Holomorphic embeddings and immersions of Stein manifolds: a survey

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Dedicated to Kang-Tae Kim for his sixtieth birthday

Abstract In this paper we survey results on the existence of holomorphic embeddings and immersions of Stein manifolds into complex manifolds. Most results pertain to proper maps into Stein manifolds. We include a new result saying that every continuous map $X \to Y$ between Stein manifolds is homotopic to a proper holomorphic embedding provided that $\dim Y > 2 \dim X$ and we allow a change of the Stein structure on X.

Keywords Stein manifold, embedding, density property, Oka manifold

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1. Introduction

In this paper we review what we known about the existence of holomorphic embeddings and immersion of Stein manifolds into complex manifolds. The emphasis is on recent results, but we also include some classical ones for the sake of historical perspective and completeness. For the definition and basic properties of the class of Stein manifolds we refer to the monographs [65, 70, 75, 55]. Recall that Stein manifolds are precisely the closed complex submanifolds of Euclidean spaces $\mathbb{C}^N$ (see Remmert [87], Bishop [20], and Narasimhan [84]; see also Theorem 2.1 below). Stein manifolds of dimension 1 are the open Riemann surfaces (see Behnke and Stein [18]). A domain in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy (Cartan and Thullen [26]).

In §2 we survey the results on proper holomorphic immersions and embeddings of Stein manifolds into Euclidean spaces. Of special interest is the minimal embedding and immersion dimension. Theorem 2.4, due to Eliashberg and Gromov [37] (1992) and Schürmann [92] (1997), settles this question for Stein manifolds of dimension > 1. On the other hand, the question whether every open Riemann surface embeds holomorphically (properly or nonproperly) into $\mathbb{C}^2$ is still wide open, and we describe its current status. We also briefly discuss the use of holomorphic automorphisms of Euclidean spaces in the construction of embeddings with rather wild behavior.

Recently, it has been discovered by Andrist et al. [14, 15, 44] that there is a big class of Stein manifolds Y which contain every Stein manifold X with $2 \dim X < \dim Y$ as a closed complex submanifold (see Theorem 3.4). In fact, this holds for every Stein manifold Y enjoying Varolin's *density property* [94, 95]: the Lie algebra of all holomorphic vector fields on Y is spanned by the $\mathbb{C}$-complete vector fields, i.e., those whose flow is an action of the additive group $(\mathbb{C}, +)$ by holomorphic automorphisms of Y (see Definition 3.3). The class of Stein manifolds with the density property is quite big and contains most complex Lie groups and homogeneous spaces, as well as many nonhomogeneus manifolds. It has been the focus of intensive research during the last decade; we refer the reader to the recent surveys [78] and [55, §4.10].

In §4 we recall a result of Drinovec Drnovšek and the author [33, 35] to the effect that every strongly pseudoconvex Stein domain X embeds properly holomorphically into an arbitrary Stein manifold Y satisfying $\dim Y > 2\dim X$. More precisely, every continuous map $\overline{X} \to Y$ which is holomorphic on X is homotopic to a proper holomorphic embedding $X \hookrightarrow Y$ (Theorem 4.1). The analogous result holds for immersions if $\dim Y \geq 2\dim X$, and also for every q-complete manifold Y with $q \in \{1, \ldots, \dim Y - 2\dim X + 1\}$, where the Stein case corresponds to $q = 1$. This summarizes a long line of previous results. There is a recent application [52] of these techniques to the *Hodge conjecture* for the highest dimensional a priori nontrivial cohomology group of a q-complete manifold. We also mention recent results on the existence of *complete* proper holomorphic embeddings and immersions of strongly pseudoconvex domains into balls.

In §5 we show how the combination of techniques in the above mentioned papers [33, 35] with those of Slapar and the author [51, 50] can be used to prove that, if X and Y are Stein manifolds and $\dim Y > 2\dim X$, then every continuous map $X \to Y$ is homotopic to a proper holomorphic embedding up to a deformation of the Stein structure on X. The analogous result holds for immersions if $\dim Y \geq 2\dim X$, and for q-complete manifolds Y with $q \leq \dim Y - 2\dim X + 1$. This result is in the spirit of the *soft Oka principle* developed in the papers [51, 50]. (An expositions of the results from [51, 50] is available in the monographs [55, §10.9] and [28, Theorem 8.43 and Remark 8.44].) A result in a similar vein, concerning embeddings of open Riemann surfaces into $\mathbb{C}^2$ up to a deformation of their conformal structure, is due to Alarcón and López [11] (a special case was proved in [27]); see also Ritter [89] for embeddings into $(\mathbb{C}^*)^2$.

I conclude this introduction with an apology for not having included any discussion of topics from Cauchy-Riemann geometry such as the boundary behavior of proper holomorphic maps, of maps between special domains such as balls and other symmetric domains, normal form of hypersurfaces, the reflection principle, etc. It is impossible to properly discuss this major subject in the present survey of limited size and with a rather different focus. A reader may wish to consult the recent survey by Pinchuk et al. [85], my old survey [41] from 1993, and the monograph by Baouendi et al. [17] from 1999.

We shall be using the following notation and terminology. By $\mathcal{O}(X)$ we denote the algebra of all holomorphic functions on a complex manifold X, and by $\mathcal{O}(X, Y)$ the space of all holomorphic maps $X \to Y$ between a pair of complex manifolds; thus $\mathcal{O}(X) = \mathcal{O}(X, \mathbb{C})$. These spaces carry the compact-open topology. This topology can be defined by a complete metric which renders them Baire spaces; in particular, $\mathcal{O}(X)$ is a Fréchet algebra. (See [55, p. 5] for more details.)

Recall that a compact set K in a complex manifold X is said to be *$\mathcal{O}(X)$-convex* if $K = \widehat{K} := \{p \in X : |f(p)| \leq \sup_K |f| \ \ \forall\, f \in \mathcal{O}(X)\}$. The *Oka-Weyl theorem* says that every holomorphic function on an open neighborhood of a compact $\mathcal{O}(X)$-convex set K in a Stein manifold X can be approximated uniformly on K by functions in $\mathcal{O}(X)$. (See [75, §4.2 and Theorem 5.6.2] or [55, Theorem 2.8.4]. This is a generalization of the classical Runge theorem which pertains to the case $X = \mathbb{C}$.)

2. Embeddings and immersions of Stein manifolds into Euclidean spaces

In this section we survey results on proper holomorphic immersions and embeddings of Stein manifolds into Euclidean spaces.

2.1. Classical results. We begin by recalling the theorems of Remmert [87], Bishop [20], and Narasimhan [84] from the period 1956–1961.

Theorem 2.1. *Assume that X is a Stein manifold of dimension n.*

(a) *If $N > 2n$ then the set of proper embedding $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(b) *If $N \geq 2n$ then the set of proper immersions $X \hookrightarrow \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(c) *If $N > n$ then the set of proper maps $X \to \mathbb{C}^N$ is dense in $\mathcal{O}(X, \mathbb{C}^N)$.*
(d) *If $N \geq n$ then the set of almost proper maps $X \to \mathbb{C}^N$ is residual in $\mathcal{O}(X, \mathbb{C}^N)$.*

A complete proof can also be found in the monograph by Gunning and Rossi [70].

Recall that a set in a Baire space (such as $\mathcal{O}(X, \mathbb{C}^N)$) is said to be *residual*, or a set *of second category*, if it is the intersection of at most countably many open everywhere dense sets. Every residual set is dense. A property of elements in a Baire space is said to be *generic* of it holds for all elements in a residual set.

The density statement for embeddings and immersions is an easy consequence of the following result which follows from the jet transversality theorem for holomorphic maps. (See Forster [39] for maps to Euclidean spaces and Kaliman and Zaidenberg [79] for the general case. A more comprehensive discussion of this topic can be found in [55, §8.8].) Note also that maps which are immersions or embeddings on a given compact set constitute an open set in the corresponding mapping space.

Proposition 2.2. *Assume that X is a Stein manifold, K is a compact set in X, and $U \Subset X$ is an open relatively compact set containing K. If Y is a complex manifold such that $\dim Y > 2 \dim X$, then every holomorphic map $f\colon X \to Y$ can be approximated uniformly on K by holomorphic embeddings $U \hookrightarrow Y$. If $2 \dim X \leq \dim Y$ then f can be approximated by holomorphic immersions $U \to Y$.*

Proposition 2.2 fails in general without shrinking the domain of the map, for otherwise it would yield nonconstant holomorphic maps of $\mathbb{C}$ to any complex manifold of dimension > 1 which is clearly false. On the other hand, it holds without shrinking the domain of the map if the target manifold Y satisfies a suitable holomorphic flexibility property; in particular, if it is an *Oka manifold.* (See [55, Chap. 5] for the definition of this class of complex manifolds and [55, Corollary 8.8.7] for the mentioned result.)

In the proof of Theorem 2.1, parts (a)–(c), we exhaust X by a sequence $K_1 \subset K_2 \subset \cdots$ of compact $\mathcal{O}(X)$-convex sets and approximate the holomorphic map $f_j\colon X \to \mathbb{C}^N$ in the inductive step, uniformly on K_j, by a holomorphic map $f_{j+1}\colon X \to \mathbb{C}^N$ whose norm $|f_{j+1}|$ is sufficiently big on $K_{j+1} \setminus K_j$. This is achieved by applying the Oka-Weil approximation theorem in order to make each component function big on a certain $\mathcal{O}(X)$-convex set whose union contains $K_{j+1} \setminus \mathring{K}_j$. If the approximation is close enough at every step then the sequence f_j converges to a proper holomorphic map $f = \lim_{j\to\infty} f_j\colon X \to \mathbb{C}^N$. If $N > 2n$ then every map f_j in the sequence can be made an embedding on K_j by Proposition 2.2 (immersion in $N \geq 2n$), and hence the limit map f is also such.

A more efficient way of constructing proper maps (immersions, embeddings) $X \to \mathbb{C}^N$ was explained by Bishop [20]. He showed that any holomorphic map $X \to \mathbb{C}^n$ from an n-dimensional Stein manifold X can be approximated uniformly on compacts in X by almost proper maps $h\colon X \to \mathbb{C}^n$ (this is part (d) in Theorem 2.1). More precisely, there is an increasing sequence $P_1 \subset P_2 \subset \cdots \subset X$ of relatively compact open sets exhausting X such that every P_j is a union of finitely many special analytic polyhedra and h maps P_j properly onto a polydisc $a_j\mathbb{D}^n \subset \mathbb{C}^n$, where $0 < a_1 < a_2 < \ldots$ and $\lim_{j\to\infty} a_j = +\infty$. (Here, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc and $\mathbb{D}^n$ is the Cartesian product of n copies of it.) We can obtain a proper map $(h, g)\colon X \to \mathbb{C}^{n+1}$ by choosing the function $g \in \mathcal{O}(X)$ big enough on each of the compact sets $L_j = \{x \in \overline{P}_{j+1} \setminus P_j : |h(x)| \le a_{j-1}\}$; this is possible by the Oka-Weil theorem since $\overline{P}_j \cup L_j$ is $\mathcal{O}(X)$-convex. One can then find proper immersions and embeddings by adding a suitable number of additional components to (h, g) (any such map is clearly proper) and using Proposition 2.2 and the Oka-Weil theorem inductively.

The first of the above mentioned procedures easily adapts to give a proof of the following interpolation theorem due to Acquistapace et al. [2, Theorem 1]. Their result also pertains to Stein spaces of bounded embedding dimension.

Theorem 2.3. [2, Theorem 1] *Assume that X is an n-dimensional Stein manifold, X' is a closed complex subvariety of X, and $\phi\colon X' \hookrightarrow \mathbb{C}^N$ is a proper holomorphic embedding for some $N > 2n$. Then the set of all proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ that extend ϕ is dense in the space of all holomorphic maps $X \to \mathbb{C}^N$ extending ϕ. The analogous result holds for proper holomorphic immersions $X \to \mathbb{C}^N$ when $N \ge 2n$.*

The interpolation theorem fails in general when $N < 2n$. Indeed, for every integer $n > 1$ there exists a proper holomorphic embedding $\phi\colon \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^{2n-1}$ which does not extend to an injective holomorphic map $f\colon \mathbb{C}^n \to \mathbb{C}^{2n-1}$ because the complement $\mathbb{C}^{2n-1} \setminus \phi(\mathbb{C}^{n-1})$ is Eisenman n-hyperbolic (see [55, Proposition 9.5.6]; this topic is discussed in §2.4 below).

The answer to the interpolation problem for embeddings in the borderline case $N = 2n$ seems unknown.

2.2. Embeddings and immersions into spaces of minimal dimension. After Theorem 2.1 was proved in the early 1960's, one of the main questions driving this theory during the next three decades was to find the smallest number $N = N(n)$ for a given $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ such that every Stein manifold X of dimension n embeds (or immerses) properly holomorphically into $\mathbb{C}^N$. The belief that Stein manifolds admit proper holomorphic embeddings below the generic dimension in Theorem 2.1(a) was based on the observation that a Stein manifold of dimension n is homotopically equivalent to a CW complex of dimension at most n (half of the real dimension of X); this follows by Morse theory (see Milnor [83]) and the existence of strongly plurisubharmonic Morse exhaustion functions on X (see Hamm [71] and [55, §3.12]). This problem, which was investigated by Forster [39], Eliashberg and Gromov [69] and others, gave rise to interesting new methods in Stein geometry. One of the ingredients which plays a key role in the solution is a certain major extension of the Oka-Grauert theory, due to Gromov whose 1989 paper [68] marks the beginning of *modern Oka theory*. (See the monograph [55] for a complete account and the paper [47] for a leisurely introduction to Oka theory.) Except in the case $n = 1$ when X is an open Riemann surface, the following answer to the question of minimal embedding and immersion dimensions was given by Eliashberg and Gromov [37] in 1992, with an improvement by one for odd values on n due to Schürmann [92].

Theorem 2.4. [37, 92] *Every Stein manifold X of dimension n immerses properly holomorphically into $\mathbb{C}^M$ with $M = \left[\frac{3n+1}{2}\right]$, and if $n > 1$ then X embeds properly holomorphically into $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$.*

Schürmann [92] also found optimal embedding dimension for Stein spaces with singularities and with bounded embedding dimension.

In view of an example of Forster [39, Proposition 3] (see also [55, Proposition 9.3.3]), the embedding dimension $N = \left[\frac{3n}{2}\right] + 1$ is minimal for every $n > 1$, and the immersion dimension $M = \left[\frac{3n+1}{2}\right]$ is minimal for every even n, while for odd n there could be two possible values. This question was discussed again in 2012 by Ho et al. [74] who gave new examples showing that the dimensions in Theorem 2.4 are optimal (except perhaps for immersions with n odd) already for Stein manifolds which are Grauert tubes around compact totally real submanifolds.

A more complete discussion of this topic and a self-contained proof of Theorem 2.4 can be found in [55, Chap. 9]; see in particular Sections 9.2–9.5. Here we only give a brief indication of the main ideas used in the proof.

One begins by choosing a sufficiently generic almost proper map $h\colon X \to \mathbb{C}^n$ (see Theorem 2.1(d)) and then tries to find the smallest possible number of functions $g_1, \ldots, g_q \in \mathcal{O}(X)$ such that the map

$$f = (h, g_1, \ldots, g_q)\colon X \to \mathbb{C}^{n+q} \tag{2.1}$$

is a proper embedding (or immersion). Starting with a big but finite number of functions $\tilde g_1, \ldots, \tilde g_{\tilde q} \in \mathcal{O}(X)$ which do the job, we try to reduce their number by applying a suitable fibrewise linear projection onto a smaller dimensional subspace, where the projection depends holomorphically on the base point. Explicitly, we look for functions $g_j = \sum_{k=1}^{\tilde q} a_{j,k}\tilde g_k$ for $j = 1, \ldots, q$ and $a_{j,k} \in \mathcal{O}(X)$ such that the map (2.1) is a proper embedding. (In order to separate those pairs of points in X which are not separated by the base map $h\colon X \to \mathbb{C}^n$, we consider coefficient functions of the form $a_{j,k} = b_{j,k} \circ h$ where $b_{j,k} \in \mathcal{O}(\mathbb{C}^n)$.) Analysis of this condition shows that the graph of the map $\alpha = (a_{j,k})\colon X \to \mathbb{C}^{q\tilde q}$ must avoid a certain complex subvariety Σ of $E = X \times \mathbb{C}^{q\tilde q}$. (In the second case mentioned above, a similar analysis holds for the map $\beta = (b_{j,k})\colon \mathbb{C}^n \to \mathbb{C}^{q\tilde q}$.)

This outline cannot be applied directly since the base map $h\colon X \to \mathbb{C}^n$ may have too complicated behavior. Instead, one proceeds by induction on strata in a suitably chosen stratification of X adjusted to h. The induction steps are of two kinds. In a step of the first kind, we find a map $g = (g_1, \ldots, g_q)$ which separates points on the (finite) fibres of h over the next bigger stratum, and it matches the map from the previous step on the union of the previous strata (a closed complex subvariety of X). A step of the second kind amounts to removing the kernel of the differential dh_x for all points in the next stratum, thereby ensuring that $df_x = dh_x \oplus dg_x$ is injective. It turns out that in both steps the exceptional subvariety Σ which must be avoided is such that the projection $\pi\colon E \setminus \Sigma \to X$ is a stratified holomorphic fibre bundle all of whose fibres are Oka manifolds. More precisely, each fibre $\Sigma_x = \Sigma \cap E_x$ is either empty or a union of finitely many affine linear subspaces of E_x of codimension > 1 provided that $q \geq \left[\frac{n}{2}\right] + 1$. The same lower bound guarantees the existence of a continuous section $\alpha\colon X \to E \setminus \Sigma$ avoiding Σ. (This lower bound is needed to separate points on the strata of maximal dimension n. Smaller values suffices to remove the kernel of the differential and to separate points on the strata of lower dimensions.) Gromov's Oka principle [68] then furnishes a holomorphic section $X \to E \setminus \Sigma$. A general Oka principle for sections of stratified holomorphic fiber bundles with Oka fibres is given by [55, Theorem 5.4.4]. We refer the reader to the original papers or to [55, §9.3–9.4] for further details.

A weak point of all classical constructions of proper holomorphic immersions and embeddings of Stein manifolds into Euclidean spaces is that they are coordinate dependent and do not generalize easily to more general target manifolds. A conceptually new method has been found recently by Ritter and the author [48]. It is based on a method of separating certain pairs of compact polynomially convex sets in $\mathbb{C}^N$ by Fatou-Bieberbach domains (biholomorphic images of $\mathbb{C}^N$) which contain one of the sets and avoid the other one. Another recently developed method which also depends on holomorphic automorphisms and applies to a much bigger class of target manifolds is discussed in §3.

2.3. Embedding open Riemann surfaces into $\mathbb{C}^2$. The constructions described so far fail to embed open Riemann surfaces into $\mathbb{C}^2$. The problem is that the subvariety Σ, which arises in the proof of Theorem 2.4, may contain a hypersurface, and hence the Oka principle for sections of $E \setminus \Sigma \to X$ fails in general due to hyperbolicity of its complement.

It is still an open problem whether every open Riemann surface embeds as a smooth closed complex curve in $\mathbb{C}^2$. (By Theorem 2.1 it properly holomorphically embeds into $\mathbb{C}^3$ and immerses with normal crossings into $\mathbb{C}^2$. Every compact Riemann surface embeds holomorphically into $\mathbb{CP}^3$ and immerses into $\mathbb{CP}^2$, but very few of them embed into $\mathbb{CP}^2$; see [67].) There are no topological obstructions to this problem — it was shown by Alarcón and López [11] that every open orientable surface S carries a complex structure J such that the Riemann surface $X = (S, J)$ admits a proper holomorphic embedding into $\mathbb{C}^2$.

There is a variety of results in the literature concerning the existence of proper holomorphic embeddings of certain special open Riemann surfaces into $\mathbb{C}^2$; the reader may wish to consult the survey in [55, §9.10–9.11]. Here we mention only a few of the currently most general known results on the subject. The first one from 2009, due to Wold and the author, concerns bordered Riemann surfaces.

Theorem 2.5. [53, Corollary 1.2] *Assume that X is a compact bordered Riemann surface with boundary of class $\mathscr{C}^r$ for some $r > 1$. If $f\colon X \hookrightarrow \mathbb{C}^2$ is a $\mathscr{C}^1$ embedding that is holomorphic in the interior $\mathring{X} = X \setminus bX$, then f can be approximated uniformly on compacts in $\mathring{X}$ by proper holomorphic embeddings $\mathring{X} \hookrightarrow \mathbb{C}^2$.*

The proof relies on techniques introduced by Wold in his papers [97, 96, 98]. One of them concerns exposing boundary points of an embedded bordered Riemann surface in $\mathbb{C}^2$; see [53] or [55, §9.9]. This technique was improved in [53]; see also the exposition in [55, §9.9]. The second one depends on methods of Andersén-Lempert theory concerning holomorphic automorphisms of complex Euclidean spaces (see [13, 49] and [55, §4.9]; this topic is dicussed in §2.4 below). A proper holomorphic embedding $\mathring{X} \hookrightarrow \mathbb{C}^2$ is obtained by first exposing a boundary point in each of the boundary curves of $f(X) \subset \mathbb{C}^2$, sending these points to infinity by a rational shear on $\mathbb{C}^2$ without other poles on $f(X)$, and then using a carefully constructed sequence of holomorphic automorphisms of $\mathbb{C}^2$ whose domain of convergence is a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ which contains the embedded complex curve $f(X) \subset \mathbb{C}^2$, but does not contain any of its boundary points. If $\phi\colon \Omega \to \mathbb{C}^2$ is a Fatou-Bieberbach map then $\phi \circ f\colon X \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding. A complete exposition of this proof can also be found in [55, §9.10].

A domain X in the Riemann sphere $\mathbb{P}^1$ is a *generalized circled domain* if every connected component of $\mathbb{P}^1 \setminus X$ is a round disc or a point. By the uniformization theorem of He and Schramm [72, 73], every domain in $\mathbb{P}^1$ with at most countably many complementary components is conformally equivalent to a generalized circled domain.

Theorem 2.6. [54, Theorem 5.1] *Let X be a generalized circled domain in $\mathbb{P}^1$. If all but finitely many punctures in $\mathbb{P}^1 \setminus X$ are limit points of discs in $\mathbb{P}^1 \setminus X$, then X embeds properly holomorphically into $\mathbb{C}^2$.*

The paper [54] contains several other more precise results on this subject.

The special case of Theorem 2.6 for plane domains $X \subset \mathbb{C}$ bounded by finitely many Jordan curves (and without punctures) is due to Globevnik and Stensønes [60]. Results on embedding certain Riemann surfaces with countably many boundary components into $\mathbb{C}^2$ were also proved by Majcen [82]; an exposition can be found in [55, §9.11]. The proof of Theorem 2.6 relies on similar techniques as that of Theorem 2.5, but it uses a considerably more involved induction scheme for dealing with infinitely many boundary components, clustering them together into suitable subsets to which the available analytic methods can be applied. By the same technique, one can show the analogous result for domains in tori $\mathbb{C}/\Gamma$, where $\mathbb{Z}^2 \cong \Gamma \subset \mathbb{C}$ is a lattice of rank 2.

There are a few other recent results concerning embeddings of open Riemann surfaces into $\mathbb{C} \times \mathbb{C}^*$ and $(\mathbb{C}^*)^2$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Ritter showed in [88] that, for every circular domain $X \subset \mathbb{D}$ with finitely many boundary components in the disc $\mathbb{D}$, each homotopy class of continuous maps $X \to \mathbb{C} \times \mathbb{C}^*$ contains a proper holomorphic map. If $\mathbb{D} \setminus X$ contains finitely many punctures, then every continuous map $X \to \mathbb{C} \times \mathbb{C}^*$ is homotopic to a proper holomorphic immersion that identifies at most finitely many pairs of points in X (Lárusson and Ritter [80]). Ritter [89] also gave an analogue of Theorem 2.5 for proper holomorphic embeddings of certain open Riemann surfaces into $(\mathbb{C}^*)^2$.

2.4. Automorphisms of Euclidean spaces and wild embeddings. There is another line of investigation that we wish to touch upon. It concerns the question how many proper holomorphic embeddings $X \hookrightarrow \mathbb{C}^N$ of a given Stein manifold X are there up to automorphisms of $\mathbb{C}^N$ (and possibly also of X). This question was motivated by certain famous results from algebraic geometry, such as the one of Abhyankar and Moh [1] and Suzuki [93] to the effect that every polynomial embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$ is equivalent to the linear embedding $z \mapsto (z, 0)$ by a polynomial automorphism of $\mathbb{C}^2$.

It is a basic fact that for any $N > 1$ the holomorphic automorphism group $\mathrm{Aut}(\mathbb{C}^N)$ is very big an complicated. This is in stark contrast to the situation for bounded or, more generally, hyperbolic domains in $\mathbb{C}^N$ which have few automorphism; see Greene et al. [66] for a survey of the latter topic. Major early work on understanding the group $\mathrm{Aut}(\mathbb{C}^N)$ was made by Rosay and Rudin [90]. This theory became very useful with the papers of Andersén and Lempert [13] and Rosay and the author [49]. The central result is that every map in a smooth isotopy of biholomorphic mappings $\Phi_t \colon \Omega = \Omega_0 \to \Omega_t$ $(t \in [0, 1])$ between Runge domains in $\mathbb{C}^N$, with Φ_0 the identity on Ω, can be approximated uniformly on compacts in Ω by holomorphic automorphisms of $\mathbb{C}^N$ (see [49, Theorem 1.1] or [55, Theorem 4.9.2]). A similar phenomenon holds on any Stein manifold with the density property; see §3. A comprehensive survey of this subject can be found in [55, Chap. 4].

By twisting a given submanifold of $\mathbb{C}^N$ with a sequence of holomorphic automorphisms of $\mathbb{C}^N$ it is possible to show that for any pair of integers $1 \leq n < N$ the set of all equivalence classes of proper holomorphic embeddings $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$, modulo automorphisms of both spaces, is uncountable. In particular, the Abhyankar-Moh theorem fails in the holomorphic category since there exist proper holomorphic embeddings $\phi\colon \mathbb{C} \hookrightarrow \mathbb{C}^2$ that are nonstraightenable by automorphisms of $\mathbb{C}^2$ [46], and also embeddings whose complement $\mathbb{C}^2 \setminus \phi(\mathbb{C})$ is Kobayashi hyperbolic [25]. More generally, for any pair of integers $1 \le n < N$ there exists a proper holomorphic embedding $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^N$ such that every nondegenerate holomorphic map $\mathbb{C}^{N-n} \to \mathbb{C}^N$ intersects $\phi(\mathbb{C}^n)$ at infinitely many points [42]. It is also possible to arrange that $\mathbb{C}^N \setminus \phi(\mathbb{C}^n)$ is Eisenman $(N-n)$-hyperbolic [22]. This gives rise to many nonequivalent embeddings $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$; see [30]. (A more comprehensive discussion of this subject can be found in [55, §4.18].)

By using nonlinearizable proper holomorphic embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^2$, Derksen and Kutzschauch gave the first known examples of nonlinearizable periodic automorphisms of $\mathbb{C}^n$ [29]. For instance, there is a nonlinearizable holomorphic involution on $\mathbb{C}^4$.

In another direction, Baader et al. [16] constructed an example of a properly embedded disc in $\mathbb{C}^2$ whose image is topologically knotted; this answered a questions of Kirby. It is unknown whether there exists a knotted proper holomorphic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$, or an unknotted embedding $\mathbb{D} \hookrightarrow \mathbb{C}^2$ of the disc.

Automorphisms of $\mathbb{C}^2$ and $\mathbb{C}^* \times \mathbb{C}$ were used in a very clever way by Wold in his landmark construction of non-Runge Fatou-Bieberbach domains in $\mathbb{C}^2$ [99] and of non-Stein long $\mathbb{C}^2$'s [100]. Each of these results solved a long-standing open problem. More recently, Wold's construction was developed further by Boc Thaler and the author [21] who showed that there is a continuum of pairwise nonequivalent long $\mathbb{C}^n$'s for any $n > 1$ which do not admit any nonconstant holomorphic or plurisubharmonic functions. (See aso [55, 4.21].) Automorphisms of Euclidean spaces have also been applied to the construction of complete holomorphic embeddings; see §4.3 below for a discussion of this topic.

3. Embeddings into Stein manifolds with the density property

3.1. Universal Stein manifolds. It is natural to ask which Stein manifolds, besides the Euclidean spaces, contain all Stein manifolds of suitable dimension as closed complex submanifolds. To facilitate the discussion, we introduce the following notions.

Definition 3.1. Let Y be a Stein manifold.

(1) Y is *universal for proper holomorphic embeddings* if every Stein manifold X with $2\dim X < \dim Y$ admits a proper holomorphic embedding $X \hookrightarrow Y$.

(2) Y is *strongly universal for proper holomorphic embeddings* if, under the assumptions in (1), every continuous map $f_0\colon X \to Y$ which is holomorphic in a neighborhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$ is homotopic to a proper holomorphic embedding $f_0\colon X \hookrightarrow Y$ by a homotopy $f_t\colon X \to Y$ $(t \in [0,1])$ such that f_t is holomorphic and arbitrarily close to f_0 on K for every $t \in [0,1]$.

(3) Y is (strongly) *universal for proper holomorphic immersions* if condition (1) (resp. (2)) holds for proper holomorphic immersions $X \to Y$ from any Stein manifold X satisfying $2\dim X \le \dim Y$.

In the terminology of Oka theory (cf. [55, Chap. 5]), a complex manifold Y is (strongly) universal for proper holomorphic embeddings if it satisfies the basic Oka property (with approximation) for maps $X \to Y$ from Stein manifolds of dimension $2\dim X < \dim Y$. The dimension hypotheses in the above definition are justified by Proposition 2.2.

The main goal is of course to find good sufficient conditions for a Stein manifold to be universal. If a manifold Y is Brody hyperbolic [24] (i.e., it does not admit any nonconstant holomorphic images of $\mathbb{C}$) then clearly no complex manifold containing a nontrivial holomorphic image of $\mathbb{C}$ can be embedded into Y. In order to get positive results, one must therefore assume that Y enjoys a suitable holomorphic flexibility (anti-hyperbolicity) property. It is natural to ask the following question.

Problem 3.2. Is every Stein Oka manifold (strongly) universal for proper holomorphic embeddings and immersions?

Recall (see e.g. [55, Theorem 5.5.1]) that every Oka manifold is strongly universal for non-proper holomorphic maps, embeddings and immersions. Indeed, the cited theorem asserts that a generic holomorphic map $X \to Y$ from a Stein manifold X into an Oka manifold Y is an immersion if $\dim Y \geq 2\dim X$, and an injective immersion if $\dim Y > 2\dim X$. However, the Oka condition does not imply universality for *proper* holomorphic maps since there are examples of (compact or noncompact) Oka manifolds without any closed complex subvarieties of positive dimension (see [55, Example 9.8.3]).

3.2. Manifolds with the (volume) density property. The following condition was introduced in 2000 by Varolin [94, 95].

Definition 3.3. A complex manifold Y enjoys the (holomorphic) *density property* if the Lie algebra generated by the $\mathbb{C}$-complete holomorphic vector fields on Y is dense in the Lie algebra of all holomorphic vector fields in the compact-open topology.

A complex manifold Y endowed with a holomorphic volume form ω enjoys the *volume density property* if the analogous density condition holds in the Lie algebra of all holomorphic vector fields on Y with vanishing ω-divergence.

The algebraic density and volume density properties were introduced by Kaliman and Kutzschebauch [77]. The class of Stein manifolds with the (volume) density property is quite big and includes most complex Lie groups and homogeneous spaces, as well as many nonhomogeneous manifolds. We refer to [55, §4.10] for a more complete discussion and an up-to-date collection of references on this subject. Another recent survey is the paper by Kaliman and Kutzschebauch [78]. Every complex manifold with the density property is an Oka manifold, and a Stein manifold with the density property is elliptic in the sense of Gromov (see [55, Proposition 5.6.23]).

The following result is due to Andrist and Wold [15] in the special case when X is an open Riemann surface, to Andrist et al. [14, Theorems 1.1, 1.2] for embeddings, and to the author [44, Theorem 1.1] for immersions in the double dimension.

Theorem 3.4. [14, 15, 44] *Every Stein manifold with the density or the volume density property is strongly universal for proper holomorphic embeddings and immersions.*

To prove Theorem 3.4, one follows the scheme of proof of the Oka principle for maps from Stein manifolds to Oka manifolds (see [55, Chapter 5]), but with a crucial addition which we now briefly describe.

Assume that $D \Subset X$ is a relatively compact strongly pseudoconvex domain with smooth boundary and $f\colon \overline{D} \hookrightarrow Y$ is a holomorphic embedding such that $f(bD) \subset Y \setminus L$, where L is a given compact $\mathcal{O}(Y)$-convex set in Y. We wish to approximate f uniformly on $\overline{D}$ by a holomorphic embedding $f'\colon \overline{D'} \hookrightarrow Y$ of a bigger strongly pseudoconvex domain

$\overline{D'} \Subset X$ to Y, where D' is either a union of D with a small convex bump B chosen such that $f(\overline{D \cap B}) \subset Y \setminus L$, or a thin handlebody whose core is the union of D and a suitable smoothly embedded totally real disc in $X \setminus D$. (The second case amounts to a change of topology of the domain, and it typically occurs when passing a critical points of a strongly plurisubharmonic exhaustion function on X.) In view of Proposition 2.2, we only need to approximate f by a holomorphic map $f' \colon \overline{D'} \to Y$ since a small generic perturbation of f' then yields an embedding. It turns out that the second case involving a handlebody easily reduces to the first one; see [55, §5.11] for this reduction. The attaching of a bump is handled by using the density property of Y which allows us to find a holomorphic map $g \colon \overline{B} \to Y \setminus L$ approximating f as closely as desired on a neighborhood of the attaching set $\overline{B \cap D}$ and satisfying $g(\overline{B}) \subset Y \setminus L$. (More precisely, we use that isotopies of biholomorphic maps between pseudoconvex Runge domains in Y can be approximated by holomorphic automorphisms of Y; see [49, Theorem 1.1] and also [55, Theorem 4.10.5] for the version pertaining to Stein manifolds with the density property.) Assuming that g is sufficiently close to f on $\overline{B \cap D}$, we can glue them into a holomorphic map $f' \colon \overline{D'} \to Y$ which approximates f on $\overline{D}$ and satisfies $f'(\overline{B}) \subset Y \setminus L$. The proof of the theorem is completed by an induction procedure in which every induction step is of the type described above. The inclusion $f'(\overline{B}) \subset Y \setminus L$ satisfied by the next map in the induction step guarantees properness of the limit embedding $X \hookrightarrow Y$. (Of course the sets $L \subset Y$ also increase and form an exhaustion of Y.)

The case of immersions requires a somewhat more precise analysis. In the induction step described above, we must ensure that the immersion $f \colon \overline{D} \to Y$ is injective (an embedding) on the attaching set $\overline{B \cap D}$ of the bump B. This can be arranged by general position provided that $\overline{B \cap D}$ is very thin. We show in [44] that it suffices to work with convex bumps such that, in suitably chosen holomorphic coordinates on a neighborhood of $\overline{B}$, the set B is a convex polyhedron and $\overline{B \cap D}$ is a very thin neighborhood of one of its faces. This means that $\overline{B \cap D}$ is small thickening of a $(2n-1)$-dimensional object in X, and hence we can easily arrange that f is injective on it. The rest of the proofs proceeds as before. This complete our sketch of proof of Theorem 3.4.

3.3. On the Schoen-Yau conjecture. The following corollary to Theorem 3.4 is related to a conjecture of Schoen and Yau [91] that the disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ does not admit any proper harmonic maps to $\mathbb{R}^2$. The case $n = 1$ (when X is an open Riemann surface) is due to Andrist and Wold [15, Theorem 5.6].

Corollary 3.5. *Every Stein manifold X of dimension n admits a proper holomorphic immersion to $(\mathbb{C}^*)^{2n}$, and a proper pluriharmonic map into $\mathbb{R}^{2n}$.*

Proof. The space $(\mathbb{C}^*)^n$ with coordinates $z = (z_1, \ldots, z_n)$ ($z_j \in \mathbb{C}^*$ for $j = 1, \ldots, n$) enjoys the volume density property with respect to the volume form

$$\omega = \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}.$$

(See Varolin [95] or [55, Theorem 4.10.9(c)].) Hence, [44, Theorem 1.2] (the part of Theorem 3.4 above concerning immersions into the double dimension) furnishes a proper holomorphic immersion $f = (f_1, \ldots, f_{2n}) \colon X \to (\mathbb{C}^*)^{2n}$. It follows that the map

$$u = (u_1, \ldots, u_{2n}) \colon X \to \mathbb{R}^{2n} \quad \text{with} \quad u_j = \log |f_j| \quad \text{for} \quad j = 1, \ldots, 2n \tag{3.1}$$

is a proper map of X to $\mathbb{R}^{2n}$ whose components are pluriharmonic functions. □

Corollary 3.5 gives a counterexample to the Schoen-Yau conjecture in every dimension and for any Stein source manifold. The first (and very explicit) counterexample was given in 1999 by Božin [23]. In 2001, Globevnik and the author [45] constructed a proper holomorphic map $f = (f_1, f_2)\colon \mathbb{D} \to \mathbb{C}^2$ whose image is contained in $(\mathbb{C}^*)^2$. In this case, the harmonic map $u = (u_1, u_2)\colon \mathbb{D} \to \mathbb{R}^2$ given by (3.1) satisfies the condition

$$\lim_{|\zeta|\to 1} \max\{u_1(\zeta), u_2(\zeta)\} = +\infty$$

which implies properness but is even stronger. Next, Alarcón and López [10] showed in 2012 that every open Riemann surface X admits a conformal minimal immersion $u = (u_1, u_2, u_3)\colon X \to \mathbb{R}^3$ with a proper (harmonic) projection $(u_1, u_2)\colon X \to \mathbb{R}^2$. In 2014, Andrist and Wold [15, Theorem 5.6] proved Corollary 3.5 in the case $n = 1$.

Comparing Corollary 3.5 with the above mentioned result of Globevnik and the author [45], one is led to ask the following question.

Problem 3.6. Let X be a Stein manifold of dimension $n > 1$. Does there exist a proper holomorphic immersion $f\colon X \to \mathbb{C}^{2n}$ such that $f(X) \subset (\mathbb{C}^*)^{2n}$?

4. Embeddings of strongly pseudoconvex Stein domains

4.1. The Oka principle for embeddings of strongly pseudoconvex domains. It is natural to ask what can be said about proper holomorphic embeddings and immersions of Stein manifolds X into arbitrary (Stein) manifolds Y.

If Y is Brody hyperbolic [24], then no complex manifold containing a nontrivial holomorphic image of $\mathbb{C}$ embeds into Y. However, if $\dim Y > 1$ then Y still admits proper holomorphic images of any bordered Riemann surface [33, 57]. For domains in higher dimensional Euclidean spaces, this line of investigation was started in 1976 by Fornæss [38] and continued in 1985 by Løw [81] and the author [40] who proved that every bounded strongly pseudoconvex domain $X \subset \mathbb{C}^n$ admits a proper holomorphic embedding into a sufficiently high dimensional ball and polydisc. The line of developments on this subject culminated in the following result of Drinovec Drnovšek and the author from 2010.

Theorem 4.1. [35, Corollary 1.2] *Let X be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold $\widetilde{X}$ of dimension n, and let Y be a Stein manifold of dimension N. If $N > 2n$ then every continuous map $f\colon \overline{X} \to Y$ which is holomorphic on X can be approximated uniformly on compacts in X by proper holomorphic embeddings $X \hookrightarrow Y$. If $N \geq 2n$ then the analogous result holds for immersions. The same holds if the manifold Y is strongly q-complete for some $q \in \{1, 2, \ldots, N - 2n + 1\}$, where the case $q = 1$ corresponds to Stein manifolds.*

In the special case when Y is a domain in a Euclidean space, this is due to Dor [31]. The papers [33, 35] include several more precise results in this direction, as well as references to the numerous previous results. Note that a continuous map $f\colon \overline{X} \to Y$ from a compact strongly pseudoconvex domain which is holomorphic on the open domain X, with values in an arbitrary complex manifold Y, can be approximated uniformly on $\overline{X}$ by holomorphic maps from small open neighborhoods of $\overline{X}$ in the ambient manifold $\widetilde{X}$, where the neighborhood depends on the map (see [34, Theorem 1.2] or [55, Theorem 8.11.4]). However, unless Y is an Oka manifold, it is impossible to approximate f uniformly on $\overline{X}$ by holomorphic maps from a fixed bigger domain $X_1 \subset \widetilde{X}$ independent of the map. For this reason, it is imperative that the initial map f in Theorem 4.1 be defined on all of $\overline{X}$.

One of the main techniques used in the proof of Theorem 4.1 is a Riemann-Hilbert type boundary value problem along with the use of special holomorphic peaking functions on X. The second tool, which enables the construction of holomorphic maps with values in any complex manifold, is the method of holomorphic sprays developed in the context of Oka theory; this is essentially a nonlinear version of the $\overline{\partial}$-method. The condition that Y be Stein or q-complete provides sufficient geometric information to construct proper maps. Here is the main idea of the proof. Choose a strongly q-convex Morse exhaustion function $\rho\colon Y \to \mathbb{R}_+$. (When $q = 1$, ρ is strongly plurisubharmonic.) By using the mentioned tools, one can approximate any given holomorphic map $f\colon \overline{X} \to Y$ by another holomorphic map $\tilde{f}\colon \overline{X} \to Y$ such that $\rho \circ \tilde{f} > \rho \circ f + c$ holds on bX for some constant $c > 0$ depending only on the geometry of ρ on a given compact set $L \subset Y$ containing $f(\overline{X})$. (Geometrically speaking, this means that we lift the image of the boundary of X in Y to a higher level of the function ρ by a prescribed amount.) At the same time, we can ensure that $\rho \circ \tilde{f} > \rho \circ f - \delta$ on X for any given $\delta > 0$, and that $\tilde{f}$ approximates f as closely as desired on a given compact $\mathscr{O}(X)$-convex set $K \subset X$. By Proposition 2.2 we can assume that our maps are embeddings. An inductive application of this technique yields a sequence of holomorphic embeddings $f_k\colon \overline{X} \hookrightarrow Y$ converging to a proper holomorphic embedding $F\colon X \hookrightarrow Y$. The same construction gives proper holomorphic immersions when $N \geq 2n$.

4.2. On the Hodge Conjecture for q-complete manifolds. A more precise analysis of the proof of Theorem 4.1 was used by Smrekar, Sukhov and the author [52] to show the following result along the lines of the Hodge conjecture.

Theorem 4.2. *If Y is a q-complete complex manifold of dimension N and finite topology such that $q < N$ and the number $N + q - 1 = 2p$ is even, then every cohomology class in $H^{N+q-1}(Y;\mathbb{Z})$ is Poincaré dual to an analytic cycle in Y consisting of proper holomorphic images of the ball $\mathbb{B}^p \subset \mathbb{C}^p$. If the manifold Y has infinite topology, the same result holds for elements of the group $\mathscr{H}^{N+q-1}(Y;\mathbb{Z}) = \lim_j H^{N+q-1}(M_j; Z)$ where $\{M_j\}_{j\in\mathbb{N}}$ is an exhaustion of Y by compact smoothly bounded domains.*

Note that $H^{N+q-1}(Y;\mathbb{Z})$ is the highest dimensional a priori nontrivial cohomology group of a q-complete manifold Y of dimension N. We do not know whether a similar result holds for lower dimensional cohomology groups of a q-complete manifold. In the special case when Y is a Stein manifold, the situation is better understood thanks to the Oka-Grauert principle, and the reader can find appropriate references in the paper [52].

4.3. Complete bounded complex submanifolds. There are interesting recent constructions of properly embedded complex submanifolds $X \subset \mathbb{B}^N$ of the unit ball of $\mathbb{C}^N$ (or of pseudoconvex domains in $\mathbb{C}^N$) which are *complete* in the sense that every curve in X terminating on the sphere $b\mathbb{B}^N$ has infinite length. Equivalently, the metric on X, induced from the Euclidean metric on $\mathbb{C}^N$ by the embedding $X \hookrightarrow \mathbb{C}^N$, is a complete metric.

The question whether there exist complete bounded complex submanifolds in Euclidean spaces was asked by Paul Yang in 1977, and the first examples were provided by Jones [76] in 1979. Recent results on this subject are due to Alarcón and the author [5], Alarcón and López [12], Drinovec Drnovšek [32], Globevnik [58, 59], Alarcón et al. [9], and Alarcón and Globevnik [8]. The last mentioned paper shows that properly embedded complete complex curves in the ball $\mathbb{B}^2$ can have any topology, while the paper [5] shows that any bordered Riemann surface admits a proper complete holomorphic immersion into $\mathbb{B}^2$ and a proper complete holomorphic embedding into $\mathbb{B}^3$. Drinovec Drnovšek [32] proved that every strongly pseudoconvex domain embeds as a complete complex submanifold of a high dimensional ball. Globevnik proved in [58] that for every $N > 1$ the ball $\mathbb{B}^N$ can be foliated by complete complex hypersurfaces given as level sets of a holomorphic function on $\mathbb{B}^N$.

The constructions in all these papers, except those of Globevnik [58, 59], rely on one of the following two methods:

(a) Riemann-Hilbert boundary values problem (see the proof of Theorem 4.1);
(b) holomorphic automorphisms of the ambient space $\mathbb{C}^N$.

Each of these methods can be used to increase the intrinsic boundary distance in an embedded or immersed submanifold. The first method has the advantage of preserving the complex structure (since one can control the placement of the image in $\mathbb{C}^N$ and keep it bounded), and the disadvantage of introducing self-intersections in the double dimension or below. The second method is precisely opposite—it does not introduce any self-intersections, but it also does not provide any control of the complex structure since one is must cut away pieces of the image manifold in order to keep it suitably bounded.

The first of these methods — the Riemann-Hilbert boundary value problem—has recently been applied very successfully in the theory of minimal surfaces in $\mathbb{R}^n$; we refer to the papers [3, 4, 6] and the references therein. On the other hand, ambient automorphisms cannot be applied in minimal surface theory since the only class of self-maps of $\mathbb{R}^n$ $(n > 2)$ mapping minimal surfaces to minimal surfaces are the rigid maps.

Globevnik's method in [58, 59] is different from both of the above. He showed that for every $N > 1$ there is a holomorphic function f on the ball $\mathbb{B}^N$ whose real part $\Re f$ is unbounded on every path of finite length that ends on $b\mathbb{B}^N$. It follows that every level set $M_c = \{f = c\}$ is a closed complete complex hypersurface in $\mathbb{B}^N$, and M_c is smooth for most values of c in view of Sard's lemma. The function f is constructed such that its real part grows sufficiently fast on a certain labyrinth $\Lambda \subset \mathbb{B}^N$, consisting of pairwise disjoint closed polygonal domains in real affine hyperplanes, such that every curve in $\mathbb{B}^N \setminus \Lambda$ which terminates on $b\mathbb{B}^N$ has infinite length. The advantage of his method is that it gives an affirmative answer to Yang's question in all dimensions and codimensions. The disadvatange is that one cannot control the topology or the complex structure of the level sets. By using instead holomorphic automorphisms in order to push a submanifold off the labyrinth Λ, Alarcón et al. [9] succeeded to obtain partial control of the topology of the embedded submanifold, and complete control in the case of complex curves [8].

By using the labyrinths constructed in [9, 58], Alarcón and the author showed in [7] that there exists a complete injective holomorphic immersion $\mathbb{C} \to \mathbb{C}^2$ whose image is everywhere dense in $\mathbb{C}^2$. The analogous result holds for any closed complex submanifold $X \subsetneqq \mathbb{C}^n$ for $n > 1$ in place of $\mathbb{C} \subset \mathbb{C}^2$. Furthermore, if X intersects the ball $\mathbb{B}^n$ and K is a connected compact subset of $X \cap \mathbb{B}^n$, then there is a Runge domain $\Omega \subset X$ containing K which admits a complete holomorphic embedding $\Omega \to \mathbb{B}^n$ whose image is dense in $\mathbb{B}^n$.

4.4. Submanifolds of the ball with exotic boundary behaviour. The boundary behavior of proper holomorphic maps between bounded pseudoconvex domains with smooth boundaries in complex Euclidean spaces has been studied extensively. It is generally believed that maps between domains of the same dimension always extend smoothly up to the boundary; we refer to the recent survey [85] for results on this subject. On the other hand, proper holomorphic maps into higher dimensional domains may have rather wild boundary behavior. For example, Globevnik [56] proved in 1987 that, given $n \in \mathbb{N}$, if $N \in \mathbb{N}$ is sufficiently large there exists a continuous map $g\colon \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^N$ which is holomorphic in $\mathbb{B}^n$ and satisfies $g(b\mathbb{B}^n) = b\mathbb{B}^N$.

It is natural to ask whether there is any relationship between the area and the boundary behavior of a submanifold. In particular, what can be said about the boundary behavior of properly embedded complex submanifolds of finite area in the ball? Recently, the author [43] constructed an example of a properly embedded holomorphic disc in the ball $\mathbb{B}^2 \subset \mathbb{C}^2$ which has arbitrarily small area (hence is the zero set of a bounded holomorphic function on $\mathbb{B}^2$ according to a result of Berndtsson [19]) and whose boundary curve is injectively immersed and everywhere dense in the sphere $b\mathbb{B}^2$. In fact, the embedding $\mathbb{D} \hookrightarrow \mathbb{B}^2$ constructed in [43] extends holomorphically across the boundary of the disc with the exception of (at least) one boundary point. Examples of immersed complex submanifolds with similar behavior were found earlier by Globevnik and Stout [61].

5. Soft Oka principle for proper holomorphic embeddings

By combining the lifting technique in the proof of Theorem 4.1 with methods from the papers by Slapar and the author [51, 50] on the *soft Oka principle*, one can prove the following result which does not seem available in the literature.

Theorem 5.1. *Let (X, J) and Y be Stein manifolds, where $J\colon TX \to TX$ denotes the complex structure operator on X. If $\dim Y > 2\dim X$ then for every continuous map $f\colon X \to Y$ there exists a Stein structure J' on X, homotopic to J, and a proper holomorphic embedding $f'\colon (X, J') \hookrightarrow Y$ homotopic to f. If $\dim Y \geq 2\dim X$ then f' can be chosen to be a proper holomorphic immersion having only simple double points. The same result holds if the manifold Y is q-complete for some $q \in \{1, 2, \ldots, \dim Y - 2\dim X + 1\}$, where $q = 1$ corresponds to Stein manifolds.*

The main result of [51] (see Theorems 1.1 and 1.2 in the cited paper) amounts to the same statement for holomorphic maps (instead of proper embeddings), but without any hypothesis on the target manifold Y. In order to obtain *proper* holomorphic maps $X \to Y$, we need a suitable geometric hypothesis on the manifold Y in view of the examples of noncompact (Oka) manifolds without any closed complex subvarieties [55, Example 9.8.3].

The results from [51, 50] were extended to 1-convex source manifolds X by Prezelj and Slapar [86]. For Stein manifolds X of complex dimension 2, the results in [51] also stipulate a change of the underlying $\mathscr{C}^\infty$ structure on X. It was later shown by Cieliebak and Eliashberg that this is not necessary if we begin with an integrable Stein structure; see [28, Theorem 8.43 and Remark 8.44]. A comprehensive analysis of exotic Stein structures on smooth orientable 4-manifolds was made by Gompf [62, 63, 64].

Sketch of proof of Theorem 5.1. In order to fully understand the proof, the reader should be familiar with [51, proof of Theorem 1.1] since the construction is quite involved and cannot be repeated here with all details. (Theorem 1.2 in the same paper gives an equivalent formulation where one does not change the Stein structure on X, but instead finds a desired holomorphic map on a Stein Runge domain $\Omega \subset X$ which is diffeotopic to X.) We explain the main step in the case $\dim Y > 2\dim X$; the theorem follows by using it inductively as in [51]. An interested reader is invited to provide the details.

Assume that $X_0 \subset X_1$ is a pair of relatively compact, smoothly bounded, strongly pseudoconvex domains in X such that there exists a strongly plurisubharmonic Morse function ρ on an open set $U \supset \overline{X_1 \setminus X_0}$ in X satisfying

$$X_0 \cap U = \{x \in U : \rho(x) < a\}, \quad X_1 \cap U = \{x \in U : \rho(x) < b\}$$

for a pair of constants $a < b$ and $d\rho \neq 0$ on $bX_0 \cup bX_1$. Also, let $L_0 \subset L_1$ be a pair of compact sets in Y. (In the induction, L_0 and L_1 are sublevel sets of a strongly q-convex exhaustion function on Y.) Let $f_0 : X \to Y$ be a continuous map whose restriction to a neighborhood of $\overline{X}_0$ is a J-holomorphic embedding satisfying $f_0(bX_0) \subset Y \setminus L_0$. The goal is to find a new Stein structure J_1 on X, homotopic to J by a smooth homotopy that is fixed in a neighborhood of $\overline{X}_0$, such that f_0 can be deformed to a map $f_1 : X \to Y$ whose restriction to a neighborhood of $\overline{X}_1$ is a J_1-holomorphic embedding satisfying

$$f_1(\overline{X_1 \setminus X_0}) \subset Y \setminus L_0, \qquad f_1(bX_1) \subset Y \setminus L_1 \tag{5.1}$$

and approximating f_0 uniformly on $\overline{X}_0$ as closely as desired. (The complex structure on the target manifold Y is kept fixed.) An obvious inductive application of this result proves Theorem 5.1 as explained in [51]. (For the case $\dim X = 2$, see also [28, Theorem 8.43 and Remark 8.44].) By subdividing the problem into finitely many steps of the same kind, it suffices to consider the following two basic cases:

(a) *The noncritical case:* $d\rho \neq 0$ on $\overline{X_1 \setminus X_0}$. In this case we say that X_1 is a *noncritical strongly pseudoconvex extension* of X_0.
(b) *The critical case:* ρ has exactly one critical point p in $\overline{X_1 \setminus X_0}$.

Let $U_0 \subset U_0' \subset X$ be a pair of small open neighborhoods of $\overline{X}_0$ such that f_0 is an embedding on U_0'. Also, let $U_1 \subset U_1' \subset X$ be small open neighborhoods of $\overline{X}_1$.

In case (a) there exists a smooth diffeomorphism $\phi : X \to X$ which is diffeotopic to the identity map on X by a diffeotopy which is fixed on $U_0 \cup (X \setminus U_1')$ such that $\phi(U_1) \subset U_0'$. The map $\tilde{f}_0 = f_0 \circ \phi : X \to Y$ is then a holomorphic embedding on the set U_1 with respect to the Stein structure $J_1 = \phi^*(J)$ on X (the pullback of J by ϕ). Applying the lifting procedure in the proof of Theorem 4.1 and up to shrinking U_1 around $\overline{X}_1$, we can homotopically deform $\tilde{f}_0$ to a continuous map $f_1 : X \to Y$ whose restriction to U_1 is a J_1-holomorphic embedding $U_1 \hookrightarrow Y$ satisfying conditions (5.1).

In case (b), the change of topology of the sublevel sets of ρ at the point p is described by attaching to the strongly pseudoconvex domain $\overline{X}_0$ a smoothly embedded totally real disc $M \subset X_1 \setminus X_0$, with $p \in M$ and $bM \subset bX_0$, whose dimension equals the Morse index of ρ at the critical point p. As shown in [36, 28, 51], M can be chosen such that $\overline{X}_0 \cup M$ has a basis of smooth strongly pseudoconvex neighborhoods (handlebodies) H which deformation retract onto $\overline{X}_0 \cup M$ such that X_1 is a noncritical strongly pseudoconvex extension of H. Furthermore, as explained in [51], we can homotopically deform the map $f_0 : X \to Y$, keeping it fixed in some neighborhood of $\overline{X}_0$, to a map that is holomorphic on H and maps $H \setminus \overline{X}_0$ to $L_1 \setminus L_0$. By Proposition 2.2 we can assume that the new map is a holomorphic embedding on H. This reduces case (b) to case (a).

In the inductive construction, we alternate the application of cases (a) and (b). If $\dim Y \geq 2 \dim X$ then the same procedure applies to immersions. □

A version of this construction, for embedding open Riemann surfaces into $\mathbb{C}^2$ (or $(\mathbb{C}^*)^2$) up to a deformation of their complex structure, can be found in the papers by Alarcón and López [11] and Ritter [89]. However, they use holomorphic automorphisms in order to push the boundary curves to infinity without introducing self-intersections of the image complex curve. (The technique in the proof of Theorem 4.1, which depends on the Riemann-Hilbert method, will in general introduce self-intersections in double dimension.)

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