The Calabi-Yau problem for Riemann surfaces with finite genus and countably many ends

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Abstract In this paper, we show that if $R$ is a compact Riemann surface and $M = R \setminus \bigcup_i D_i$ is a domain in $R$ whose complement is a union of countably many pairwise disjoint smoothly bounded closed discs $D_i$, then $M$ is the complex structure of a complete bounded minimal surface in $\mathbb{R}^3$. We prove that there is a complete conformal minimal immersion $X : M \to \mathbb{R}^3$ extending to a continuous map $X : M \to \mathbb{R}^n$ such that $X(bM) = \bigcup_i X(bD_i)$ is a union of pairwise disjoint Jordan curves. This extends a recent result for bordered Riemann surfaces.

Keywords Riemann surface, minimal surface, Calabi-Yau problem

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1. Introduction

A classical problem in the theory of minimal surfaces in Euclidean spaces is the conformal Calabi-Yau problem, asking which open Riemann surfaces, $M$, admit a complete conformal minimal immersion $X : M \to \mathbb{R}^n$ ($n \geq 3$) with bounded image. (Recall that a continuous map $X : M \to \mathbb{R}^n$ is said to be complete if the image of any divergent curve in $M$ has infinite Euclidean length. If $X$ is an immersion, this is equivalent to asking that the Riemannian metric $X^* (ds^2)$ on $M$, induced by the Euclidean metric $ds^2$ on $\mathbb{R}^n$ via $X$, is a complete metric.) This problem originates in a conjecture of E. Calabi from 1965 that such immersions do not exist (see Kobayashi and Eells [24, p. 170] or Chern [15, p. 212]). Groundbreaking counterexamples to Calabi’s conjecture were given by Jorge and Xavier [23] in 1980 (a complete immersed minimal disc in $\mathbb{R}^3$ with a bounded coordinate function), Nadirashvili [28] in 1996 (a complete bounded immersed minimal disc in $\mathbb{R}^3$), and many others. In particular, there are examples in the literature of complete bounded minimal surfaces in $\mathbb{R}^3$ with any topological type (see Ferrer, Martín, and Meeks [18]). The related asymptotic Calabi-Yau problem (see S.-T. Yau [37, p. 360]) asks about the asymptotic behaviour of such surfaces near their ends. We refer to the recent papers [3, 10] for the history and literature on these problems.

The main result of this paper is the following.

**Theorem 1.1.** Let $R$ be a compact Riemann surface. If $M = R \setminus \bigcup_{i=0}^\infty D_i$ is a domain in $R$ whose complement is a countable union of pairwise disjoint closed discs $D_i$ (diffeomorphic images of the closed disc $\overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$), then $M$ is the complex structure of a complete bounded minimal surface in $\mathbb{R}^3$.

More precisely, there exists a continuous map $X : \overline{M} \to \mathbb{R}^n$ such that $X : M \to \mathbb{R}^n$ is a complete conformal minimal immersion and $X(bM) = \bigcup_i X(bD_i)$ is the union of pairwise disjoint Jordan curves. If $n = 4$ then $X : M \to \mathbb{R}^4$ can be chosen an immersion with simple double points, and if $n \geq 5$ then $X : \overline{M} \to \mathbb{R}^n$ can be chosen an embedding.
The discs \( D_i \) in the theorem are assumed to have boundaries of class \( C^k \) for some \( k > 1 \), possibly noninteger. The Jordan curves \( X(bD_i) \subset \mathbb{R}^n \) are everywhere nonrectifiable, but we show that \( X \) can be chosen such that they have Hausdorff dimension one.

The analogue of Theorem 1.1 in the case when \( M \) is the complement of finitely many pairwise disjoint discs in a compact Riemann surface was obtained in [3, Theorem 1.1]; this also follows by a simplification of the proof of Theorem 1.1. Such \( M \) is a bordered Riemann surface whose boundary, \( bM \), consists of finitely many Jordan curves. The surfaces in our theorem still have finite genus but countably many ends.

Theorem 1.1 is proved in Sect. 3 by an inductive application of [3, Lemma 4.1]; see Lemma 2.3. This lemma shows that one can increase the intrinsic diameter of a conformal minimal immersion \( M \to \mathbb{R}^n \) from a compact bordered Riemann surface by an arbitrarily big amount while keeping the map arbitrarily uniformly close to the original one.

Our proof gives several additions to the theorem. In particular, \( X \) can be chosen with vanishing flux. Alternatively, a minor modification of the proof enables us to prescribe the flux of \( X \) on any given finite family of classes in the first homology group \( H_1(M,\mathbb{Z}) \); however, we do not know whether \( X \) can be chosen with arbitrarily prescribed flux map. On the other hand, if we do not insist on controlling the flux of \( X \), then we can choose any conformal minimal immersion \( X_0 : \mathbb{R} \setminus D_0 \to \mathbb{R}^n \) and find for any given number \( \epsilon > 0 \) a map \( X \) as in the theorem which is uniformly \( \epsilon \)-close to \( X_0 \) on \( M \).

Theorem 1.1, together with the uniformization theorem of He and Schramm [21] from 1993 solving Koebe’s conjecture (see also [22]), implies the following corollary.

**Corollary 1.2.** Every proper domain in the Riemann sphere \( \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \) with at most countably many boundary components, none of which are points, admits a complete conformal minimal immersion into \( \mathbb{R}^3 \) with bounded image, and an embedding with these properties into \( \mathbb{R}^5 \). The analogous result holds for domains in any complex torus.

Indeed, He and Schramm proved in [21] that every domain in Corollary 1.2 is conformally equivalent to a circle domain, that is, a domain in \( \mathbb{C} \) whose complement is a union of at most countably many pairwise disjoint closed round discs. Hence, Theorem 1.1 applies to such domains.

We wish to take this occasion for recording an observation regarding the conformal Calabi-Yau problem for Riemann surfaces of finite topological type, i.e., of finite genus and with finitely many ends. It is classical (see Stout [33] and Crane [17]) that every such surface, \( M \), is conformally equivalent to a bordered Riemann surface \( M' \) with real analytic boundary curves and with finitely many points removed. If in addition \( M \) is a bordered Riemann surface bounded by Jordan curves, then any conformal map \( \phi : M \to M' \) extends to a homeomorphism \( \phi : \overline{M} \to \overline{M'} \) by the seminal theorem of Carathéodory [14] from 1913. Furthermore, if \( bM \) is smooth, then \( \phi \) extends to a smooth diffeomorphism \( \phi : M \to M' \) by Painlevé’s theorem from 1887 [31, 32]. Improvements concerning the boundary regularity of \( \phi \) were made by many authors. In particular, it was shown by Warschawski [34] in 1935 that if \( bM \) is of class \( C^{k,\alpha} \) for some \( k \in \mathbb{Z}_+ \) and \( 0 < \alpha < 1 \), then \( \phi \) is of the same class \( C^{k,\alpha} \) on \( \overline{M} \). We refer to Goluzin [20] for more information. For the corresponding boundary regularity results for minimal surfaces, see Nitsche [30, 29].

By a puncture end, or simply a puncture of an open Riemann surface, we mean an end that is conformally isomorphic to the punctured disc. Since every bounded harmonic function extends harmonically across an isolated point, there are no bounded complete conformal
The Calabi-Yau problem

minimal immersions from a Riemann surface with a puncture. Together with [3, Theorem 1.1] (the analogue of Theorem 1.1 for finitely many discs), this gives the following corollary.

**Corollary 1.3.** An open Riemann surface, \( M \), of finite topological type admits a bounded complete conformal minimal immersion \( M \to \mathbb{R}^3 \) if and only if \( M \) has no punctures.

On the other hand, Colding and Minicozzi [16, Corollary 0.13] proved that a complete embedded minimal surface of finite topology in \( \mathbb{R}^3 \) is necessarily proper in \( \mathbb{R}^3 \), hence unbounded; this was extended to surfaces of finite genus and countably many ends by Meeks, Pérez, and Ros [26, Theorem 1.3]. Hence, Theorem 1.1 and its corollaries expose a major difference between the immersed and the embedded conformal Calabi-Yau problem.

The techniques developed in the papers [7, 3, 9] also furnish an analogue of Lemma 2.3 for immersed holomorphic curves in \( C^n (n \geq 2) \), null holomorphic curves in \( C^n \) for \( n \geq 3 \), and holomorphic Legendrian curves in \( C^{2n+1} \) for any \( n \geq 1 \). Recall that null curves are holomorphic immersions \( \mathbf{Z} = (Z_1, \ldots, Z_n) : M \to C^n \) from an open Riemann surface \( M \) satisfying the nullity condition \( (dZ_1)^2 + (dZ_2)^2 + \cdots + (dZ_n)^2 = 0 \). Every holomorphic curve in \( C^n \) is also a minimal surface by Wirtinger’s theorem [35], while the real and the imaginary part of a null holomorphic curve in \( C^n \) are conformal minimal surfaces in \( \mathbb{R}^n \). By following the proof of Theorem 1.1 and using the analogue of Lemma 2.3 for the appropriate classes of holomorphic curves, one obtains the following result.

**Theorem 1.4.** Let \( M \) be an open Riemann surface as in Theorem 1.1. Then, for any \( n \geq 2 \) there exists a continuous map \( \mathbf{Z} : M \to C^n \) such that \( \mathbf{Z} : M \to C^n \) is a complete holomorphic immersion and \( \mathbf{Z}(\partial M) = \bigcup Z(bD_i) \) is the union of pairwise disjoint Jordan curves \( Z(bD_i) \). If \( n \geq 3 \) then \( \mathbf{Z} \) can be chosen an embedding and such that \( \mathbf{Z} : M \to C^n \) is a complete null holomorphic embedding, or (if \( n \) is odd) a complete holomorphic Legendrian embedding.

Theorem 1.4 contributes to the collection of results concerning Yang’s problem [36] from 1977 which asks about the existence and boundary behaviour of bounded complete complex submanifolds of complex Euclidean spaces. For recent developments on this subject, we refer to the papers [2, 11, 12, 19, 13] and the authors’ survey [10].

The situation for Riemann surfaces more general than those in Theorem 1.1 is not understood yet, and we mention the following open problems in this direction. The analogous questions make sense for holomorphic (null, Legendrian) curves.

**Problem 1.5.** (A) Let \( M \) be an open Riemann surface which admits a nonconstant bounded harmonic function and has no punctures. Does \( M \) admit a bounded complete conformal minimal immersion into \( \mathbb{R}^3 \)?

(B) In particular, if \( K \) is a compact set in \( C \) of positive capacity and without isolated points, does \( C \setminus K \) admit a bounded complete conformal minimal immersion into \( \mathbb{R}^3 \)?

(C) Is there an example of a complete bounded minimal surface in \( \mathbb{R}^3 \) whose underlying complex structure is \( C \setminus K \), where \( K \) is a Cantor set in \( C \)?

Recall (see [23, 4, 5, 6]) that every nonconstant bounded harmonic function \( h : M \to (a, b) \subset \mathbb{R} \) on an open Riemann surface \( M \) is a component function of a complete conformal minimal immersion \( X = (X_1, X_2, h) : M \to \mathbb{R}^3 \) whose range is therefore contained in the slab \( \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : a < x_3 < b \} \). Note also that the complement of a compact set \( K \subset C \) of positive capacity admits nonconstant bounded holomorphic (hence harmonic) functions. Hence, the above questions are very natural.
2. Preliminaries

Given a compact smooth manifold \( M \) with nonempty boundary \( bM \) and an interior point \( p_0 \in M = M \setminus bM \), we denote by \( \mathscr{C} = \mathscr{C}(M, p_0) \) the set of all paths \( \gamma : [0, 1] \to M \) with \( \gamma(0) = p_0 \) and \( \gamma(1) \in bM \). Given a continuous map \( X : M \to \mathbb{R}^n \), we define

\[
\text{dist}_X(p_0, bM) = \inf \{ \text{length}(X \circ \gamma) : \gamma \in \mathscr{C} \} = [0, +\infty],
\]

where \( \text{length}(\lambda) \) denotes the Euclidean length of a path \( \lambda : [0, 1] \to \mathbb{R}^n \), i.e., the supremum of the lengths of all subdivision paths \( 0 = t_0 < t_1 < \ldots < t_m = 1 \) of the interval \([0, 1]\). The same definition applies to a compact domain \( M \) with countably many boundary components as in Theorem 1.1. The number \( \text{dist}_X(p_0, bM) \) is called the intrinsic diameter of \( M \) with respect to the map \( X \) and the point \( p_0 \). A change of the base point changes the intrinsic diameter by a constant.

Assume now that \( X : M \to \mathbb{R}^n \) is a smooth immersion and let \( g = X^*ds^2 \) be the induced Riemannian metric on \( M \). Given a piecewise \( \mathcal{C}^1 \) path \( \gamma : [0, 1] \to M \), we have

\[
\text{length}(X \circ \gamma) = \text{length}_g(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_g \, dt.
\]

It is well known and easily seen that \( \text{dist}_X(p_0, bM) = \text{dist}_g(p_0, bM) \) is the infimum of the lengths of piecewise \( \mathcal{C}^1 \) paths in the family \( \mathscr{C}(M, p_0) \). Indeed, we can replace any path \( \gamma \) in \( M \) by a piecewise smooth path which is not longer than \( \gamma \) by taking a suitably fine subdivision \( 0 = t_0 < t_1 < \ldots < t_m = 1 \) of \([0, 1]\) and replacing each segment \( C_i = \{ \gamma(t) : t_{i-1} < t < t_i \} \) by the geodesic arc connecting the points \( \gamma(t_{i-1}) \) and \( \gamma(t_i) \).

We shall also consider two bigger classes \( \mathscr{C}_d = \mathscr{C}_d(M, p_0) \subset \mathscr{C}_d = \mathscr{C}_d(M, p_0) \) of piecewise \( \mathcal{C}^1 \) paths in \( M \) with the given initial point \( p_0 \in M \). The first one, \( \mathscr{C}_d(M, p_0) \), consists of all \( \text{divergent paths} \) \( \gamma : [0, 1] \to M \) with \( \gamma(0) = p_0 \), i.e., such that \( \gamma(t) \) leaves any compact subset of \( M \) as \( t \) approaches 1. (However, the limit of \( \gamma(t) \) as \( t \to 1 \) need not exist.) The bigger class \( \mathscr{C}_d(M, p_0) \) consists of all paths \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = p_0 \) and, as \( t \to 1 \), \( \gamma(t) \) clusters on \( bM \), i.e., there is a sequence

\[
0 < t_1 < t_2 < \cdots < t_1 \text{ with } \lim_{j \to \infty} t_j = 1 \text{ and } \lim_{j \to \infty} \gamma(t_j) = p \in bM.
\]

We call any such path \text{quasidivergent}.

We claim that we get the same intrinsic diameter \( \text{dist}_g(p_0, bM) \) by using paths in the bigger family \( \mathscr{C}_d \), and hence also by using paths in \( \mathscr{C}_d \).

**Lemma 2.1.** Let \( g \) be a smooth Riemannian metric on a compact manifold with boundary \( M \), and let \( p_0 \in M \). For every path \( \gamma \in \mathscr{C}_d \), we have that \( \text{length}_g(\gamma) \geq \text{dist}_g(p_0, bM) \).

**Proof.** Fix \( \epsilon > 0 \). Let \( U \subset M \) be a neighbourhood of \( bM \) such that every point \( q \in U \) can be connected to a point \( p \in bM \) by an arc of length less than \( \epsilon \). Given \( \gamma \in \mathscr{C}_d \), there is \( t_0 \in [0, 1) \) such that \( \gamma(t_0) \in U \). Choose an arc \( C \subset U \) with the beginning point \( \gamma(t_0) \), terminal point \( p \in bM \), and with \( \text{length}_g(C) < \epsilon \). Let \( \lambda : [0, 1] \to M \) be a path in \( \mathcal{C} \) such that \( \lambda(t) = \gamma(t) \) for \( t \in [0, t_0] \), and \( \lambda(t) \) for \( t \in [t_0, 1] \) is a parametrisation of \( C \). Then,

\[
\text{dist}_g(p_0, bM) \leq \text{length}_g(\lambda) < \text{length}_g(\gamma) + \epsilon.
\]

Letting \( \epsilon \to 0 \) we obtain \( \text{length}_g(\gamma) \geq \text{dist}_g(p_0, bM) \). \( \square \)
Lemma 2.2. Let $M$ be a compact smooth manifold with nonempty boundary $bM$, and let $X : M \to \mathbb{R}^n$ be a $C^1$ immersion. Given a point $p_0 \in M$ and a number $\eta > 0$, there exists a number $\epsilon > 0$ such that for every continuous map $Y : M \to \mathbb{R}^n$ with $\|X - Y\|_{C^0(M)} := \max\{|X(p) - Y(p)| : p \in M\} < \epsilon$ we have that
\[
\inf\{\text{length}(Y \circ \gamma) : \gamma \in C_q(M, p_0)\} \geq \text{dist}_X(p_0, bM) - \eta.
\]

Note that $\text{length}(Y \circ \gamma)$ denotes the Euclidean length of the path $Y \circ \gamma : [0, 1) \to \mathbb{R}^n$. The same result holds for maps to an arbitrary Riemannian manifold in place of $(\mathbb{R}^n, ds^2)$.

**Proof.** We may assume that $M$ is a compact domain with boundary in a somewhat bigger smooth manifold $\tilde{M}$, and that $X$ extends to a smooth immersion $\tilde{X} : \tilde{M} \to \mathbb{R}^n$. Let $N \to \tilde{M}$ denote the smooth normal bundle of the immersion $\tilde{X} : \tilde{M} \to \mathbb{R}^n$, so $\dim N = n$.

We identify $\tilde{M}$ with the zero section of $N$. By the tubular neighbourhood theorem, $\tilde{X}$ extends to a smooth immersion $F : N \to \mathbb{R}^n$ which agrees with $\tilde{X}$ on the zero section $M$. Then, $g = F^*(ds^2)$ is a smooth Riemannian metric on $N$ whose restriction to $\tilde{M}$ is the metric $X^*(ds^2)$, and the map $F : (N, g) \to (\mathbb{R}^n, ds^2)$ is a local isometry. Let $\text{dist}_g$ denote the distance function on $N$ induced by the Riemannian metric $g$.

Let $\eta > 0$ be as in the lemma. There is a neighbourhood $U \subset N$ of $M$ such that
\[
\text{dist}_{g,U}(p_0, bM) > \text{dist}_{g,M}(p_0, bM) - \eta/2 = \text{dist}_X(p_0, bM) - \eta/2,
\]
where $\text{dist}_{g,U}(p_0, bM)$ is the infimum of the distances from $p_0$ to $bM$ over all paths in $U$, while $\text{dist}_{g,M}(p_0, bM)$ denotes the same quantity over all paths in $M$ (cf. [2,1]).

Since $F : N \to F(N) \subset \mathbb{R}^n$ is a local isometry and $M$ is compact, there is a number $\epsilon_0 > 0$ such that for every point $p \in M$ the closed ball
\[
B_g(p, \epsilon_0) := \{q \in N : \text{dist}_g(p, q) \leq \epsilon_0\}
\]
is contained in $U$, and $F$ maps $B(p, \epsilon_0)$ isometrically onto the closed Euclidean ball $B(X(p), \epsilon_0) \subset \mathbb{R}^n$. By decreasing $\epsilon_0$ if necessary we may assume that $0 < \epsilon_0 < \eta/2$.

Given a continuous map $Y : M \to \mathbb{R}^n$ satisfying
\[
\max_{p \in M} |Y(p) - X(p)| < \epsilon \leq \epsilon_0,
\]
the above implies that there is a unique continuous map $\tilde{Y} : M \to U$ such that
\[
Y = F \circ \tilde{Y} \quad \text{and} \quad \text{dist}_g(p, \tilde{Y}(p)) = |X(p) - Y(p)| < \epsilon \quad \text{for all} \ p \in M.
\]

Let $\gamma \in C_{cd}$ be any quasidivergent path in $M$ with $\gamma(0) = p_0$. Fix a number $\epsilon$ with $0 < \epsilon < \epsilon_0/2$. There is a boundary point $p \in bM$ and $t_0 \in (0, 1)$ such that
\[
\text{dist}_g(\gamma(t_0), p) < \epsilon.
\]

By (2.5) we have that $Y \circ \gamma = F \circ \tilde{\gamma}$, where the path $\tilde{\gamma} = \tilde{Y} \circ \gamma : [0, 1] \to U$ satisfies
\[
\text{dist}_g(\gamma(t), \tilde{\gamma}(t)) < \epsilon \quad \text{for all} \ t \in [0, 1).
\]

Since $F : (N, g) \to (\mathbb{R}^n, ds^2)$ is a local isometry, we also have that
\[
\text{length}_g(\tilde{\gamma}) = \text{length}(F \circ \tilde{\gamma}) = \text{length}(Y \circ \gamma).
\]

By (2.6), (2.7), and the triangle inequality, the point $\tilde{\gamma}(t_0) = \tilde{Y}(\gamma(t_0))$ satisfies
\[
\text{dist}_g(\tilde{\gamma}(t_0), p) \leq \text{dist}_g(\tilde{\gamma}(t_0), \gamma(t_0)) + \text{dist}_g(\gamma(t_0), p) < \epsilon + \epsilon \leq \epsilon_0 < \eta/2.
\]
By adding to the path $\tilde{\gamma} : [0, t_0] \to N$ an arc in the ball $B_g(p, \epsilon)$ of $g$-length $< \eta/2$ connecting the point $\tilde{\gamma}(t_0)$ to $p \in bM$, we obtain a path $\lambda : [0, 1] \to U$ from $\lambda(0) = p_0$ to $\lambda(1) = p \in bM$ such that

$$\text{length}_g(\lambda) < \text{length}_g(\tilde{\gamma}) + \eta/2.$$  

We obviously have $\text{length}_g(\lambda) \geq \text{dist}_{g, U}(p_0, bM)$. Together with (2.3) we obtain

$$\text{length}_g(\tilde{\gamma}) > \text{length}_g(\lambda) - \eta/2 \geq \text{dist}_{g, U}(p_0, bM) - \eta/2 > \text{dist}_X(p_0, bM) - \eta.$$

In view of (2.8) it follows that $\text{length}(Y \circ \gamma) > \text{dist}_X(p_0, bM) - \eta$. Since $\gamma \in \mathcal{C}_{cd}(M, p_0)$ was arbitrary, this proves the lemma.

The following lemma (see [3, Lemma 4.1]) is the main ingredient in the proof of Theorem 1.1. It enables us to make the intrinsic diameter of an immersed conformal minimal surface arbitrarily big by a deformation that is arbitrarily small in the Hausdorff sense. Condition (iii) in the lemma follows from a general position theorem; see [3, Theorem 4.5].

**Lemma 2.3.** Let $M$ be a compact bordered Riemann surface, let $n \geq 3$ be an integer, and let $X : M \to \mathbb{R}^n$ be a conformal minimal immersion of class $\mathcal{C}^1$. Given a point $p_0 \in M$, an integer $d \in \mathbb{Z}_+$, and numbers $\epsilon > 0$ (small) and $\mu > 0$ (big), there is a conformal minimal immersion $Y : M \to \mathbb{R}^n$ satisfying the following conditions.

(i) $|Y(p) - X(p)| < \epsilon$ for all $p \in M$.
(ii) $\text{dist}_Y(p_0, bM) > \mu$.
(iii) $Y|_{bM} : bM \to \mathbb{R}^n$ is injective.
(iv) $\text{Flux}_Y = \text{Flux}_X$.

**3. Proof of Theorem 1.1**

We shall construct a map $X : M \to \mathbb{R}^n$ satisfying the conclusion of the theorem and such that the Jordan curves $X(bD_k)$, $k = 0, 1, 2, \ldots$, have Hausdorff dimension one. Moreover, we shall ensure that the complete conformal minimal immersion $X : M \to \mathbb{R}^n$ has vanishing flux.

Let $Z_+ = \{0, 1, 2, \ldots\}$. Assume that $R$ is a compact Riemann surface and

$$M = R \setminus \bigcup_{i=0}^{\infty} D_i,$$

where $\{D_i\}_{i \in Z_+}$ is a countable family of pairwise disjoint closed discs in $R$. Let

$$M_i = R \setminus \bigcup_{k=0}^{i} \hat{D}_k, \quad i = 0, 1, 2, \ldots.$$  

Clearly, $M_i$ is a compact bordered Riemann surface with boundary $bM_i = \bigcup_{k=0}^{i} bD_k$, and

$$M_0 \supset M_1 \supset M_2 \supset \cdots \supset \bigcap_{i=1}^{\infty} M_i = \overline{M}.$$  

Denote by $d$ a Riemannian distance function on $R$. Let $p_0 \in M$, and let $X_0 : M_0 \to \mathbb{R}^n$ be a conformal minimal immersion of class $\mathcal{C}^1$ with vanishing flux and such that $X_0|_{bM_0} : bM_0 \to \mathbb{R}^n$ is injective; the existence of such is guaranteed in [3]. Pick numbers $\epsilon_0 > 0$ and $\tau_0 \in \mathbb{N} = \{1, 2, 3, \ldots\}$. An inductive application of Lemmas 2.2 and 2.3
furnishes a sequence of conformal minimal immersions $X_i : M_i \to \mathbb{R}^n$, numbers $\epsilon_i > 0$, and integers $\tau_i > i$ satisfying the following conditions for every $i \in \mathbb{N}$.

(a.) $\text{dist}_{X_i}(p_0, bM_i) > i$.

(b.) $X_i : bM_i \to \mathbb{R}^n$ is injective.

(c.) $\sup_{p \in M_i} |X_i(p) - X_{i-1}(p)| < \epsilon_i - 1$.

(d.) For every continuous map $Y : M_i \to \mathbb{R}^n$ with $\|Y - X_i\|_{\mathcal{C}(M_i)} < 2\epsilon_i$, we have

$$\inf\{|\text{length}(Y \circ \gamma) : \gamma \in \mathcal{C}_{qd}(M_i, p_0)| \} > \text{dist}_{X_i}(p_0, bM_i) - 1 > i - 1.$$

(See Lemma 2.2)

We have $0 < \epsilon_i < \frac{1}{2} \min \{\epsilon_{i-1}, \delta_i, \tau_i^{-1}\}$, where

$$\delta_i := \frac{1}{\tau_i} \inf\{|X_i(p) - X_i(q)| : p, q \in bM_i, d(p, q) > \frac{1}{\tau_i}\} > 0.$$

(f.) We have $\tau_i > \tau_{i-1}$ and for each $k \in \{0, \ldots, i\}$ there is a set $A_{i,k} \subset X_i(bD_k)$ consisting of $\tau_{i+1}^k$ points such that

$$\max\{\text{dist}(p, A_{i,k}) : p \in X_i(bD_k)\} < \frac{1}{\tau_i},$$

where $\text{dist}(p, A_{i,k}) = \min\{|p - q| : q \in A_{i,k}\}$ is the Euclidean distance in $\mathbb{R}^n$.

(g.) $X_i$ has vanishing flux. Let us explain the induction step. Assume that for some $i \in \mathbb{N}$ we have maps $X_0, \ldots, X_{i-1}$ and numbers $\epsilon_0, \ldots, \epsilon_{i-1}$ and $\tau_0, \ldots, \tau_{i-1}$ satisfying these conditions for the respective values of the index. (This holds for $i = 1$ by using $X_0$, $\epsilon_0$, and $\tau_0$, and the above conditions are void except for (a.), (b.), and (g.); the second part of (f.) also holds true if we choose $\tau_0 \in \mathbb{N}$ sufficiently large.) Lemma 2.3 applied to $X_{i-1}|_{M_i}$ furnishes the next conformal minimal immersion $X_i : M_i \to \mathbb{R}^n$ satisfying conditions (a.), (b.), (c.), and (g.); note that $X_{i-1}|_{M_i}$ is flux vanishing since so is $X_i$ by (g.). Pick $\tau_i \in \mathbb{N}$ so large that (f.) is satisfied; it suffices to choose $\tau_i > \tau_{i-1} + \sum_{k=0}^i \text{length}(X_i(bD_k))$. Pick a number $\epsilon_i > 0$ satisfying condition (e.); such exists since $X_i|_{bM_i}$ is injective by (b.). Finally, decreasing $\epsilon_i > 0$ if necessary we may assume that (d.) holds as well in view of Lemma 2.2. The induction may proceed.

Conditions (e.) and (e.) imply that the sequence $X_i$ converges uniformly on $\overline{M}$ (3.1) to a continuous map $X = \lim_{i \to \infty} : \overline{M} \to \mathbb{R}^n$ whose restriction to $M$ is a conformal minimal immersion $X : M \to \mathbb{R}^n$, provided that each $\epsilon_i > 0$ is chosen sufficiently small. More precisely, for every $p \in \overline{M}$ we have that

$$|X(p) - X_i(p)| \leq \sum_{k=i}^{\infty} |X_{k+1}(p) - X_k(p)| < \sum_{k=i}^{\infty} \epsilon_k < 2\epsilon_i.$$  

(3.3)

We can extend $X$ from $\overline{M}$ to a continuous map $X : M_i \to \mathbb{R}^n$ such that the above inequality holds for all $p \in M_i$.

Conditions (a.) and $0 < \epsilon_i < \epsilon_{i-1}/2$ (see (e.)) ensure that $X : M \to \mathbb{R}^n$ is complete. Indeed, consider any divergent path $\gamma : [0, 1) \to M$ with $\gamma(0) = p_0$. There is an increasing sequence $0 < t_1 < t_2 < \cdots < 1$ with $\lim_{j \to \infty} t_j = 1$ such that $\lim_{j \to \infty} \gamma(t_j) = p \in bM$ (cf. (2.2)). Then, $p \in bD_{i_0}$ for some $i_0 \in \mathbb{Z}_+$, and hence $p \in bM_i$ for all $i \geq i_0$. It
follows that $\gamma$ is a quasidivergent path in the bordered Riemann surface $M_i$ for any $i \geq i_0$. Conditions (a$_i$), (d$_i$), and the estimate (3.3) imply for any $i \geq i_0$ that

$$\text{length}(X(\gamma)) > \text{dist}_X(p_0, bM_i) - 1 > i - 1.$$  

Letting $i \to +\infty$ shows that $\text{length}(X(\gamma)) = +\infty$. Conditions (b$_i$), (c$_i$), and (e$_i$) easily imply that the limit map $X : M \to \mathbb{R}^n$ is injective on $bM = \bigcup_{i \in \mathbb{Z}_+} bD_i$ (see [3] proof of Theorem 1.1) for the details), whereas (g$_i$) ensures that $X$ has vanishing flux. Finally, in order to check that all the Jordan curves $X(bD_k)$, $k = 0, 1, 2, \ldots$, have Hausdorff dimension one, pick such a $k$. By (3.3), (e$_i$), and (f$_i$), we have for each $i > k$ that

$$\max\{\text{dist}(p, A_{i,k}) : p \in X(bD_k)\} < \frac{2}{\tau_i^i}.$$  

Since the set $A_{i,k}$ consists of precisely $\tau_i^{i+1}$ points and $\tau_i^{i+1}(2/\tau_i^i)^{1+1/i} = 2^{1+1/i} \leq 4$ for every integer $i > k$, inequality (3.4) implies that the Hausdorff measure $H^1(X(bD_k))$ is finite, and hence the Hausdorff dimension of $X(bD_k)$ is at most one (cf. [2] Lemma 2.2) or [11 Sect. 4.1]; see e.g. [27] for an introduction to the Hausdorff measure). On the other hand, $X(bD_k)$ is homeomorphic to the circle $T = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, and hence its Hausdorff dimension is at least one, so it is one.

Furthermore, by using also the general position argument for minimal surfaces at every step of the proof (see [8 Theorem 4.1]) we can ensure that the limit map $X : M \to \mathbb{R}^n$ is an immersion with simple double points if $n = 4$ and an embedding if $n \geq 5$. These additions are obtained in exactly the same way as in [3] proof of Theorem 1.1.

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References

The Calabi-Yau problem


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