \{g_i(b_i)\} \subset g_i(X) \subset \mathbb{C}^n$ is polynomially convex. By the Andersén-Lempert theorem (see [14, Theorem 2.1] or [12, Theorem 4.12.1]) we can find an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ which approximates the identity map as closely as desired on $g_i(K_i')$, it fixes each of the points $a_1, \ldots, a_{i-1} \in g_i(K_{i-1})$, and it satisfies $\phi(g_i(b_i)) = a_i$. If the approximation is close enough, then the proper holomorphic embedding

$$f_i = \phi \circ g_i = \phi \circ \theta \circ f_{i-1}\colon X \hookrightarrow \mathbb{C}^n$$

satisfies conditions (i), (ii) and (iii) for the index i. Indeed, (i) and (ii) are obvious, and (iii) follows by observing that

$$f_i(K_{i-1}) \subset r'\mathbb{B}, \quad f_i(bK_i') \subset \mathbb{C}^n \setminus R\,\overline{\mathbb{B}}, \quad \text{and } f_i(K_i') \cap \widetilde{L}_{k_0} = \varnothing$$

provided that ϕ approximates the identity sufficiently closely on $g_i(K_i')$. Thus, any path in X starting in K_{i-1} and ending in $X \setminus K_i'$ is mapped by f_i to a path in $\mathbb{C}^n \setminus \widetilde{L}_{k_0}$ starting in $r'\mathbb{B}$ and ending in $\mathbb{C}^n \setminus R\,\overline{\mathbb{B}}$, hence its length is bigger than $1/\epsilon_{i-1}$ by the choice of $\widetilde{L}_{k_0}$.

We now choose a compact $\mathscr{O}(X)$-convex set $K_i \subset X$ containing $K_i' \cup \{b_i\}$ in its interior. Furthermore, K_i can be chosen as big as desired, thereby ensuring that the sequence of these sets will exhaust X. By choosing $\epsilon_i > 0$ small enough we obtain conditions (iv) and (v). Indeed, since the sets $K_{i-1} \subset K_i'$ are contained in the interior of K_i, uniform approximation on K_i gives approximation in the $\mathscr{C}^1$-norm on K_i' by the Cauchy estimates.

This finishes the induction step and hence completes the proof of Theorem 1.1.

3. Proof of Theorem 1.4 and Corollary 1.5

We begin with the proof of Theorem 1.4.

Let X be a closed complex submanifold of $\mathbb{C}^n$ for some $n > 1$, and let $f_0 \colon X \hookrightarrow \mathbb{C}^n$ denote the inclusion map. Let $K \subset X \cap \mathbb{B}^n$ be a connected compact subset, and let $A = \{a_j\}_{j\in\mathbb{N}}$ be a countable subset of $\mathbb{B}^n$. Pick a compact connected $\mathscr{O}(X)$-convex set $K_0 \subset X \cap \mathbb{B}^n$ containing K and a number $0 < \epsilon_0 < 1$. Similarly to what has been done in the proof of Theorem 1.1, we shall inductively construct the following:

(a) an increasing sequence of connected compact $\mathscr{O}(X)$-convex subsets of X,
$$K_1 \subset K_2' \subset K_2 \subset K_3' \subset K_3 \subset \cdots$$
such that $K_{i-1} \subset \mathring{K}_i'$ and $K_i' \subset \mathring{K}_i \subset X$ hold for all $i \in \mathbb{N}$,

(b) a sequence of proper holomorphic embeddings $f_i \colon X \hookrightarrow \mathbb{C}^n$ $(i \in \mathbb{N})$,

(c) a sequence $(b_i)_{i\in\mathbb{N}} \subset X$ without repetition such that $b_i \in K_i$ for every $i \in \mathbb{N}$, and

(d) a decreasing sequence of numbers $\epsilon_i > 0$,

such that the following conditions hold for every $i \in \mathbb{N}$:

(i) $\|f_i - f_{i-1}\|_{K_{i-1}} < \epsilon_{i-1}$,

(ii) $a_j = f_i(b_j) \in f_i(K_i)$ for $j = 1, \ldots, i$ and $f_i(b_j) = f_{i-1}(b_j)$ for $j = 1, \ldots, i-1$,

(iii) $\mathrm{dist}_{f_i}(K_{i-1}, X \setminus K_i') > 1/\epsilon_{i-1}$ (see (1.1)),

(iv) $0 < \epsilon_i < \epsilon_{i-1}/2$,

(v) if $g \colon X \to \mathbb{C}^n$ is a holomorphic map such that $\|g - f_i\|_{K_i} < 2\epsilon_i$, then g is an injective immersion on K_{i-1} and $\mathrm{dist}_g(K_{i-1}, X \setminus K_i) > 1/(2\epsilon_{i-1})$, and

(vi) $f_i(K_i) \subset \mathbb{B}^n$.

The main novelty with respect to the the proof of Theorem 1.1 is condition (vi) which implies that the connected domain

$$\Omega = \bigcup_{i=1}^{\infty} K_i \subset X \tag{3.1}$$

may be a proper subset of X. Note that Ω is pseudoconvex and Runge in X since each set K_i is $\mathscr{O}(X)$-convex. Granted these conditions, we see as in the proof of Theorem 1.1 that the limit map $f := \lim_{i\to\infty} f_i \colon \Omega \to \mathbb{C}^n$ exists and is a complete holomorphic embedding whose image $f(\Omega)$ contains the countable set A; moreover, we have $f(\Omega) \subset \mathbb{B}^n$ in view of (vi). Thus, to complete the proof of Theorem 1.4 it remains to establish the induction.

For the inductive step we assume that for some $i \in \mathbb{N}$ we have already found maps f_j, sets $K_j' \subset K_j$, and numbers $\epsilon_j > 0$ satisfying the stated conditions for $j = 0, \ldots, i-1$. (This is vacuous for $i = 1$.) The next map f_i will be obtained in two steps, each obtained by a composition with a suitably chosen holomorphic automorphism of $\mathbb{C}^n$.

Write $\mathbb{B} = \mathbb{B}^n$. By compactness of the set K_{i-1} and property (vi) for the index $i-1$ there is a number $0 < r < 1$ such that

$$f_{i-1}(K_{i-1}) \subset r\mathbb{B}. \tag{3.2}$$

Pick a number $R \in (r, 1)$. Let $L = \bigcup_{k=1}^{\infty} L_k \subset R\mathbb{B} \setminus r\overline{\mathbb{B}}$ be a labyrinth as in the proof of Theorem 1.1. Set $\widetilde{L}_k = \bigcup_{j=1}^{k} L_k$ for all $k \in \mathbb{N}$. Pick $k_0 \in \mathbb{N}$ such that the length of any path $\lambda \colon [0,1] \to \mathbb{C}^n \setminus \widetilde{L}_{k_0}$ with $|\lambda(0)| = r$ and $|\lambda(1)| = R$ is bigger than $1/\epsilon_{i-1}$. Reasoning as in the proof of Theorem 1.1, we find a holomorphic automorphism $\theta \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying

(I) $|\theta(f_{i-1}(x)) - f_{i-1}(x)| < \epsilon_{i-1}/2$ for all $x \in K_{i-1}$,

(II) $\theta(a_j) = a_j$ for $j = 1, \ldots, i-1$,

(III) $a_i \notin \theta(f_{i-1}(X))$, and

(IV) $\theta(f_{i-1}(X)) \cap \widetilde{L}_{k_0} = \varnothing$.

Moreover, by (3.2) we may choose θ close enough to the identity on $f_{i-1}(K_{i-1})$ so that

(V) $g_i(K_{i-1}) \subset r\mathbb{B}$, where $g_i := \theta \circ f_{i-1} \colon X \hookrightarrow \mathbb{C}^n$.

Since g_i is a proper holomorphic embedding, there is a connected compact $\mathscr{O}(X)$-convex set $K_i' \subset X$ such that $K_{i-1} \subset \mathring{K}_i'$ and

$$g_i(bK_i') \subset \mathbb{B} \setminus R\overline{\mathbb{B}}. \tag{3.3}$$

(For example, fixing a number $\rho \in (R, 1)$, we may choose K_i' such that $g_i(K_i')$ is the connected component of the set $g_i(X) \cap \rho\overline{\mathbb{B}}$ which contains $g_i(K_{i-1})$.) Properties (3.2), (IV), (V), and (3.3) ensure that

$$\mathrm{dist}_{g_i}(K_{i-1}, X \setminus K_i') > 1/\epsilon_{i-1}. \tag{3.4}$$

Let U be the connected component of $\mathbb{B} \cap g_i(X)$ containing the set $g_i(K_i')$. Set $V := g_i^{-1}(U) \subset X$ and note that $K_i' \subset V$. Pick a point $b_i \in V \setminus K_i'$; then $g_i(b_i) \in U \setminus g_i(K_i')$. Choose a smooth embedded arc $\gamma \subset V \setminus \mathring{K}_i'$ having an endpoint p in K_i' and being otherwise disjoint from K_i'. Then, $g_i(\gamma)$ is an embedded arc in $\mathbb{B}$ having $g_i(p) \in g_i(K_i')$ as an endpoint and being otherwise disjoint from $g_i(K_i')$. Since the set $\mathbb{B} \setminus g_i(K_i')$ is path connected and contains the point a_i in view of (III), there exists a homeomorphism

$$F \colon g_i(K_i' \cup \gamma) \to g_i(K_i') \cup F(g_i(\gamma)) \subset \mathbb{C}^n$$

which equals the identity on a neighborhood of $g_i(K_i')$ such that the arc $F(g_i(\gamma))$ is contained in $\mathbb{B}$, has $g_i(p)$ and a_i as endpoints, and is otherwise disjoint from $g_i(K_i')$. Since K_i' is $\mathscr{O}(X)$-convex, the set $g_i(K_i') \subset \mathbb{C}^n$ is polynomially convex. In this situation, [13, Proposition, p. 560] (on *combing hair by holomorphic automorphisms*; see also [12, Corollary 4.13.5, p. 148]) enables us to approximate F uniformly on $g_i(K_i' \cup \gamma)$ by a holomorphic automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ such that

$$\phi(a_j) = a_j \ \text{ for } j = 1, \ldots, i-1 \quad \text{and} \quad \phi(g_i(b_i)) = a_i. \tag{3.5}$$

Consider the proper holomorphic embedding

$$f_i := \phi \circ g_i = \phi \circ \theta \circ f_{i-1} \colon X \hookrightarrow \mathbb{C}^n.$$

If the approximation of F by ϕ is close enough uniformly on $g_i(K_i' \cup \gamma)$ then the inclusion (3.3) and the maximum principle guarantee that

$$f_i(K_i' \cup \gamma) = \phi(g_i(K_i' \cup \gamma)) \subset \mathbb{B}.$$

Hence there is a connected compact $\mathscr{O}(X)$-convex subset $K_i \subset X$ such that $K_i' \cup \gamma \subset \mathring{K}_i$ and $f_i(K_i) \subset \mathbb{B}$. Assuming that the approximation of F by ϕ is close enough, the inequality (3.4) ensures that $\mathrm{dist}_{f_i}(K_{i-1}, X \setminus K_i') > 1/\epsilon_{i-1}$, and so the same holds when replacing K_i' by the bigger set K_i. Summarizing, the map f_i satisfies conditions (i), (iii), and (vi). Moreover, conditions (II), (3.5), and the fact that $b_i \in \gamma \subset K_i$ guarantee condition (ii). Finally, conditions (iv) and (v) hold provided that $\epsilon_i > 0$ is chosen small enough.

This concludes the proof of Theorem 1.4.

Corollary 1.5 is a particular case of the following result.

Corollary 3.1. *Every open connected orientable smooth surface S of finite topology admits a complex structure J such that the open Riemann surface $R = (S, J)$ admits a complete holomorphic embedding $f\colon R \hookrightarrow \mathbb{C}^2$ whose image $f(R)$ lies in the ball $\mathbb{B}^2$ and contains any given countable subset of $\mathbb{B}^2$. In particular, $f(R)$ can be made dense in $\mathbb{B}^2$.*

Proof. Let S be an open connected orientable smooth surface of finite topology, and let $A \subset \mathbb{B}^2$ be a countable subset. Let J_0 be a complex structure on S such that the open Riemann surface $R_0 = (S, J_0)$ admits a proper holomorphic embedding $\phi\colon R_0 \hookrightarrow \mathbb{C}^2$; such J_0 exists by [8] (see also [5] for the case of surfaces of infinite topology). Up to composing with an homothety we may assume that all the topology of $X := \phi(R_0)$ is contained in $\mathbb{B}^2$, meaning that $X \cap \mathbb{B}^2$ is homeomorphic to X and $X \setminus \mathbb{B}^2$ consists of finitely many pairwise disjoint closed annuli, each one bounded by a Jordan curve in $b\mathbb{B}^2 = \{z \in \mathbb{C}^2 \colon |z| = 1\}$. Theorem 1.4 applied to $X \subset \mathbb{C}^2$ and any compact subset K of $X \cap \mathbb{B}^2$ which is a strong deformation retract of X gives a Runge domain $\Omega \subset X$ containing K and a complete holomorphic embedding $f\colon \Omega \to \mathbb{B}^2$ with $A \subset f(\Omega)$. Since $K \subset \Omega$, K is homeomorphic to X, and Ω is Runge in X, we have that also Ω is homeomorphic to X, and hence to $R_0 = (S, J_0)$. Thus, there is a complex structure J on S such that $R = (S, J)$ is biholomorphic to Ω. The open Riemann surface R and the complete holomorphic embedding $f\colon R \to \mathbb{C}^2$ satisfy the conclusion of the corollary. □

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