**H-principle for complex contact structures on Stein manifolds**

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**Abstract** In this paper we prove that every formal complex contact structure on a Stein manifold $X$ is homotopic to a holomorphic contact structure on a Stein domain $Ω ⊂ X$ which is diffeotopic to $X$. We also prove a parametric h-principle in this setting, analogous to Gromov’s h-principle for contact structures on smooth open manifolds. On Stein threefolds we obtain a complete homotopy classification of formal complex contact structures. Our methods also furnish a parametric h-principle for germs of holomorphic contact structures along totally real submanifolds of class $C^2$ in arbitrary complex manifolds.

**Keywords** Stein manifold, complex contact structure, h-principle

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1. Introduction

A *complex contact manifold* is a pair $(X, ξ)$, where $X$ is a complex manifold of (necessarily) odd dimension $2n + 1 ≥ 3$ and $ξ$ is a completely nonintegrable holomorphic hyperplane subbundle (a *contact bundle*) of the holomorphic tangent bundle $TX$, meaning that the O’Neill tensor $ξ × ξ → TX/ξ$, $(v, w) ↦ [v, w]$ mod $ξ$, is nondegenerate. Note that $ξ = ker α$ where $α$ is a holomorphic 1-form on $X$ with values in the complex line bundle $L = TX/ξ$ (the normal bundle of $ξ$) which realises the quotient projection

$$0 → ξ ↪ TX → α → L → 0. \tag{1.1}$$

Thus, $α$ is a holomorphic section of the twisted cotangent bundle $TX ⊗ L$. The contact condition is equivalent to $α ∧ (dα)^n ≠ 0$ at every point of $X$. A theorem of Darboux [9] says that, locally at any point, $ξ$ is holomorphically contactomorphic to the standard contact bundle $ξ_{std} = ker α_{std}$ on $C^{2n+1}$ given by the 1-form $α_{std} = dz + \sum_{j=1}^{n} x_j dy_j$, where $(x, y, z)$ are complex coordinates on $C^{2n+1}$. (See also [30] or [23, p. 67] for the real case, and [2] Theorem A.2] for the holomorphic case.)

We denote by $Cont_{hol}(X)$ the space of all holomorphic contact forms on $X$, endowed with the compact-open topology. In this paper we consider the existence and homotopy classification of complex contact forms on Stein manifolds of dimension $2n + 1 ≥ 3$.

We begin by recalling a few general observations due to LeBrun and Salamon [33, 34]. If $α ∈ Cont_{hol}(X)$, then $ω = α ∧ (dα)^n$ is a holomorphic $(2n + 1)$-form on $X$ with values in the line bundle $L^{n+1} = L^\otimes(n+1)$, i.e., an element of $H^0(X, K_X ⊗ L^{n+1})$ where $K_X = Λ^{2n+1}TX$ is the canonical bundle of $X$.Being nowhere vanishing, $ω$ defines a holomorphic trivialisation of the line bundle $K_X ⊗ L^{n+1}$, so we conclude that

$$K_X^{−1} = K_X^* \cong L^{n+1}. \tag{1.2}$$
Similarly, \((da)^n|_ξ\) is a nowhere vanishing section of the line bundle \((Λ^{2n}ξ)^* \otimes L^n\) (i.e., \(dα|_ξ\) is an \(L\)-valued complex symplectic form on the bundle \(ξ\)), so we have that

\begin{equation}
Λ^{2n}ξ \cong L^n = (TX/ξ)^n.
\end{equation}

In particular, on a contact 3-fold \((X,ξ)\) we have \(Λ^2ξ \cong TX/ξ\). It is easily seen that conditions \([1.2]\) and \([1.3]\) are equivalent to each other. These facts impose strong restrictions on the existence of complex contact structures, especially on compact manifolds; see the survey by Beauville [3] and the introduction to [1].

Assume now that \(X\) is a Stein manifold of dimension \(2n + 1 \geq 3\). For a generic holomorphic 1-form \(α\) on \(X\), the equation \(α \wedge (da)^n = 0\) defines (a possibly empty) complex hypersurface \(Σ_α \subset X\), and \(α\) is a contact form on the Stein manifold \(X \setminus Σ_α\). This observation shows that there exist a plethora of Stein contact manifolds, but does not answer the question whether a given Stein manifold (or a given diffeomorphism class of Stein manifolds) admits a contact structure. More precisely, when is a complex hyperplane subbundle \(ξ \subset TX\) satisfying \([1.3]\) homotopic to a holomorphic contact subbundle?

The following notion is motivated by Gromov’s h-principle for real contact structures on smooth open manifolds (see \([30]\)).

**Definition 1.1.** Let \(X\) be a complex manifold of dimension \(2n + 1 \geq 3\). A formal complex contact structure on \(X\) is a pair \((α,β)\), where \(α\) is a smooth \((1,0)\)-form on \(X\) with values in a complex line bundle \(L \to X\) satisfying \([1.2]\), \(β\) is a smooth \((2,0)\)-form on \(X\) with values in \(L\), and

\begin{equation}
α \wedge β^n = α \wedge β \wedge \cdots \wedge β \neq 0 \quad \text{holds at each point of } X.
\end{equation}

Note that \(α\) is a nowhere vanishing section of the vector bundle \(T^*X \otimes L\) of rank \(\dim X\); such always exists if \(X\) is a Stein manifold of dimension \(\geq 2\). A \((2,0)\)-form \(β\) satisfying \([1.4]\) is an \(L\)-valued complex symplectic form on the complex \(2n\)-plane bundle \(ξ = \ker α \subset TX\), and \(α \wedge β^n\) is a topological trivialisation of \(K_X \otimes L^{n+1}\).

We denote by \(\text{Cont}_\text{hol}(X)\) the space of all formal contact structures on \(X\), endowed with the \(C^∞\) compact-open topology. We have the natural inclusion

\begin{equation}
\text{Cont}_\text{hol}(X) \hookrightarrow \text{Cont}_\text{for}(X), \quad α \mapsto (α, dα).
\end{equation}

The following is our first main result; it is proved in Sect. 6. (See also Theorem 6.1)

**Theorem 1.2.** Let \(X\) be a Stein manifold of odd dimension. Given \((α_0, β_0) \in \text{Cont}_\text{for}(X)\), there is a Stein domain \(Ω \subset X\), diffeotopic to \(X\), and a homotopy \((α_t, β_t) \in \text{Cont}_\text{for}(X)\) \((t \in [0, 1])\) such that \(α_t|_Ω \in \text{Cont}_\text{hol}(Ω)\) and \(β_1|_Ω = dα_1|_Ω\). Furthermore, if \(α_0, α_1 \in \text{Cont}_\text{hol}(X)\) are connected by a path in \(\text{Cont}_\text{for}(X)\), they are also connected by a path of holomorphic contact forms on some Stein domain \(Ω \subset X\) diffeotopic to \(X\).

A domain \(Ω \subset X\) is said to be diffeotopic to \(X\) if there is a smooth family of diffeomorphisms \(h_t : X \xrightarrow{∞} h_t(X) \subset X\) \((t \in [0, 1])\) such that \(h_0 = \text{Id}_X\) and \(h_1(X) = Ω\). If \(J\) denotes the complex structure operator on \(X\), then \(J_t = h_t^*(J)\) is a homotopy of complex structures on \(X\) with \(J_0 = J\) and \(J_1 = h_1^*(J|_Ω)\).

By Cieliebak and Eliashberg [8] Theorem 8.43 and Remark 8.44], the domain \(Ω\) and the diffeotopy \(\{h_t\}_{t \in [0, 1]}\) in Theorem 1.2 can be chosen such that the domain \(h_t(X) \subset X\) is Stein (equivalently, the manifold \((X, J_t)\) with \(J_t = h_t^*(J)\) is Stein) for every \(t \in [0, 1]\).
We also prove a parametric version of Theorem 1.2 (see Theorem 6.1) which says that a continuous compact family of formal complex contact structures on $X$ can be deformed to a continuous family of holomorphic contact structures on a Stein domain $\Omega \subset X$ diffeotopic to $X$, and the deformation may be kept fixed for those values of the parameter for which the given formal structure is already a holomorphic contact structure.

For real contact structures, Gromov’s h-principle [30] says that the inclusion (1.5) of the space of smooth contact forms into the space of formal contact forms is a weak homotopy equivalence on any smooth open manifold. In particular, every formal contact structure is homotopic to an honest contact structure. (See also Eliashberg and Mishachev [14, Sect. 10.3].) The situation is more complicated for closed manifolds as was discovered later by Bennequin [4] and Eliashberg [11, 13]. In particular, the h-principle for real contact structures fails on the 3-sphere, but it holds for the class of overtwisted contact structures on any compact orientable 3-manifold; see [11, Theorem 1.6.1]. This was extended to manifolds of dimensions $\geq 5$ by Borman, Eliashberg, and Murphy in 2015 [6].

Our results in the present paper seem to be the first analogues in the holomorphic category of the above mentioned Gromov’s h-principle. At this time we are unable to construct holomorphic contact forms on the whole Stein manifold under consideration. The main, and seemingly highly nontrivial problem arising in the proof, is the following. (Note that the analogous approximation problem for integrable holomorphic subbundles — holomorphic foliations — is also open in general; see [17, Problem 9.16.8].)

**Problem 1.3.** Given a holomorphic contact form $\alpha$ on an open neighbourhood of a compact convex set $K \subset \mathbb{C}^{2n+1}$, is it possible to approximate $\alpha$ uniformly on $K$ by holomorphic contact forms on $\mathbb{C}^{2n+1}$? Is such approximation also possible for any continuous family of holomorphic contact forms $\alpha_p$ with parameter $p \in P$ in a compact Hausdorff space?

This issue does not appear in the smooth case since one can pull back a contact structure on a neighbourhood $U$ of a compact convex set $K \subset \mathbb{R}^{2n+1}$ to a contact structure on $\mathbb{R}^{2n+1}$ by a diffeomorphism $\mathbb{R}^{2n+1} \to U$ which equals the identity near $K$.

By following the proof of Theorem 1.2 and using the gluing lemma for biholomorphic maps [17, Theorem 9.7.1]), we obtain the following result which is proved in Sect. 6.

**Theorem 1.4.** If Problem 1.3 has an affirmative answer, then every formal complex contact structure on a Stein manifold $X$ is homotopic to a holomorphic contact structure on $X$. Furthermore, if the parametric version of Problem 1.3 has an affirmative answer, then the inclusion (1.5) is a weak homotopy equivalence.

We now consider more carefully the case when $X$ is a Stein manifold with $\dim X = 3$. Let $L$ be a holomorphic line bundle on $X$ satisfying (1.2), i.e., such that $K_X \otimes L^2$ is a trivial line bundle. (Recall that every complex vector bundle on a Stein manifold carries a compatible structure of a holomorphic vector bundle by the Oka-Grauert principle; see [17, Theorem 5.3.1].) Note that $T^*X \otimes L$ admits a nowhere vanishing holomorphic section $\alpha$ (see [17, Corollary 8.3.2]), i.e., an $L$-valued holomorphic 1-form on $X$. Let $\xi = \ker \alpha \subset TX$. Then, $K_X \cong \Lambda^2\xi^* \otimes (TX/\xi)^* \cong \Lambda^2\xi^* \otimes L^*$. Since $K_X \cong (L^*)^2$ by the assumption, we see that $\Lambda^2\xi^* \otimes L$ is a trivial bundle (see (1.3)). A trivialisation of $\Lambda^2\xi^* \otimes L$ is a 2-form $\beta$ on $\xi$ with values in $L$ such that $\omega = \alpha \wedge \beta$ is a trivialisation of $K_X \otimes L^2$, i.e., $(\alpha, \beta) \in \text{Cont}_{\text{hol}}(X)$. Hence, the necessary condition (1.2) for the existence of an $L$-valued formal contact structure on $X$ is also sufficient when $X$ is Stein and $\dim X = 3$. 


We denote by $\text{Cont}_{\text{for}}(X, L)$ the subset of $\text{Cont}_{\text{for}}(X)$ given by pairs of $L$-valued forms $(\alpha, \beta) \in \text{Cont}_{\text{for}}(X)$. Clearly, $\text{Cont}_{\text{for}}(X, L)$ is a union of connected components of $\text{Cont}_{\text{for}}(X)$. We claim that the connected components of $\text{Cont}_{\text{for}}(X, L)$ coincide with the homotopy classes of topological trivialisations of $K_X \otimes L^2$. One direction is obvious: given a homotopy $(\alpha_t, \beta_t) \in \text{Cont}_{\text{for}}(X, L)$ with $t \in [0, 1]$, the family $\alpha_t \wedge \beta_t$ is a homotopy of trivialisations of $K_X \otimes L^2$. Conversely, assume that $(\alpha_0, \beta_0), (\alpha_1, \beta_1) \in \text{Cont}_{\text{for}}(X, L)$ and there is a homotopy $\omega_t$ of trivialisations of $K_X \otimes L^2$ with $\omega_0 = \alpha_0 \wedge \beta_0$ and $\omega_1 = \alpha_1 \wedge \beta_1$. Since $\dim X = 3$ and $X$ is Stein, it is homotopy equivalent to a 3-dimensional CW complex. A simple topological argument in the line of [17, proof of Corollary 8.3.2] then shows that $\alpha_0$ and $\alpha_1$ can be connected by a homotopy $\alpha_t$ of nowhere vanishing sections of $T^* X \otimes L$. Let $\xi_t = \ker \alpha_t \subset TX$ for $t \in [0, 1]$. Then, $\omega_t = \alpha_t \wedge \tilde{\beta}_t$ where $\tilde{\beta}_t$ is a trivialisation of $\Lambda^2 \xi_t^* \otimes L$ and $\tilde{\beta}_0 = \beta_0$. At $t = 1$ we have $\omega_1 = \alpha_1 \wedge \beta_1 = \alpha_1 \wedge \tilde{\beta}_1$, and it follows that $\tilde{\beta}_1|_{\xi_1} = \beta_1|_{\xi_1}$. This proves the claim.

Recall that the isomorphism classes of complex (or holomorphic) line bundles on a Stein manifold $X$ are in bijective correspondence with the elements of $H^2(X; \mathbb{Z})$ by Oka’s theorem (see [17, Theorem 5.2.2]). The above observations yield the following homotopy classification of formal complex contact structures on Stein threefolds.

**Proposition 1.5.** If $X$ is a Stein manifold of dimension 3, then the connected components of the space $\text{Cont}_{\text{for}}(X)$ of formal complex contact structures on $X$ are in one-to-one correspondence with the following pairs of data:

1. an isomorphism class of a complex line bundle $L$ on $X$ satisfying $L^2 \cong (K_X)^{-1}$, i.e., an element $c \in H^2(X; \mathbb{Z})$ with $2c = c_1(TX)$, and
2. a choice of a homotopy class of trivialisations of the line bundle $K_X \otimes L^2$, that is, an element of $[X, \mathbb{C}^*] = [X, S^1] = H^1(X; \mathbb{Z})$.

In particular, if $H^1(X; \mathbb{Z}) = 0$ and $H^2(X; \mathbb{Z}) = 0$ then the space $\text{Cont}_{\text{for}}(X)$ is connected; this holds in particular for $X = \mathbb{C}^3$.

Theorem 1.2 and Proposition 1.5 imply the following corollary.

**Corollary 1.6.** Let $X$ be a Stein manifold of dimension 3. Given a holomorphic line bundle $L$ on $X$ such that $(K_X)^{-1} \cong L^2$, there is a Stein domain $\Omega \subset X$ diffeotopic to $X$ and a holomorphic contact subbundle $\xi \subset T\Omega$ such that $T\Omega/\xi \cong L|_{\Omega}$. Furthermore, given a pair of holomorphic $L$-valued contact forms $\alpha_0, \alpha_1$ on $X$ such that $\alpha_1 \wedge d\alpha_1/\alpha_0 \wedge d\alpha_0 : X \to \mathbb{C}^*$ is null homotopic, there is a Stein domain $\Omega \subset X$ as above and a homotopy $\alpha_t \in \text{Cont}_{\text{hol}}(\Omega)$ ($t \in [0, 1]$) connecting $\alpha_0|_{\Omega}$ to $\alpha_1|_{\Omega}$.

Since in the above corollary we must pass to a Stein subdomain of $X$ when constructing contact structures and their homotopies, the following problem remains open.

**Problem 1.7.** Let $X$ be a Stein manifold of dimension 3 with $H^1(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) = 0$. Is the space $\text{Cont}_{\text{hol}}(X)$ connected? In particular, is $\text{Cont}_{\text{hol}}(\mathbb{C}^3)$ connected?

**Remark 1.8.** Corollary 1.6 gives a homotopy classification of contact forms on Stein 3-folds, but not necessarily of contact bundles. A holomorphic contact bundle $\xi$ on $X$ is determined by a holomorphic 1-form $\alpha$ up to a nonvanishing factor $f \in \mathcal{O}(X, \mathbb{C}^*)$. Since $f\alpha \wedge d(f\alpha) = f^2\alpha \wedge d\alpha$, this changes the trivialisation of $K_X \otimes L^2$ by $f^2$. (More generally, if $\dim X = 2n+1$ then the trivialisation of $K_X \otimes L^{n+1}$ given by $\alpha \wedge (d\alpha)^n$ changes by the factor $f^{n+1}$.) Hence, a homotopy class of holomorphic contact bundles on a Stein 3-fold $X$ is uniquely determined by a pair $(c, d)$, where $c \in H^2(X; \mathbb{Z})$ satisfies $2c = c_1(TX)$. 

and \(d \in H^1(X;\mathbb{Z})/2H^1(X;\mathbb{Z})\). By Corollary 1.6 every such pair is represented by a holomorphic contact bundle on a Stein domain \(\Omega \subset X\) diffeotopic to \(X\). \(\square\)

We do not have a comparatively good classification results for \(\text{Cont}_{\text{hol}}(X)\) on Stein manifolds of dimension five or more. Granted the necessary conditions (1.2), (1.3) for the normal bundle \(L\), the existence and classification of complex symplectic forms \(\beta\) on the \(2n\)-plane bundle \(\xi = \ker \alpha\) amounts to the analogous problem for sections of an associated fibre bundle with the fibre \(GL_{2n}(\mathbb{C})/Sp_{2n}(\mathbb{C})\). We do not pursue this issue here.

One may wonder to what extent is it possible to control the choice of the domain \(\Omega \subset X\) in Theorem 1.9 and Corollary 1.6. In our proof, \(\Omega\) arises as thin Stein neighbourhood of an embedded CW complex in \(X\) which represents its Morse complex, so it carries all topology of \(X\). However, since a Mergelyan type approximation theorem is used in the construction, we do not know how big \(\Omega\) can be. We describe the construction more precisely at the end of this introduction and supply references.

The method actually gives much more. Assume that \(X\) is an odd dimensional complex manifold (not necessarily Stein) and \(W \subset X\) is a tamely embedded CW complex of dimension at most \(\dim X\). (A suitable notion of tameness was introduced by Gompf [25, 26].) Let \((\alpha, \beta)\) be a formal contact structure on \(X\). After a small topological adjustment of \(W\) in \(X\), there is a holomorphic contact form \(\tilde{\alpha} \in \text{Cont}_{\text{hol}}(\Omega)\) on a Stein thickening \(\Omega \subset X\) of \(W\) such that \((\tilde{\alpha}, d\tilde{\alpha})\) is homotopic to \((\alpha, \beta)\) in \(\text{Cont}_{\text{for}}(\Omega)\).

This is illustrated most clearly by looking at holomorphic contact structures in neighbourhoods of totally real submanifolds. A real submanifold \(M\) of class \(\mathcal{C}^1\) in a complex manifold \(X\) is said to be totally real if the tangent space \(T_xX\) at any point \(x \in M\) (a real vector subspace of \(T_xX\)) does not contain any complex line. By Grauert [27], such \(M\) admits a basis of tubular Stein neighbourhoods in \(X\), the so called Grauert tubes. Note that every smooth \(n\)-manifold \(M\) is a totally real submanifold of a Stein \(n\)-manifold: take the compatible real analytic structure on \(M\), let \(M^\mathbb{C}\) be its complexification, and choose \(X\) to be a Grauert tube around \(M\) in \(M^\mathbb{C}\).

The following is the 1-parametric h-principle for germs of complex contact structure along a totally real submanifold; see Theorem 4.1 for the fully parametric case.

**Theorem 1.9.** Let \(M\) be a totally real submanifold of class \(\mathcal{C}^2\) in a complex manifold \(X\). Every formal complex contact structure \((\alpha_0, \beta_0) \in \text{Cont}_{\text{for}}(X)\) is homotopic in \(\text{Cont}_{\text{for}}(X)\) to a holomorphic contact structure \((\alpha, d\alpha)\) in a tubular Stein neighbourhood of \(M\) in \(X\). Furthermore, any two holomorphic contact forms \(\alpha_0, \alpha_1\) in a neighbourhood of \(M\) which are formally homotopic along \(M\) are also homotopic by a family of holomorphic contact forms \(\alpha_t \in \text{Cont}_{\text{hol}}(\Omega)\) \((t \in [0, 1])\) in a Stein neighbourhood \(\Omega \subset X\) of \(M\).

In dimension 3 we have the following simpler statement in view of Proposition 1.5

**Corollary 1.10.** Let \(X\) be a 3-dimensional complex manifold and \(M \subset X\) be a totally real submanifold of class \(\mathcal{C}^2\). Then, germs of complex contact forms on \(X\) along \(M\) are classified up to homotopy by pairs consisting of a complex line bundle \(L\) over a neighbourhood of \(M\) satisfying \(L^2|_M \cong (K_X)^{-1}|_M\) and an element of \(H^1(M;\mathbb{Z})\).

When \(M\) is a totally real submanifold of maximal dimension \(n\) in a complex \(n\)-manifold \(X\), we have \(TX|_M = TM^\mathbb{C} = TM \oplus TM\) (since the complex structure operator \(J\) on \(TX\) induces an isomorphism of the tangent bundle \(TM\) onto the normal bundle of \(M\) in \(X\)). Replacing \(X\) by a Grauert tube around \(M\), it follows that \(c_1(TX) = c_1(TX|_M) = \).
$c_1(TM^\mathbb{C})$, so the canonical class of $X$ only depends on $M$. We shall see in Example 1.12 that this is not the case in general for totally real submanifolds of lower dimension.

**Example 1.11.** Let $X$ be a Grauert tube around the 3-sphere $S^3$ in its complexification. Then, $H^1(X;\mathbb{Z}) = H^1(S^3;\mathbb{Z}) = 0$ and $H^2(X;\mathbb{Z}) = H^2(S^3;\mathbb{Z}) = 0$. By Corollary 1.10 there is a unique homotopy class of germs of complex contact structures around $S^3$ in $X$. We get it for instance by taking a totally real embedding of $S^3$ into $\mathbb{C}^3$ (see [22] Theorem 1.4) or [29] p. 193] and using the standard complex contact form $dz + xdy$ on $\mathbb{C}^3$.

It was shown by Eliashberg [11] that there exist countably many homotopy classes of smooth contact structures on $S^3$. By choosing them real analytic, we can complexify them to obtain holomorphic contact structures in neighbourhoods of $S^3$ in $X$. By what has been said above, these are homotopic to each other as holomorphic contact bundles. □

**Example 1.12.** Let $Y$ be a Grauert tube around the 2-sphere $S^2$ in its complexification. We have $TY|_{S^2} = TS^2 \oplus TS^2$ which is trivial, so $TY$ is holomorphically trivial by the Oka-Grauert principle. Let $\pi: X \to Y$ be a holomorphic line bundle; the isomorphism classes of such bundles correspond to the elements of $H^2(Y;\mathbb{Z}) = H^2(S^2;\mathbb{Z}) = \mathbb{Z}$. Considering $Y$ as the zero section of $X$, we can view $Y$ as the normal bundle $N_{Y,X}$ of $Y$ in $X$. Since $TY$ is trivial, the adjunction formula for the canonical bundle gives

$$K_X|_Y \cong K_Y \otimes (N_{Y,X})^{-1} = X^{-1}.$$  

For each choice of the bundle $X \to Y$ with even Chern number $c_1(X) \in H^2(Y;\mathbb{Z}) = \mathbb{Z}$, $(K_X)^{-1}$ has a unique holomorphic square root $L$ with $c_1(L) = \frac{1}{2}c_1(X)$. By Corollary 1.10 there is a holomorphic $L$-valued contact form on a neighbourhood of $S^2$ in $X$. A Stein tube around $S^2$ in the trivial bundle $X = Y \times \mathbb{C}$ can be represented as a domain in $\mathbb{C}^3$, for example, as a tube around the standard 2-sphere $S^2 \subset \mathbb{R}^3 \subset \mathbb{C}^3$. The examples with nonzero Chern classes clearly cannot be represented as domains in $\mathbb{C}^3$. □

**Example 1.13.** Let $X$ be a 3-dimensional Grauert tube around an embedded circle $S^1 \subset X$. In this case $H^2(X;\mathbb{Z}) = H^2(S^3;\mathbb{Z}) = 0$, and by Corollary 1.10 the homotopy classes of holomorphic contact forms along $S^1$ are classified by $H^1(X;\mathbb{Z}) = H^1(S^3;\mathbb{Z}) = \mathbb{Z}$. We can see them explicitly on $X = \mathbb{C}^* \times \mathbb{C}^2$ as follows. Let $(x, y, z)$ be complex coordinates on $\mathbb{C}^3$. Set $S^1 = \{(x, 0, 0) \in \mathbb{C}^3 : |x| = 1\}$. For each $k \in \mathbb{Z}$ let

$$\alpha_k = \begin{cases} dz + \frac{x^{k+1}}{x+1}dy & \text{if } k \neq -1, \\ \frac{1}{2} i (dz + xdy) & \text{if } k = -1. \end{cases}$$

Then $\alpha_k \wedge d\alpha_k = x^kdx \wedge dy \wedge dz$ for every $k \in \mathbb{Z}$, so the homotopy class of the corresponding framing of the trivial bundle $X \times \mathbb{C} \to X$ equals $k$. By Remark 1.8, the contact bundle $\xi_k = \ker \alpha_k$ on $\mathbb{C}^* \times \mathbb{C}^2$ is homotopic to $\xi_0$ if $k$ is even, and to $\xi_1 \cong \xi_{-1}$ is $k$ is odd. The bundles $\xi_0$ and $\xi_1$ are not homotopic to each other through contact bundles.

Note that the form $\alpha_k$ for $k \neq -1$ is the pullback of $\alpha_0 = dz + xdy$ (the standard contact form on $\mathbb{C}^3$) by the covering map $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^2$, $(x, y, z) \mapsto (x^{k+1}/(k+1), y, z)$. In order to understand $\alpha_{-1}$, consider the contact form on $\mathbb{C}^3$ given by

$$\beta = \cos x \; dz + \sin x \; dy.$$  

It defines the standard structure on $\mathbb{C}^3$, because it is the pullback of $dz - ydx$ by the automorphism $(x, y, z) \mapsto (x, y \cos x - z \sin x, y \sin x + z \cos x)$. Let $F: \mathbb{C}^3 \to \mathbb{C}^* \times \mathbb{C}^2$ denote the universal covering map $F(x, y, z) = (e^{ix}, y, z)$. A calculation shows that
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\[ \beta = F^* \alpha', \] where \( \alpha' \) is the contact form on \( \mathbb{C}^* \times \mathbb{C}^2 \) given by

\[ \alpha' = \frac{1}{2} \left( \frac{x}{1} + 1 \right) dz + \frac{1}{2i} \left( x - \frac{1}{x} \right) dy, \quad \alpha' \wedge d\alpha' = \frac{1}{ix} dx \wedge dy \wedge dz. \]

Then, \( \alpha_{-1} \) is homotopic to \( \alpha' \) through the family of contact forms on \( \mathbb{C}^* \times \mathbb{C}^2 \) defined by

\[ \sigma_t = \frac{1}{\sqrt{2(1+t^2)}} \left( \left( tx + \frac{1}{x} \right) dz + \left( x - \frac{1}{x} \right) e^{-int/2} dy \right), \quad t \in [0,1]. \]

We have \( \sigma_0 = \alpha_{-1}, \sigma_1 = \alpha', \) and \( \sigma_t \wedge d\sigma_t = e^{-int/2} x^{-1} dx \wedge dy \wedge dz \) for all \( t \in [0,1] \).

**Example 1.14.** The previous example can be generalised to \((\mathbb{C}^*)^2 \times \mathbb{C}\) and \((\mathbb{C}^*)^3\) which are complexifications of the 2-torus and the 3-torus, respectively. Let us consider the latter. Denote by \( T^k \) the \( k \)-dimensional torus, the product of \( k \) copies of \( S^1 \). The domain \( X = (\mathbb{C}^*)^3 \) is a Stein tube around the standard totally real embedding \( T^3 \hookrightarrow \mathbb{C}^3 \) onto the distinguished boundary of the polidisc. We have \( H^2(X;\mathbb{Z}) = H^2(T^3;\mathbb{Z}) = \mathbb{Z}^3 \) and \( H^1(X;\mathbb{Z}) = H^1(T^3;\mathbb{Z}) = \mathbb{Z} \) (see Rotman [37, p. 404]). Clearly, \( K_X \) is trivial, and since \( H^2(X;\mathbb{Z}) \) is a free abelian group, it’s only square root is the trivial bundle. Hence by (1.2) all contact forms on \( X \) have values in the trivial bundle, and we have \( \mathbb{Z}^3 \)-many homotopy classes of trivialisations. Consider the following family of contact forms on \( X = (\mathbb{C}^*)^3 \), where \( (k,l,m) \in \mathbb{Z}^3 \):

\[ \alpha_{k,l,m} = \begin{cases} z^m dz + \frac{1}{k+1} x^{k+1} y^l dy & \text{if } k \neq -1, \\ \frac{1}{k} z^m dz + xy^l dy & \text{if } k = -1, \end{cases} \]

We have that \( \alpha_{k,l,m} \wedge d\alpha_{k,l,m} = x^k y^l z^m dx \wedge dy \wedge dz \), so this family provides all possible homotopy classes of framings of the trivial bundle \( X \times \mathbb{C} \).

The above examples suggest that in many natural cases one can find globally defined holomorphic contact forms representing all homotopy classes in Proposition 1.5.

**Problem 1.15.** Is it possible to represent every homotopy class of formal complex contact structures on an affine algebraic manifold by an algebraic contact form?

Our proofs of Theorems 1.9 and 4.1 proceed by triangulating the manifold \( M \) and inductively deforming a formal contact structure \((\alpha, \beta)\) to an almost contact structure along \( M \). We show that the open partial differential relation of first order, controlling the almost contact condition on a totally real disc, is ample in the coordinate directions; see Lemma 2.1. Hence, Gromov’s h-principle [29, 31] can be applied to extend an almost contact structure from the boundary of a cell to the interior, provided that it extends as a formal contact structure; see Lemma 2.2. Finally, approximating an almost contact form \( \alpha \) on \( M \) sufficiently closely in the fine \( \mathcal{C}^\infty \) topology by a holomorphic 1-form \( \tilde{\alpha} \) ensures that \( \tilde{\alpha} \) is a contact form in a neighbourhood of \( M \) in \( X \). The same arguments apply to families of such structures, thereby yielding the parametric h-principle in Theorem 4.1.

A similar method is used to prove Theorems 1.2 and 6.1 (see Sect. 6). The inductive step amounts to extending a holomorphic contact form \( \alpha \) from a neighbourhood of a compact strongly pseudoconvex domain \( W \) in \( X \) across a handle whose core is a totally real disc \( M \) attached with its boundary sphere \( bM \) to \( bW \). More precisely, \( M \setminus bM \subset X \setminus W \), the attachment is \( J \)-orthogonal along \( bM \), where \( J \) denotes the almost complex structure on \( X \), and \( bM \) is a Legendrian submanifold of the strongly pseudoconvex hypersurface \( bW \) with its smooth contact structure given by complex tangent planes. The union \( W \cup M \) then admits a basis of tubular Stein neighbourhoods (see [12] and [13]). Assuming that \( \alpha \) extends to \( M \)
as a formal contact structure, Lemma 4.3 furnishes an almost contact extension. Finally, by Mergelyan’s theorem we can approximate \( \alpha \) in the \( \mathcal{C}^1 \) topology on \( W \cup M \) by a holomorphic contact form \( \tilde{\alpha} \) on a Stein neighbourhood of \( W \cup M \).

With these analytic tools in hand, Theorems 1.2 and 6.1 are proved by following the scheme developed by Eliashberg [12] in his landmark construction of Stein manifold structures on any smooth almost complex manifold \( (X,J) \) with the correct handlebody structure. (The special case \( \dim X = 2 \) is rather different and was explained by Gompf [24] [25] [26], but this is not relevant here.) A more precise explanation of Eliashberg’s construction was given by Slapar and the author [20] [21] in their proof of the soft Oka principle for maps from any Stein manifold \( X \) to an arbitrary complex manifold \( Y \). Expositions are also available in the monographs by Cieliebak and Eliashberg [8] Chap. 8 and the author [17] Secs. 10.9–10.11.

Finally, the proof of Theorem 1.4 (see Sect. 6) follows the induction scheme used in Oka theory; see [17] Sect. 5. Besides the tools already mentioned above, an additional ingredient is a gluing lemma for holomorphic contact forms (see Lemma 6.2).

2. Germs of complex contact structures on domains in \( \mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1} \)

We denote the complex variables on \( \mathbb{C}^n \) by \( z = (z_1, \ldots, z_n) \) with \( z_i = x_i + iy_i \) for \( i = 1, \ldots, n \), where \( i = \sqrt{-1} \). We shall consider \( \mathbb{R}^n \) as the standard real subspace of \( \mathbb{C}^n \).

Let \( D \) be a compact set in \( \mathbb{R}^{2n+1} (n \in \mathbb{N}) \) which is the closure of a domain with piecewise \( \mathcal{C}^1 \) boundary. In this section we consider the problem of approximating a complex contact structure \( \alpha \), defined on a neighbourhood of a compact subset \( \Gamma \subset bD \), by a complex contact structure \( \tilde{\alpha} \) defined on a neighbourhood of \( D \) in \( \mathbb{C}^{2n+1} \), provided that \( \alpha \) admits a formal contact extension to \( D \) in the sense of Definition 1.1. (For applications in this paper, it suffices to consider the case when \( D \) is the standard handle \( D^m \times D^d \subset \mathbb{R}^{2n+1} \) of some index \( m \in \{1, \ldots, 2n+1\} \) and \( d = 2n+1-m \), where \( D^m \subset \mathbb{R}^m \) and \( D^d \subset \mathbb{R}^d \) are closed unit balls in the respective spaces, and \( \Gamma = bD^m \times D^d \) is the attaching set of the handle.) We will show that the parametric h-principle holds in this problem (see Lemma 2.1).

We begin with preliminaries. Let \( l \in \mathbb{N} \), and let \( K \) be a closed set in a complex manifold \( X \). A function \( f \) of class \( \mathcal{C}^l \) on an open neighbourhood \( U \subset X \) of \( K \) is said to be \( \partial \)-flat to order \( l \) on \( K \) if the jet of \( \partial f \) of order \( l-1 \) vanishes at each point of \( K \). In any system of local holomorphic coordinates \( z = (z_1, \ldots, z_n) : V \to \mathbb{C}^n \) on \( X \) centred at a point \( x_0 \in K \), this means that the value and all partial derivatives of order up to \( l-1 \) of the functions \( \partial f / \partial z_j = \frac{1}{2} (\partial f_{x_j} + i \partial f_{y_j}) \) \( (j = 1, \ldots, n) \) vanish at each point \( x \in K \cap V \). In particular, such \( f \) satisfies the Cauchy-Riemann equations at every point \( x \in K \cap V \):

\[
\frac{\partial f}{\partial z_j}(x) = \frac{\partial f}{\partial x_j}(x) = -i \frac{\partial f}{\partial y_j}(x), \quad j = 1, \ldots, n.
\]

If \( f \) is smooth of class \( \mathcal{C}^\infty \) and the above holds for all \( l \in \mathbb{N} \), then \( f \) is said to be \( \overline{\partial} \)-flat (to infinite order) on \( K \).

Assume now that \( D \subset \mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1} \) is a compact domain with piecewise \( \mathcal{C}^1 \) boundary in \( \mathbb{R}^{2n+1} \). It is classical (see e.g. [22] Lemma 4.3) or [8] Proposition 5.55]) that every function \( f : D \to \mathbb{C} \) of class \( \mathcal{C}^l \) extends to a \( \mathcal{C}^l \) function \( \tilde{f} : \mathbb{C}^{2n+1} \to \mathbb{C} \) which is \( \overline{\partial} \)-flat to order \( l \) on \( D \). When \( f \) is of class \( \mathcal{C}^\infty \), we can obtain such an extension explicitly.
by first extending \( f \) to a smooth function on \( \mathbb{R}^{2n+1} \) and setting
\[
F(x + iy) = \sum_{|I| \leq l} \frac{1}{I!} \frac{\partial^{|I|} f}{\partial x^{I}}(x) i^{I} y^{I} = f(x) + i \sum_{i=1}^{2n+1} \frac{\partial f}{\partial x_i}(x)y_i + O(|y|^2).
\]
Here, \( I = (i_1, \ldots, i_{2n+1}) \in \mathbb{Z}^{2n+1}_+ \), \( |I| = i_1 + \cdots + i_{2n+1}, \frac{\partial^{|I|} f}{\partial x^{I}}(x) = \frac{\partial^{i_1} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_{2n+1}}}, \) and \( y^{I} = y_1^{i_1} \cdots y_n^{i_{2n+1}}. \) If \( f \) is only of class \( \mathcal{C}^l \) then a \( \overline{\partial} \)-flat extension is obtained by applying Whitney’s jet-extension theorem \([39]\) to the jet on the right hand side above.

A smooth differential \((1,0)\)-form
\[
\alpha = \sum_{i=1}^{2n+1} a_i(z) dz_i
\]
on a neighbourhood of \( D \) in \( \mathbb{C}^{2n+1} \) is said to be \( \overline{\partial} \)-flat to order \( l \) on \( D \) is every coefficient function \( a_i \) is such. Every smooth \((1,0)\)-form defined on \( D \subset \mathbb{R}^{2n+1} \) extends to a \( \overline{\partial} \)-flat \((1,0)\)-form on \( \mathbb{C}^{2n+1} \) by taking \( \overline{\partial} \)-flat extensions of its coefficient. Assume that \( \alpha \) is such. In view of the CR equations we have for each \( x \in D \) that
\[
d\alpha(x) = \partial \alpha(x) = \sum_{1 \leq i < j \leq 2n+1} \left( \frac{\partial a_j}{\partial x_i}(x) - \frac{\partial a_i}{\partial x_j}(x) \right) dz_i \wedge dz_j.
\]
Write \( p_{i,j}(x) = \frac{\partial a_j}{\partial x_i}(x) \) and set
\[
\beta_{i,j}(x) := p_{j,i}(x) - p_{i,j}(x) = \frac{\partial a_j}{\partial x_i}(x) - \frac{\partial a_i}{\partial x_j}(x).
\]
With this notation, we have for all \( x \in D \) that
\[
d\alpha(x) = \beta(x) = \sum_{1 \leq i < j \leq 2n+1} \beta_{i,j}(x) dz_i \wedge dz_j,
\]
and
\[
(d\alpha)^n(x) = \beta^n(x) = \sum_{i=1}^{2n+1} b_i(x) dz_1 \wedge \cdots \wedge \hat{dz_i} \cdots \wedge dz_{2n+1},
\]
where \( \hat{dz_i} \) indicates that this term is omitted. Every coefficient \( b_i(x) \) in \((2.5)\) is a homogeneous polynomial of order \( n \) in the coefficients \( \beta_{i,k} \) of \( \beta = d\alpha \) \((2.2)\), obtained as follows. Let \( P = \{A_1, \ldots, A_n\} \) be a partition of the set \( \{1, 2, \ldots, 2n+1\} \setminus \{i\} \) into a union of \( n \) pairs \( A_k = (i_k, j_k) \) \((k = 1, \ldots, n)\), with \( i_k < j_k \). Then,
\[
b_i(x) = n! \sum_{P \subset \{1, 2, \ldots, 2n+1\} \setminus \{i\}} \prod_{(i_k, j_k) \in P} \beta_{i_k,j_k}(x) = n! \sum_{P \subset \{1, 2, \ldots, 2n+1\} \setminus \{i\}} \prod_{(i_k, j_k) \in P} (p_{j_k,i_k}(x) - p_{i_k,j_k}(x))
\]
for all \( x \in D \). Finally, from \((2.2)\) and \((2.5)\) we obtain for all \( x \in D \) that
\[
\alpha(x) \wedge (d\alpha)^n(x) = \alpha(x) \wedge \beta^n(x) = \left( \sum_{i=1}^{2n+1} (-1)^{i-1} a_i(x) b_i(x) \right) dz_1 \wedge \cdots \wedge dz_{2n+1}.
\]
A smooth \((1,0)\)-form \( \alpha \) on \( \mathbb{C}^{2n+1} \), defined on a neighbourhood of \( D \subset \mathbb{R}^{2n+1} \) and \( \overline{\partial} \)-flat on \( D \) to the first order, is said to be an almost contact form on \( D \) if
\[
\alpha \wedge (d\alpha)^n \neq 0 \text{ at every point of } D.
\]
Note that $d\alpha|_D = \partial\alpha|_D$. Approximating $\alpha$ sufficiently closely in the $\mathcal{C}^1$ topology on $D$ by a holomorphic 1-form $\tilde{\alpha}$ gives a holomorphic contact structure $\tilde{\xi} = \ker \tilde{\alpha}$ on a neighbourhood of $D$ in $\mathbb{C}^{2n+1}$. If the coefficients of $\alpha$ are real analytic, then the complexification of $\alpha$ defines a holomorphic contact structure near $D$.

We see from $(2.3)$, $(2.6)$, and $(2.7)$ that the condition $(2.8)$ depends only on the first order jet of the restrictions $a|_D$ of the coefficients of $\alpha$ to $D$, so it defines an open set in the space of 1-jets of 1-forms on $D$. More precisely, we may view $a|_D$ as a smooth section $x \mapsto (x, a_1(x), \ldots, a_{2n+1}(x))$ of the trivial bundle $E = D \times \mathbb{C}^{2n+1} \to D$. Let $E^{(1)} \to E$ be the bundle of 1-jets of sections of $E \to D$. The fibre of $E^{(1)}$ over a point $(x, a) \in E = D \times \mathbb{C}^{2n+1}$ (with $a = (a_1, \ldots, a_{2n+1})$) consists of all matrices $p = (p_{ij}) \in \mathbb{C}^{2n+1} \times \mathbb{C}^{2n+1}$, where $a_j : D \to \mathbb{C}^{2n+1}$ and $p : D \to \mathbb{C}^{2n+1} \times \mathbb{C}^{2n+1}$. Such a section is said to be holonomic if $p(x)$ is the 1-jet of $a(x)$ for each $x \in D$, that is, $p_{ij}(x) = \frac{\partial a_i}{\partial \xi^j}(x)$ for all $i, j = 1, \ldots, 2n + 1$. Let $\mathcal{R}$ be the open subset of $E^{(1)}$ defined by

$$R = \left\{(x, a, p) \in E^{(1)} : \sum_{i=1}^{2n+1} (-1)^{i-1} a_i b_i \neq 0\right\},$$

where each $b_i$ is determined by $p = (p_{ij})$ according to the formula $(2.6)$ (ignoring the base point $x$). Thus, $\mathcal{R}$ is an open subset of $E^{(1)}$ which controls the contact condition for $\mathcal{B}$-flat 1-forms along $D$.

**Lemma 2.1.** The partial differential relation $\mathcal{R}$ defined by $(2.9)$ is ample in the coordinate directions (in the sense of M. Gromov $(31)(29)$).

**Proof.** Choose an index $i \in \{1, \ldots, 2n + 1\}$. Write $p = (p_1, \ldots, p_{2n+1})$ and $p_i = (p_j, \ldots, p_{2n+1}) \in \mathbb{C}^{2n+1}$ for $j = 1, \ldots, 2n + 1$. Consider a restricted 1-jet of the form $e = (x, a, p_1, \ldots, \tilde{p}_i, \ldots, p_{2n+1})$ where the vector $p_i$ is omitted. Set

$$R_e = \left\{p_i \in \mathbb{C}^{2n+1} : (x, a, p_1, \ldots, p_i, 0, \ldots, p_{2n+1}) \in \mathcal{R}\right\},$$

The differential relation $\mathcal{R}$ is said to be ample in the coordinate directions if every set $\mathcal{R}_e$ of this type is either empty, or else the convex hull of each of its connected components equals $\mathbb{C}^{2n+1}$. In the case at hand, we see from $(2.6)$ and $(2.7)$ that the function

$$h(a, p) = \sum_{j=1}^{2n+1} (-1)^{j-1} a_j b_j(p),$$

where $b_j = b_j(p)$ is determined by $(2.6)$, is affine linear in $p_i = (p_{i+1}, \ldots, p_{2n+1})$. Indeed, every $p_{i, j}$ appears at most once in each of the products in $(2.6)$. Since

$$\mathcal{R}_e = \left\{p_i \in \mathbb{C}^{2n+1} : h(a, p_1, \ldots, p_i, 0, \ldots, p_{2n+1}) \neq 0\right\},$$

it follows that $\mathcal{R}_e$ is either empty or else the complement of a complex affine hyperplane in $\mathbb{C}^{2n+1}$; in the latter case its convex hull equals $\mathbb{C}^{2n+1}$. This proves Lemma 2.1.

In order to apply this lemma, we need the following observation. Let $\alpha$ be a 1-form $(2.1)$ with smooth coefficients $a = (a_1, \ldots, a_{2n+1}) : D \to \mathbb{C}^{2n+1}$, and let

$$\beta(x) = \sum_{1 \leq i < j \leq 2n+1} \beta_{i,j}(x) dz_i \wedge dz_j, \quad x \in D,$$
be a smooth 2-form on $D$. (At this point we consider forms with values in the trivial line bundle.) Note that the linear projection $\mathbb{C}^{(2n+1)^2} \ni (p_{i,j}) \mapsto (\beta_{i,j} = p_{j,i} - p_{i,j}) \in \mathbb{C}^n(2n+1)$ is surjective and hence a Serre fibration, i.e., it enjoys the homotopy lifting property. In particular, we may write $\beta_{i,j} = p_{j,i} - p_{i,j}$ for some smooth functions $p_{i,j}$ on $D$. Let $p = (p_{i,j}) : D \to \mathbb{C}^{(2n+1)^2}$. It then follows from the definition of the differential relation $\mathcal{R}$ (see (2.9)) that $(\alpha, \beta)$ is a formal contact structure on $D$ (see Definition 1.1), i.e.,

\begin{equation}
\alpha \wedge \beta^n \neq 0 \quad \text{on } D,
\end{equation}

if and only if the map $x \mapsto (x, a(x), p(x))$ is a (not necessarily holonomic) section of $\mathcal{R}$.

A seminal result of M. Gromov says that sections of an ample open differential relation $\mathcal{R}$ of first order satisfy all forms of the h-principle (see [29] Sect. 2.4, [14] Sect. 18.2, or [38] Theorem 4.2). This means that every section of $\mathcal{R}$ is homotopic through sections of $\mathcal{R}$ to a holonomic section, the homotopy can be chosen fixed on a compact subset of the base domain where the given section is already holonomic, and a similar statement holds for families of sections, where the homotopy is kept fixed on the set of holonomic sections. The basic technical result is the following; we state it for the case at hand. (See for instance [31] Lemma 3.1.3, p. 339) which is stated for the special case when $D$ is a compact cube and $\Gamma = \partial D$; the general case follows by induction on a suitable triangulation of the pair $(D, \Gamma)$. A brief survey is also available in [17, Sect. 1.10].)

**Lemma 2.2.** Let $D \subset \mathbb{R}^{2n+1}$ be a compact domain with piecewise $C^1$ boundary, and let $\Gamma \subset \partial D$ be the closure of an open subset of $\partial D$ with piecewise $C^1$ boundary. Assume that $\alpha$ is a smooth $\partial$-flat $(1, 0)$-form on $D$ (see (2.1)) and $\beta$ is a smooth $(2, 0)$-form on $D$ (2.11) such that (2.12) holds and also $d\alpha(x) = \beta(x)$ for all $x \in \Gamma$, i.e.,

$$
\beta_{i,j}(x) = \frac{\partial \alpha_i}{\partial x_j}(x) - \frac{\partial \alpha_j}{\partial x_i}(x) \quad \text{for all } x \in \Gamma \text{ and } i, j = 1, \ldots, 2n + 1.
$$

Given $\epsilon > 0$ there is a homotopy $(\alpha_t, \beta_t) \ (t \in [0, 1])$ of pairs of forms of the same type satisfying the following conditions.

(i) $(\alpha_0, \beta_0) = (\alpha, \beta)$.
(ii) $\alpha_t(x) \wedge \beta_t(x)^n \neq 0$ for all $x \in D$ and $t \in [0, 1]$.
(iii) $|\alpha_t(x) - \alpha(x)| < \epsilon$ for all $x \in D$ and $t \in [0, 1]$.
(iv) The homotopy is fixed for $x \in \Gamma$.
(v) $\beta_1 = d\alpha_1$ holds at all points of $D$, i.e., $\alpha_1$ is an almost contact form on $D$.

Assume furthermore that $P$ is a compact Hausdorff space, $Q \subset P$ is a closed subspace, and $\{(\alpha_p, \beta_p)\}_{p \in P}$ is a continuous family of data as above such that for every $p \in Q$ we have that $d\alpha_p = \beta_p$ on $D$. Then, there is a homotopy $(\alpha_{p,t}, \beta_{p,t}) \ (t \in [0, 1])$ which is fixed (independent of $t$) for every $p \in Q$ and satisfies conditions (i)–(v) for every $p \in P$.

In condition (iii) we use the Euclidean norm for the coefficient vector of the form $\alpha_t - \alpha$, i.e., $\alpha_t$ is uniformly $\epsilon$-close to $\alpha = \alpha_0$ on $D$ for all $t \in [0, 1]$. Note however that in general $\alpha_1$ cannot be chosen $C^1$-close to $\alpha$.

### 3. Asymptotically holomorphic and almost contact forms

We now introduce a general notion of an almost contact form along a closed subset $M$ in a complex manifold $X$ (see Definition 3.3). This is necessary since we shall be applying coordinate changes which are asymptotically holomorphic on $M$, but not necessarily
holomorphic. For simplicity we discuss scalar valued forms, although the same notions apply to differential forms with values in any holomorphic line bundle on $X$. However, Lemma 3.2 and Corollary 3.5 only apply to scalar valued forms and will be used locally.

A smooth differential $m$-form $\alpha$ on a complex manifold $X$ decomposes uniquely as the sum $\alpha = \sum_{p+q=m} \alpha^{p,q}$ of its $(p, q)$-homogeneous parts. In local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on $X$ we have

$$\alpha^{p,q} = \sum a_{I,J} \, dz_1 \wedge \cdots \wedge dz_p \wedge \bar{dz}_1 \wedge \cdots \wedge \bar{dz}_q$$

for some smooth coefficient functions $a_{I,J}$. In particular, for a 1-form $\alpha$ we have

$$\alpha = \sum_{i=1}^n a_i \, dz_i + \sum_{i=1}^n b_i \, d\bar{z}_i = \alpha^{1,0} + \alpha^{0,1}.$$  

The exterior derivative of $X$ splits as $d = \partial + \bar{\partial}$. If $\alpha$ is a 1-form then

$$(d\alpha)^{2,0} = \partial \alpha^{1,0}, \quad (d\alpha)^{1,1} = \partial \alpha^{0,1} + \bar{\partial} \alpha^{1,0}, \quad (d\alpha)^{0,2} = \bar{\partial} \alpha^{0,1}.$$  

**Definition 3.1.** Let $M$ be a closed subset of a complex manifold $X$.  

(a) A smooth $m$-form $\alpha$, defined on a neighbourhood of $M$ in $X$, is of type $(m, 0)$ on $M$ if $\alpha|_M = \alpha^{m,0}|_M$.  

The space of all such forms on variable neighbourhoods of $M$ is denoted $E^{m,0}(M, X)$.  

(b) A smooth 1-form $\alpha$, defined on a neighbourhood of $M$ in $X$, is asymptotically holomorphic (of order 1) on $M$ if for every point $x_0 \in M$ there is a holomorphic coordinate system on $X$ around $x_0$ in which $\alpha$ has the form (3.1) and the following conditions hold for $i = 1, \ldots, n$:

$$\bar{\partial} a_i(x_0) = 0, \quad b_i(x_0) = 0, \quad db_i(x_0) = 0.$$  

The space of all such forms on variable neighbourhoods of $M$ is denoted $AH^1(M, X)$.  

The first two conditions in (3.2) are equivalent to $\alpha \in E^{1,0}(M, X)$ and $\bar{\partial} \alpha^{1,0}|_M = 0$, so $d\alpha^{1,0}|_M = \partial \alpha^{1,0}|_M$. The last condition in (3.2) implies $d\alpha^{0,1}|_M = 0$, but the converse is not true since $\bar{\partial} \alpha^{0,1}|_M = 0$ holds under the weaker condition $\frac{\partial a_i}{\partial \bar{z}_k} = \frac{d b_i}{d \bar{z}_k}$ on $M$ for all $i, k = 1, \ldots, n$. In particular, we have that

$$AH^1(M, X) \subset \{ \alpha \in E^{1,0}(M, X) : d\alpha^{1,0}|_M = 0, \quad \bar{\partial} \alpha^{1,0}|_M = 0, \quad d\alpha \in E^{2,0}(M, X) \}.$$  

Assume now that $X$ and $Y$ are complex manifolds and $F : X \to Y$ is smooth map. Let $M$ be a closed subset of $X$. We say $F$ is $\bar{\partial}$-flat (or asymptotically holomorphic) to order $k \in \mathbb{N}$ on $M$ if, in any pair of holomorphic coordinates on the two manifolds, we have

$$D^{k-1}(\bar{\partial} F)|_M = 0,$$

where $D^{k-1}$ is the total derivative of order $k - 1$ applied to the components $\partial F_i/\partial \bar{z}_j$ of $\bar{\partial} F$. Chain rule shows that this notion is independent of the choice of coordinates.

The following lemma shows in particular that conditions (3.2) defining the class $AH^1(M, X)$ is invariant under $\bar{\partial}$-flat coordinate changes.

**Lemma 3.2.** Assume that $X$ and $Y$ are complex manifolds and $F : X \to Y$ is $C^2$ map which is $\bar{\partial}$-flat to order 2 on a closed subset $M \subset X$. Set $M' = \overline{F(M)} \subset Y$. If $\alpha \in AH^1(M', Y)$ then $F^* \alpha \in AH^1(M, X)$ and

$$d(F^* \alpha)|_M = \partial ((F^* \alpha)^{1,0})|_M = F^* (\partial \alpha^{1,0}|_M).$$
Proof. Fix a point \( x_0 \in M \subset X \) and let \( y_0 = F(x_0) \in M' \subset Y \). By the assumption there are holomorphic coordinates \( w = (w_1, \ldots, w_n) \) on a neighborhood \( U \) of \( y_0 \) in \( Y \) such that
\[
\alpha = \sum_{i=1}^{n} a_i \, dw_i + \sum_{i=1}^{n} b_i \, d\bar{w}_i = \alpha^{1,0} + \alpha^{0,1},
\]
where the coefficients satisfy the following conditions (see (3.2)):
\[
\bar{\partial} a_i(y_0) = 0, \quad b_i(y_0) = 0, \quad db_i(y_0) = 0.
\]
The pullback form \( \tilde{\alpha} = F^* \alpha \) on \( F^{-1}(U) \subset X \) equals
\[
\tilde{\alpha} = \sum_{i=1}^{n} \left[ (a_i \circ F) \, dF_i + (b_i \circ F) \, d\bar{F}_i \right]
= \sum_{i=1}^{n} \left[ (a_i \circ F) \, \partial F_i + (b_i \circ F) \, \partial \bar{F}_i \right] + \sum_{i=1}^{n} \left[ (a_i \circ F) \, \bar{\partial} F_i + (b_i \circ F) \, \bar{\partial} \bar{F}_i \right]
= \tilde{\alpha}^{1,0} + \tilde{\alpha}^{0,1}.
\]
At the point \( x_0 \in M \) we have \( b_i \circ F(x_0) = 0 \) and \( \bar{\partial} F_i(x_0) = 0 \) for all \( i \), and hence
\[
\tilde{\alpha}^{1,0}(x_0) = \sum_{i=1}^{n} a_i(y_0) \partial F_i(x_0) = F^*(\alpha^{1,0})(x_0), \quad \tilde{\alpha}^{0,1}(x_0) = 0.
\]
Furthermore, since \( db_i(y_0) = 0 \) and \( d(\bar{\partial} F_i)(x_0) = 0 \) for all \( i \), a simple calculation shows that the coefficients of \( \tilde{\alpha}^{0,1} \) in any holomorphic coordinate system on \( X \) around \( x_0 \) vanish to second order at \( x_0 \). Finally, consider the \((1,1)\)-form
\[
\bar{\partial} \tilde{\alpha}^{1,0} = \sum_{i=1}^{n} \left[ \bar{\partial} (a_i \circ F) \wedge \partial F_i + (a_i \circ F) \bar{\partial} \partial F_i + \bar{\partial} (b_i \circ F) \wedge \partial \bar{F}_i + (b_i \circ F) \bar{\partial} \bar{\partial} F_i \right].
\]
We have that
\[
\bar{\partial} (a_i \circ F)(x_0) = \sum_{k=1}^{m} \left( \frac{\partial a_i}{\partial w_k}(y_0) \bar{\partial} F_k(x_0) + \frac{\partial a_i}{\partial \bar{w}_k}(y_0) \bar{\partial} (\bar{F}_k)(x_0) \right) = 0,
\]
so the first term in the above sum for \( \bar{\partial} \tilde{\alpha}^{1,0} \) vanishes at \( x_0 \). The other terms vanish as well since \( F \) is \( \bar{\partial} \)-flat to the second order at \( x_0 \). This shows that \( \tilde{\alpha} = F^* \alpha \) is asymptotically holomorphic at \( x_0 \). Since the point \( x_0 \in M \) was arbitrary, this completes the proof. \( \square \)

**Definition 3.3.** Let \( X^{2n+1} \) be a complex manifold and \( M \) be a closed subset of \( X \).

(a) A pair \((\alpha, \beta)\) with \( \alpha \in \mathcal{E}^{1,0}(M, X) \) and \( \beta \in \mathcal{E}^{2,0}(M, X) \) (see Definition 3.1) is a formal complex contact structure on \( M \) if
\[
\alpha \wedge \beta^n = \alpha^{1,0} \wedge (\beta^{2,0})^n \neq 0 \quad \text{holds at every point of } M.
\]
We denote by \( \text{Cont}_{\text{for}}(M, X) \) the space of formal contact structures on \( M \subset X \).

(b) An asymptotically holomorphic 1-form \( \alpha \in \text{AH}^1(M, X) \) (see Definition 3.1(b)) is an almost contact form on \( M \) if
\[
\alpha \wedge (d\alpha)^n \neq 0 \quad \text{holds at every point of } M.
\]
We denote the space of almost contact forms on \( M \) by \( \text{AC}(M, X) \).
Remark 3.4. Note that for every \((\alpha, \beta) \in \text{Cont}_{\text{for}}(M, X)\) the pair \((\alpha^{1,0}, \beta^{2,0})\) is a formal contact structure on an open neighbourhood of \(M\) in \(X\) (since \((3.4)\) is an open condition). Likewise, \(\text{AC}(M, X)\) is an open subset of \(\text{AH}^1(M, X)\) in the fine \(\mathcal{C}^1\) topology on \(M\). For \(\alpha \in \text{AH}^1(M)\), the almost contact condition \((3.5)\) is equivalent to
\[
\alpha^{1,0} \wedge (d\alpha^{1,0})^n = \alpha^{1,0} \wedge (\partial\alpha^{1,0})^n \neq 0 \quad \text{on } M.
\]
Hence, this notion generalises the one introduced in Sect. \([2]\) as in particular \((2.8)\).

The next corollary follows immediately from the definitions and Lemma \([3.2]\).

Corollary 3.5. Suppose that \(X\) and \(Y\) are complex manifolds of dimension \(2n + 1\), \(M\) is a closed subset of \(X\), and \(F: X \to Y\) is a diffeomorphism which is \(\overline{\partial}\)-flat to order \(2\) on \(M\).

(a) If \((\alpha, \beta) \in \text{Cont}_{\text{for}}(F(M), Y)\) then \((F^*\alpha, F^*\beta) \in \text{Cont}_{\text{for}}(M, X)\).

(b) If \(\alpha \in \text{AC}(F(M), Y)\) then \(F^*\alpha \in \text{AC}(M, X)\).

4. Complex contact structures near totally real submanifolds

In this section we prove the following parametric h-principle for complex contact structures along any totally real submanifold \(M\) of class \(\mathcal{C}^2\) in a complex manifold \(X^{2n+1}\). This subsumes the basic h-principle given by Theorem \([1.9]\).

Theorem 4.1. Let \(M\) be a topologically closed totally real submanifold of class \(\mathcal{C}^2\) (possibly with boundary) in a complex manifold \(X^{2n+1}\). Assume that \(P\) is a compact Hausdorff space and \(Q \subset P\) is a closed subspace. Let \((\alpha_p, \beta_p) \in \text{Cont}_{\text{for}}(X)\) \((p \in P)\) be a continuous family of formal complex contact structures with values in a holomorphic line bundle \(L\) on \(X\) (see Definition \([3.3]\)) such that for every \(p \in Q\), \(\alpha_p \in \text{Cont}_{\text{hol}}(X)\) and \(\beta_p = d\alpha_p\). Then, there exist a Stein neighbourhood \(\Omega \subset X\) of \(M\) and a homotopy \((\alpha_{p,t}, \beta_{p,t}) \in \text{Cont}_{\text{for}}(X)\) \((p \in P, t \in [0, 1])\) satisfying the following conditions.

(a) \((\alpha_{p,0}, \beta_{p,0}) = (\alpha_p, \beta_p)\) for all \(p \in P\).

(b) The homotopy is fixed for all \(p \in Q\).

(c) \(\alpha_{p,1|\Omega} \in \text{Cont}_{\text{hol}}(\Omega)\) and \(\beta_{p,1} = d\alpha_{p,1}\) on \(\Omega\) for all \(p \in P\).

The proof is based on Lemma \([2.2]\) and the results from Sect. \([3]\) along with some well known results concerning totally real submanifolds which we now recall.

Assume that \(M\) is a topologically closed totally real submanifold of class \(\mathcal{C}^k\) \((k \in \mathbb{N})\), possibly with boundary, in a complex manifold \(X\). Every function \(f \in \mathcal{C}^k(M)\) extends to a function \(F \in \mathcal{C}^k(X)\) which is \(\mathcal{C}^\infty\) smooth in \(X \setminus M\) and \(\overline{\partial}\)-flat to order \(k - 1\) on \(M\) (cf. \([3.3]\)):
\[
D^{k-1}(\overline{\partial}F)|_M = 0.
\]
(See \([32]\) Lemma 4.3 or \([5]\) Lemma 4, p. 148.) The analogous extension theorem holds for maps \(f: M \to Y\) of class \(\mathcal{C}^k\) to an arbitrary complex manifold — such \(f\) extends to a map \(F: U \to Y\) on an open tubular Stein neighbourhood \(U \subset X\) of \(M\) such that \(F\) is \(\overline{\partial}\)-flat to order \(k - 1\) on \(M\). Indeed, the graph of \(f\) admits a Stein neighbourhood in \(X \times Y\) according to Grauert \([27]\), so the proof reduces to the case of functions by applying the embedding theorem for Stein manifolds into Euclidean spaces and the Docquier-Grauert tubular neighbourhood theorem \([10]\). (See e.g. \([17]\) proof of Corollary 3.5.6.)

Let \(T^C M\) denote the complexified tangent bundle of \(M\), considered as a complex vector subbundle of \(TX|_M\) of rank \(m = \dim_{\mathbb{R}} M\). The quotient bundle \(\nu_M = TX|_M/T^C M\) is
the complex normal bundle of $M$ in $X$; it can be realised as a complex vector subbundle of $TX|_M$ such that $TX|_M = T^C M \oplus \nu_M$. Given a diffeomorphism $f : M_0 \to M_1$ between totally real submanifolds $M_0 \subset X$ and $M_1 \subset Y$, where $X$ and $Y$ are complex manifolds of the same dimension, we say that the complex normal bundles $\pi_i : \nu_i \to M_i$ ($i = 0, 1$) are isomorphic over $f$ if there exists an isomorphism of complex vector bundles $\phi : \nu_0 \to \nu_1$ satisfying $\pi_1 \circ \phi = f \circ \pi_0$. (We refer to [19, Sect. 2] for further details on this subject.)

The following result is implicitly contained in [19, proof of Theorem 1.2].

**Proposition 4.2.** Let $X$ and $Y$ be complex manifolds of the same dimension $n$, and let $f : M_0 \to M_1$ be a diffeomorphism of class $\mathcal{C}^k$ ($k \in \mathbb{N}$) between $\mathcal{C}^k$ totally real submanifolds $M_0 \subset X$ and $M_1 \subset Y$. If the complex normal bundles $\pi_i : \nu_i \to M_i$ ($i = 0, 1$) are isomorphic over $f$, then $f$ extends to a $\mathcal{C}^k$ diffeomorphism $F : U \to F(U) \subset Y$ on a neighbourhood $U \subset X$ of $M_0$ such that $F$ is $\partial$-flat to order $k$ on $M$. Such extension always exists if $M_0$ (and hence $M_1$) is contractible, or if $M_0$ has maximal dimension $n$.

**Proof of Theorem 4.1.** For simplicity of exposition we consider the nonparametric case (with $P$ a singleton and $Q = \emptyset$); the parametric case follows by the same arguments.

We proceed in two steps. In the first step, we deform the given formal contact structure to one that is almost contact on $M$. Here we use the h-principle furnished by Lemma 4.3 and the results in Sect. 3. In the second step we approximate the almost contact form on $M$ by a holomorphic contact form in a neighbourhood of $M$.

The first step is accomplished by the following lemma.

**Lemma 4.3** (H-principle for almost contact structures on totally real submanifolds). Let $M$ be a closed totally real submanifold of class $\mathcal{C}^2$ (possibly with boundary) in a complex manifold $X^{2n+1}$. Given $(\alpha_0, \beta_0) \in \text{Cont}_{\text{for}}(X)$, there is a homotopy $(\alpha_t, \beta_t) \in \text{Cont}_{\text{for}}(M, X)$ ($t \in [0, 1]$) such that $(\alpha_0, \beta_0)$ is the given initial pair, $\alpha_1 \in \text{AC}(M, X)$, and $\beta_1|_M = d\alpha_1|_M = \partial \alpha_1|_M$. If $M$ has nonempty piecewise $\mathcal{C}^1$ boundary $bM$ and we have $\alpha_0|_{bM} \in \text{AC}(bM, X)$ and $\beta_0|_{bM} = d\alpha_0|_{bM}$, then the homotopy $(\alpha_t, \beta_t)$ may be chosen fixed on $bM$. The analogous result holds in the parametric case.

Assume for a moment that Lemma 4.3 holds and let us complete the proof of Theorem 4.1. In view of Remark 3.4 there is an neighbourhood $U \subset X$ of $M$ such that $(\alpha_t^{1,0}, \beta_t^{2,0}) \in \text{Cont}_{\text{for}}(U)$ for $t \in [0, 1]$. Hence, we may assume that $\alpha_t = \alpha_t^{1,0}$ and $\beta_t = \beta_t^{2,0}$ in $U$. By the hypothesis we also have $\alpha_1 \in \text{AC}(M, X)$ and $\beta_1|_M = \partial \alpha_1|_M$, and by a homotopic deformation (shrinking $U$ if necessary) we may assume that $\beta_1 = \partial \alpha_1$ on $U$.

In the next step, we find a smaller neighbourhood $U' \subset U$ of $M$ and a homotopy in $\text{Cont}_{\text{for}}(U')$ from $(\alpha_1, \partial \alpha_1)$ to $(\tilde{\alpha}, \partial \tilde{\alpha})$ where $\tilde{\alpha} \in \text{Cont}_{\text{hol}}(U')$. This can be done by approximating $\alpha_1$ sufficiently closely in the fine $\mathcal{C}^1$ topology on $M$ by a holomorphic 1-form $\tilde{\alpha}$ in a neighbourhood of $M$ and setting

$$\tilde{\alpha}_t = (1-t)\alpha_1 + t\tilde{\alpha}, \quad \tilde{\beta}_t = \partial \tilde{\alpha}_t = (1-t)\partial \alpha_1 + t\partial \tilde{\alpha}$$

for $t \in [0, 1]$. Holomorphic approximation results for functions in the fine topology on totally real manifolds are well known, see for instance Manne, Øvrelid and Wold [35] and the survey [15]. These results also apply to sections of holomorphic vector bundles as shown in [17] proof of Theorem 2.8.4.

Finally, the homotopy in $\text{Cont}_{\text{for}}(U')$ from $(\alpha_0, \beta_0)$ to $(\tilde{\alpha}, \partial \tilde{\alpha})$, constructed above, can be extended to all of $X$ in a standard way by using a cut-off function on $X$ in the parameter
of the homotopy, thereby yielding a homotopy in $\text{Cont}_{for}(X)$ which equals the given one in a smaller Stein neighbourhood $\Omega \subset U'$ of $M$ and it agrees with $(\alpha_0, \beta_0)$ on $X \setminus U'$.

Assuming that Lemma 4.3 holds, this completes the proof of Theorem 4.1. The parametric case follows the same pattern and we omit the details. □

**Proof of Lemma 4.3.** Choose a triangulation of $M$ and let $M_k$ denote its $k$-dimensional skeleton, i.e., the union of all cells of dimension at most $k$. Assume inductively that for some $k < m = \dim M$ we have already found a homotopy in $\text{Cont}_{for}(M, X)$ from $(\alpha_0, \beta_0)$ to $(\alpha, \beta) \in \text{Cont}_{for}(M, X)$ satisfying

$$\alpha \in AC(M_k, X), \quad \beta|_{M_k} = d\alpha|_{M_k}, \quad \alpha \wedge (d\alpha)^n|_{M_k} \neq 0.$$ 

The inductive step amounts to deforming $(\alpha, \beta)$ by a homotopy in $\text{Cont}_{for}(M, X)$ that is fixed on $M_k$ to another pair $(\tilde{\alpha}, \tilde{\beta}) \in \text{Cont}_{for}(M, X)$ such that

$$\tilde{\alpha} \in AC(M_{k+1}, X), \quad \tilde{\beta}|_{M_{k+1}} = d\tilde{\alpha}|_{M_{k+1}}, \quad \tilde{\alpha} \wedge (d\tilde{\alpha})^n|_{M_{k+1}} \neq 0.$$ 

This can be done by applying Lemma 2.2 successively on each $(k+1)$-dimensional cell $C^{k+1}$ in the given triangulation of $M$; we now explain the details.

Let $L \to X$ be the holomorphic line bundle such that $\alpha_0, \beta_0$ have values in $L$. Note that $L$ is holomorphically trivial over a neighbourhood of the cell $C^{k+1}$ by the Oka-Grauert principle, so we may consider all our $L$-valued differential forms to be scalar valued there. The cell $C^{k+1}$ is diffeomorphic to a compact contractible domain $D^{k+1} \subset \mathbb{R}^{k+1}$ as in Lemma 2.2. We identify $\mathbb{R}^{k+1}$ with $\mathbb{R}^{k+1} \times \{0\} \subset \mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1}$. Since $M$ is totally real and of class $\mathcal{C}^2$, any diffeomorphism $F: C^{k+1} \to D^{k+1}$ of class $\mathcal{C}^2$ extends to a diffeomorphism $F$ from a neighbourhood of $C^{k+1}$ in $X$ onto a neighbourhood of $D^{k+1}$ in $\mathbb{C}^{2n+1}$ which is $\bar{\partial}$-flat to order 2 on $C^{k+1}$ (see Proposition 4.2). The inverse $G = F^{-1}$ is then $\bar{\partial}$-flat to order 2 on $D^{k+1}$. By Corollary 3.5 we have that

(i) $(G^*\alpha, G^*\beta) \in \text{Cont}_{for}(D_{k+1}, \mathbb{C}^{2n+1})$,

(ii) $G^*\alpha \in AC(bD^{k+1}, \mathbb{C}^{2n+1})$, and

(iii) $G^*\beta = d(G^*\alpha)$ holds at all points of $bD^{k+1}$.

By Lemma 2.2 we can deform $(G^*\alpha, G^*\beta)$ by a homotopy in $\text{Cont}_{for}(D^{k+1}, \mathbb{C}^{2n+1})$ that is fixed on $bD^{k+1}$ to an element $(\alpha', \beta') \in \text{Cont}_{for}(D^{k+1}, \mathbb{C}^{2n+1})$ such that $\alpha' \in AC(D^{k+1}, \mathbb{C}^{2n+1})$ and $\beta' = d\alpha'$ on $D_{k+1}$. (Lemma 2.2 applies verbatim if $k+1 = m = 2n+1$. If $k+1 < 2n+1$, we can apply it on $D^{k+1} \times r\mathbb{D}^{2n-k}$ for some $r > 0$, where $\mathbb{D}^{2n-k}$ is the closed ball around the origin in $\mathbb{R}^{2n-k}$. We can extend $G^*\alpha$ to an element of $AH^1(D^{k+1} \times r\mathbb{D}^{2n-k}, \mathbb{C}^{2n+1})$ whose restriction to $bD^{k+1} \times r\mathbb{D}^{2n-k}$ belongs to $AC(bD^{k+1} \times r\mathbb{D}^{2n-k}, \mathbb{C}^{2n+1})$ and apply Lemma 2.2 to this extension.) By Corollary 3.5 we have $F^*\alpha' \in AC(C^{k+1}, X)$ and $d(F^*\alpha') = F^*\beta'$ on $C^{k+1}$. We also use $F^*$ to transfer the homotopy in $\text{Cont}_{for}(D_{k+1}, \mathbb{C}^{2n+1})$, connecting $(G^*\alpha, G^*\beta)$ to $(\alpha', \beta')$, to a homotopy in $\text{Cont}_{for}(C^{k+1}, X)$ which is fixed on $bC^{k+1}$ and connects $(\alpha, \beta)$ to $(F^*\alpha', F^*\beta')$.

This completes the basic induction step. Applying this procedure successively on each $(k+1)$-cell in the given triangulation of $M$ yields a desired almost complex structure $\tilde{\alpha} \in AC(M_{k+1}, X)$. In the final step when $k+1 = m$ we obtain an element of $AC(M, X)$.

Clearly all steps can be carried out with a continuous dependence on a parameter, and by using cut-off functions on the parameter space we can ensure that the homotopy is fixed for the parameter values $p \in Q$. This yields the corresponding parametric h-principle. □
5. Extending a complex contact structure across a totally real handle

Recall that a compact set in a complex manifold $X$ is called a Stein compact if it admits a basis of open Stein neighbourhoods in $X$. The following lemma provides a key induction step in the proof of Theorems 1.2, 1.4, and 6.1.

**Lemma 5.1.** Let $K$ and $S = K \cup M$ be Stein compacts in a complex manifold $X^{2n+1}$, where $M = \overline{S} \setminus K$ is an embedded totally real submanifold of class $\mathcal{C}^2$. Let $(\alpha, \beta) \in \text{Cont}_{\text{for}}(X)$ be a formal contact structure with values in a holomorphic line bundle $L$. Assume that there is an open neighbourhood $U \subset X$ of $K$ such that $\alpha|_U \in \text{Cont}_{\text{hol}}(U)$ and $\beta|_U = d\alpha|_U$. Then, there exist a neighbourhood $\Omega_0 \subset U$ of $K$, a Stein neighbourhood $\Omega \subset X$ of $S$, and a homotopy $(\alpha_t, \beta_t) \in \text{Cont}_{\text{for}}(X) \ (t \in [0,1])$ satisfying the following conditions.

(i) $(\alpha_0, \beta_0) = (\alpha, \beta)$ on $\Omega_0$.
(ii) $\alpha_t|_{\Omega_0} \in \text{Cont}_{\text{hol}}(\Omega_0)$ and $\beta_t|_{\Omega_0} = d\alpha_t|_{\Omega_0}$ for all $t \in [0,1]$.
(iii) $\alpha_t$ approximates $\alpha$ as closely as desired uniformly on $K$ and uniformly in $t \in [0,1]$.
(iv) $\alpha_1|_{\Omega} \in \text{Cont}_{\text{hol}}(\Omega)$ and $\beta_1|_{\Omega} = d\alpha_1|_{\Omega}$.

The analogous result holds for a continuous family $\{(\alpha_p, \beta_p)\}_{p \in P} \subset \text{Cont}_{\text{for}}(X)$ where $P$ is a compact Hausdorff space; the homotopy may be kept fixed for the parameter values in a closed subset $Q \subset P$ such that $\alpha_p \in \text{Cont}_{\text{hol}}(X)$ for all $p \in Q$.

**Proof.** Let $U \subset X$ be a neighbourhood of $K$ as in the statement of the lemma; in particular, $\alpha|_U \in \text{Cont}_{\text{hol}}(U)$. Choose a smoothly bounded closed domain $M_0 \subset M$ such that $bM_0 \subset U$. By Lemma 4.3 we can deform $(\alpha, \beta)$ through a family of formal contact structures $(\alpha_t, \beta_t) \in \text{Cont}_{\text{for}}(X)$ such that the deformation is fixed on a neighbourhood of the compact set $K' := K \cup M \setminus M_0 \subset U$, and at $t = 1$ we have that $\alpha_1|_{M_0} \in AC(M_0, X)$ and $\beta_1|_{M_0} = d\alpha_1|_{M_0}$. Note that $\alpha_1$ is holomorphic on a neighbourhood of $K'$ (where it equals $\alpha_0$) and is asymptotically holomorphic along $M$.

By the Mergelyan approximation theorem, we can approximate $\alpha_1$ and its 1-jet along $M$ as closely as desired in the $\mathcal{C}^1$ topology on $S = K \cup M$ by an $L$-valued holomorphic 1-form $\tilde{\alpha}_1$ defined on a neighbourhood of $S$. We refer to [15] Theorem 20 for the relevant version of Mergelyan’s theorem. (In the cited source the reader can also find references to the previous works; see in particular Manne, Øvrelid and Wold [35]. The proof of [15] Theorem 20 easily adapts to provide jet-approximation; see Chenoweth [7] Proposition 7.) Although the cited results are stated for functions, they also hold for sections of holomorphic vector bundles over Stein domains as shown in [17] proof of Theorem 2.8.4). If the approximation of $\alpha_1$ by $\tilde{\alpha}_1$ is close enough on $S$, the family $(1-t)\alpha_1 + t\tilde{\alpha}_1 \ (t \in [0,1])$ is a homotopy of holomorphic contact forms in a neighbourhood of $K'$, and its restriction to $M_0$ is a homotopy in the space $AC(M_0)$ of almost contact forms on $M_0$.

By combining the homotopies from these two steps, we get a homotopy $(\alpha_t, \beta_t)$ in a neighbourhood $V \subset X$ of $S = K \cup M$ satisfying the conclusion of the lemma. Finally, by inserting a smooth cutoff function on $X$ into the parameter of the homotopy, we can glue the resulting homotopy with $(\alpha_0, \beta_0) = (\alpha, \beta)$ outside a Stein neighbourhood $\Omega \subset V$ of $S$.

It is clear that the same proof applies in the parametric situation. The main ingredients are the parametric version of Lemma 2.2 and a parametric version of Mergelyan’s theorem from [15] Theorem 20. The latter is easily obtained from the basic (nonparametric) case by applying a continuous partition of unity on the parameter space. (Compare with the proof of the parametric Oka-Weil theorem in [17] Theorem 2.8.4.)
6. Proofs of the main results

Proof of Theorem 1.2 We follow the scheme explained in the paper [21] by Slapar and the author; see in particular the proof of Theorem 1.2 in the cited source. Complete expositions of this construction can also be found in [8] Chap. 8 and [17] Secs. 10.9–10.11.

Choose a strongly plurisubharmonic Morse exhaustion function \( \rho: X \to \mathbb{R}_+ \). Let \( p_0, p_1, p_2, \ldots \in X \) be the critical points of \( \rho \) with \( \rho(p_0) < \rho(p_1) < \cdots \); thus \( p_0 \) is a minimum of \( \rho \). Choose numbers \( c_j \in \mathbb{R} \) satisfying

\[
\rho(p_0) < c_0 < \rho(p_1) < c_1 < \rho(p_2) < c_2 < \ldots .
\]

For each \( j = 0, 1, \ldots \) we set \( X_j = \{ x \in X : \rho(x) < c_j \} \). Note that \( \rho \) has a unique critical point \( p_j \) in \( X_j \setminus X_{j-1} \) for each \( j = 1, 2, \ldots \). (If \( \rho \) has only finitely many critical points \( p_0, \ldots, p_m \), the process described in the sequel will stop after \( m + 1 \) steps and the domain \( X_m = \{ \rho < c_m \} \) is diffeotopic to \( X \). This is always the case if \( X \) is an affine algebraic manifold.) By choosing the number \( c_0 \) close enough to \( \rho(p_0) \) we can arrange by a homotopy in \( \text{Cont}_\text{hol}(X) \) that \( \alpha_0 \) is a holomorphic contact form in a neighbourhood of the set \( \overline{X}_0 = \{ \rho \leq c_0 \} \) and \( \beta_0 = d\alpha_0 \) there.

Fix a number \( \epsilon > 0 \). We shall inductively construct the following objects:

(a) an increasing sequence of relatively compact, smoothly bounded, strongly pseudoconvex domains \( W_0 \subset W_1 \subset W_2 \subset \cdots \) in \( X \), with \( W_0 = X_0 \),

(b) a sequence of formal contact structures \( (\alpha_j, \beta_j) \in \text{Cont}_\text{hol}(X) \) \( (j = 1, 2, \ldots) \) with values in the given holomorphic line bundle \( L \to X \), and

(c) a sequence of smooth diffeomorphisms \( h_j: X \to X \) \( (j = 0, 1, \ldots) \) with \( h_0 = \text{Id}_X \),

satisfying the following conditions for all \( j = 1, 2, \ldots \).

(i) The set \( \overline{W}_{j-1} \) is \( \mathcal{O}(W_j) \)-convex.

(ii) There is an open neighbourhood \( U_j \subset X \) of \( W_j \) such that \( \alpha_j|U_j \in \text{Cont}_\text{hol}(U_j) \) and \( d\alpha_j = \beta_j \) in \( U_j \). (This already holds for \( j = 0 \).)

(iii) There is a homotopy \( (\alpha_j, t) \in \text{Cont}_\text{hol}(X) \) \( (t \in [0, 1]) \) such that \( (\alpha_j, 0) = (\alpha, \beta) \), and for every \( t \in [0, 1] \), \( \alpha_j, t \) is a holomorphic contact form in a neighbourhood of \( \overline{W}_{j-1} \) with \( d\alpha_j, t = \beta_j \) there.

(iv) \( \sup_{j \in \mathbb{N}} |(\alpha_j, t) - (\alpha_j, 1)| \leq \epsilon 2^{-j} \) where the difference of forms is measured with respect to a fixed pair of hermitian metrics on \( T^*X \) and \( L \).

(v) \( h_j(X_j) = W_j \) and \( h_j = \text{Id}_X \) on \( X \setminus X_{j+1} \) (hence, \( h_j(X_{j+1}) = X_{j+1} \)).

(vi) \( h_j = g_j \circ h_{j-1} \) where \( g_j : X \to X \) is a diffeomorphism which maps \( X_j \) onto \( W_j \) and

is diffeotopic to \( \text{Id}_X \) by a diffeotopy that equals \( \text{Id}_X \) on \( \overline{W}_{j-1} \cup (X \setminus X_{j+1}) \).

Granted such sequences, the domain \( \Omega = \bigcup_j W_j \subset X \) is Stein in view of condition (i), the limit \( \tilde{\alpha} = \lim_{j \to \infty} \alpha_j \) exists and is a holomorphic contact form on \( \Omega \) in view of (ii) and (iv), and the individual homotopies in (iii) can be put together into a homotopy in \( \text{Cont}_\text{hol}(\Omega) \) from \( (\alpha_0, \beta_0) \) to \( (\tilde{\alpha}, d\tilde{\alpha}) \) (see conditions (iii) and (iv)). Furthermore, conditions (v) and (vi) ensure that the sequence \( h_j \) converges to a diffeomorphism \( h = \lim_{j \to \infty} h_j : X \to \Omega \) satisfying the conclusion of Theorem 1.2. With a bit more care in the choice of \( W_j \) at each step, we can ensure that \( \Omega \) is smoothly bounded and strongly pseudoconvex. In general we cannot choose \( \Omega \) to be relatively compact, unless \( X \) admits an exhaustion function \( p : X \to \mathbb{R} \) with at most finitely many critical points. In the latter case, the above process clearly terminates in finitely many steps and yields a holomorphic contact form on a bounded strongly pseudoconvex domain \( \Omega \subset X \) diffeotopic to \( X \).
We now describe the induction step. To the strongly pseudoconvex domain \( W_{j-1} \) we attach the disc \( M_j := h_{j-1}(D_j) \), where \( D_j \subset X_j \setminus X_{j-1} \) (with \( bD_j \subset bX_{j-1} \)) is the unstable disc at the critical point \( p_j \in X_j \setminus X_{j-1} \). By [21] Lemma 3.1 we can isotopically deform \( M_j \) to a smooth totally real disc in \( X \setminus W_{j-1} \) attached to \( bW_{j-1} \) along the Legendrian sphere \( bM_j \subset bW_{j-1} \). Lemma 5.1 provides the next element \((\alpha_j, \beta_j) \in \text{Cont}_{\text{for}}(X)\), and a homotopy \((\alpha_{j,t}, \beta_{j,t}) \in \text{Cont}_{\text{for}}(X) \) (\( t \in [0, 1] \)) satisfying condition (iii), such that \( \alpha_j \) is a holomorphic contact form in a thin strongly pseudoconvex handlebody \( W_j \supset W_{j-1} \cup M_j \) and \( \beta_j = d\alpha_j \) there. The next diffeomorphism \( h_j = g_j \circ h_{j-1} \) satisfying conditions (v) and (vi) is then furnished by Morse theory. This concludes the proof.

Conditions (v) and (vi) show that the domain \( \Omega \) is diffeotopic to \( X \). By a more precise argument in the induction step one can also ensure the existence a diffeotopy \( h_t : X \to h_t(X) \subset X \) from \( h_0 = \text{Id}_X \) to a diffeomorphism \( h_1 = h : X \to \Omega \) through a family of Stein domains \( h_t(X) \subset X \); see [8] Theorem 8.43 and Remark 8.44]. This depends on the stronger technical result given by [8] Theorem 8.5, p. 157].

The same proof gives the following parametric extension of Theorem 1.2.

**Theorem 6.1.** Assume that \( X \) is a Stein manifold of dimension \( 2n + 1 \geq 3 \) and \( Q \subset P \) are compact spaces. Let \((\alpha_p, \beta_p) \in \text{Cont}_{\text{for}}(X)\) be a continuous family of formal contact structures such that for every \( p \in Q \), \((\alpha_p, \beta_p = d\alpha_p)\) is a holomorphic contact structure. Then there are a Stein domain \( \Omega \subset X \) diffeotopic to \( X \) and a homotopy \((\alpha_{p,t}, \beta_{p,t}) \in \text{Cont}_{\text{for}}(\Omega) \) (\( p \in P, t \in [0, 1] \)) which is fixed for all \( p \in Q \) such that \((\alpha_{p,1}, \beta_{p,1} = d\alpha_{p,1})\) is a holomorphic contact structure on \( \Omega \) for every \( p \in P \).

To see this, we follow the proof of Theorem 1.2 and note that, in the inductive step, the domain \( W_j \) (a Stein neighbourhood of \( \overline{W_j} \cup M_j \)) can be chosen such that Lemma 5.1 provides the next family \( \{ (\alpha_{p,j}, \beta_{p,j}) \} \}_{p \in P} \in \text{Cont}_{\text{for}}(X) \) satisfying condition (iii), where \( \alpha_{p,j} \) is a holomorphic contact form in \( W_j \) and \( \beta_{p,j} = d\alpha_{p,j} \) there for all \( p \in P \). (Here it is important to have a compact family of 1-forms.)

In the proof of Theorem 1.4 we shall need the following gluing lemma for holomorphic contact forms on Cartan pairs. (The analogous gluing lemma for nonsingular holomorphic foliations given by exact holomorphic 1-forms is [16] Theorem 4.1.)

**Lemma 6.2 (Gluing lemma for holomorphic contact forms).** Let \((A, B)\) be a Cartan pair in a complex manifold \( X^{2n+1} \) (see [17] Definition 5.7.1). Assume that \( \alpha, \beta \) are holomorphic contact forms in open neighbourhoods of \( A \) and \( B \), respectively. If \( \beta \) is sufficiently uniformly close to \( \alpha \) on a fixed neighbourhood of \( C = A \cap B \), then there exists a holomorphic contact form \( \tilde{\alpha} \) on a neighbourhood of \( A \cup B \) which approximates \( \alpha \) uniformly on \( A \) and approximates \( \beta \) uniformly on \( B \).

**Proof.** Let \( \alpha \) and \( \beta \) be holomorphic contact forms in open neighbourhoods \( A' \supset A \) and \( B' \supset B \), respectively. Set \( C' = A' \cap B' \) and define

\[
\alpha_t = (1 - t)\alpha + t\beta \quad \text{in} \ C' \quad \text{for} \ t \in [0, 1].
\]

Assuming that \( \beta \) is sufficiently uniformly close to \( \alpha \) on \( C' \), \( \alpha_t \) is a contact form on a smaller neighbourhood of \( C = A \cap B \) for every \( t \in [0, 1] \). By the proof of Gray’s stability theorem (see [28] or [23] p. 60] for the smooth case) we find

1. a neighbourhood \( C'' \subset C' \) of \( C \),
(2) an isotopy of biholomorphic maps $\phi_t: C'' \to \phi_t(C'') \subset C' (t \in [0, 1])$ with $\phi_0 = \text{Id}$ and $\phi_t$ close to the identity for all $t \in [0, 1]$, and

(3) a family of nowhere vanishing holomorphic functions $\lambda_t: C'' \to C^*$ close to 1, with $\lambda_0 = 1$,

satisfying $\phi_t^* \lambda_t = \lambda_t \phi_t$ on $C''$ for every $t \in [0, 1]$. In particular, we have

$$\phi_t^* \beta = \lambda_t \alpha$$ on $C''$.

Assuming that $\phi_1$ is sufficiently uniformly close to the identity on $C''$ (which holds if $\beta$ is close enough to $\alpha$ on $C'$), we can apply the splitting lemma [17, Theorem 9.7.1] to obtain

$$\phi_1 \circ \phi_A = \phi_B$$
on a neighbourhood of $C$, where $\phi_A$ and $\phi_B$ are biholomorphic maps close to the identity on open neighbourhoods of $A$ and $B$, respectively. On a neighbourhood of $C$ we then have

$$(\lambda_1 \circ \phi_A) \cdot \phi_A^* \alpha = \phi_A^* (\lambda_1 \alpha) = \phi_A^* (\phi_1^* \beta) = (\phi_1 \circ \phi_A)^* \beta = \phi_B^* \beta.$$ This shows that the holomorphic contact forms $\phi_A^* \alpha, \phi_B^* \beta$, defined on neighbourhoods of $A$ and $B$, respectively, have the same kernel on a neighbourhood of $C$, and hence they define a holomorphic contact structure $\xi$ on a neighbourhood of $A \cup B$. Assuming as we may that the function $\lambda_1 \circ \phi_A$ is sufficiently close to 1 on a neighbourhood of $C$, we can solve a multiplicative Cousin problem on the Cartan pair $(A,B)$ and correct the above 1-forms by the respective factors to obtain a holomorphic 1-form $\tilde{\alpha}$ on a neighbourhood of $A \cup B$, with $\ker \tilde{\alpha} = \xi$, which approximates $\alpha$ and $\beta$ on $A$ and $B$, respectively. \hfill \Box

**Proof of Theorem 1.4** We follow the inductive scheme used in Oka theory; see for instance [17, proof of Theorem 5.4.4].

We use the notation established in the proof of Theorem 1.2. The main (and in fact the only) difference from that proof is that we can now extend a holomorphic contact form (by approximation) from a neighbourhood of the sublevel set $X_{j-1} = \{ \rho \leq c_{j-1} \}$ to a neighbourhood of $\bar{X}_j = \{ \rho \leq c_j \}$, provided it extends as a formal contact structure.

The first step, namely the extension to a handlebody $W_{j-1}$ around $\bar{X}_{j-1} \cup M_j$ (where $M_j$ is a totally real disc which provides the change of topology at the critical point $p_j \in X_j \setminus X_{j-1}$) is furnished by the proof of Theorem 1.2. We may arrange the process so that $X_j$ is a noncritical strongly pseudoconvex extension of $W_{j-1}$ (see [17, Sect. 5.10]). This implies that we can obtain $X_j$ from $W_{j-1}$ by attaching finitely many convex bumps (see [17, Lemma 5.10.3]). We now successively extend the contact form (by approximation) across each bump. At every step we have a Cartan pair $(A,B)$, where $B$ is a convex bump attached to a compact strongly pseudoconvex domain $A$ along the set $C = A \cap B$. (The sets $C \subset B$ are convex in some holomorphic coordinates on a neighbourhood of $B$ in $X$.) We also have a holomorphic contact form $\alpha$ on a neighbourhood of $A$. Assuming that Problem 1.3 has an affirmative answer, we can approximate $\alpha$ uniformly on a neighbourhood of $C$ by a holomorphic contact form $\beta$ on a neighbourhood of $B$. Assuming that the approximation is close enough, Lemma 6.2 furnishes a holomorphic contact form $\tilde{\alpha}$ on neighbourhood of $A \cup B$ which approximates $\alpha$ uniformly on $A$. In finitely many steps of this kind we approximate the given holomorphic contact form on $\bar{W}_{j-1}$ by a holomorphic contact form on a neighbourhood of $\bar{X}_j$. Hence, this process furnishes in the limit a holomorphic contact form on all of $X$. The same argument applies in the parametric case provided that the parametric version of Problem 1.3 has an affirmative answer. \hfill \Box
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