IMMERSIONS OF OPEN RIEMANN SURFACES INTO THE RIEMANN SPHERE

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Abstract. In this paper we show that the space of holomorphic immersions from any given open Riemann surface, M, into the Riemann sphere $\mathbb{CP}^1$ is weakly homotopy equivalent to the space of continuous maps from M to the complement of the zero section in the tangent bundle of $\mathbb{CP}^1$. It follows in particular that this space has 2^k path components, where k is the number of generators of the first homology group $H_1(M,\mathbb{Z}) = \mathbb{Z}^k$. We also prove a parametric version of the Mergelyan approximation theorem for maps from Riemann surfaces into an arbitrary complex manifold, a result used in the proof of our main theorem.

1. The main result

In this paper, M always stands for an open Riemann surface. Our aim is to determine the weak homotopy type of the space $\mathscr{I}(M,\mathbb{CP}^1)$ of holomorphic immersions $M \to \mathbb{CP}^1$ into the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

We begin by identifying the space of *formal immersions* of M to $\mathbb{CP}^1$. Let $E = T\mathbb{CP}^1 \setminus \{0\} \xrightarrow{\pi} \mathbb{CP}^1$ denote the tangent bundle of $\mathbb{CP}^1$ with the zero section removed, a holomorphic $\mathbb{C}^*$-bundle over $\mathbb{CP}^1$. Here, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Choose a nowhere vanishing holomorphic vector field V on M. (Recall that every holomorphic vector bundle over an open Riemann surface is holomorphically trivial [7, Theorem 5.3.1]. A choice of V corresponds to a trivialisation of the tangent bundle of M.) A holomorphic immersion $f : M \to \mathbb{CP}^1$ lifts to a holomorphic map $\tilde f : M \to E$ with $\pi \circ \tilde f = f$, defined by

$$\tilde f(x) = df_x(V_x) \in T_{f(x)}\mathbb{CP}^1 \setminus \{0\} = E_{f(x)}, \quad x \in M. \tag{1.1}$$

Let Φ denote the map

$$\mathscr{I}(M,\mathbb{CP}^1) \xrightarrow{\Phi} \mathscr{O}(M,E) \subset \mathscr{C}(M,E) \tag{1.2}$$

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sending $f \in \mathscr{I}(M, \mathbb{CP}^1)$ to $\Phi(f) = \tilde{f} \in \mathscr{O}(M, E) \subset \mathscr{C}(M, E)$. We call $\mathscr{C}(M, E)$ the space of *formal immersions* of M into $\mathbb{CP}^1$. These mapping spaces carry the compact-open topology.

Our main result is the following; however, see also the more precise version given by Theorem 5.1.

Theorem 1.1. *For every open Riemann surface, M, the map Φ (1.2) from the space of holomorphic immersions $M \to \mathbb{CP}^1$ to the space of formal immersions satisfies the parametric h-principle, and hence is a weak homotopy equivalence.*

Being weak homotopy equivalence means that Φ induces a bijection

$$\pi_0(\mathscr{I}(M, \mathbb{CP}^1)) \longrightarrow \pi_0(\mathscr{C}(M, E)) = [M, E]$$

of path components of the two spaces and, for each $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and any base point $f_0 \in \mathscr{I}(M, \mathbb{CP}^1)$, an isomorphism

$$\pi_k(\Phi) : \pi_k(\mathscr{I}(M, \mathbb{CP}^1), f_0) \xrightarrow{\cong} \pi_k(\mathscr{C}(M, E), \Phi(f_0))$$

of the corresponding fundamental groups. Here, $[M, E]$ denotes the set of homotopy classes of continuous maps $M \to E$.

Since E is a fibre bundle with Oka fibre $\mathbb{C}^*$ over an Oka base $\mathbb{CP}^1$, E is an Oka manifold (cf. [7, Theorem 5.6.5]), and hence the natural inclusion $\mathscr{O}(M, E) \hookrightarrow \mathscr{C}(M, E)$ is a weak homotopy equivalence by the Oka principle. Thus, we may consider Φ either as a map to $\mathscr{O}(M, E)$, or to $\mathscr{C}(M, E)$.

Let us now identify the path components of the spaces $\mathscr{I}(M, \mathbb{CP}^1)$ and $\mathscr{C}(M, E)$. Denote by $\mathbb{Z}$ the ring of integers. The fundamental group of E equals $\pi_1(E) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and is generated by any simple loop in a fibre $E_x \cong \mathbb{C}^*$ of E (see Lemma 2.1). The first homology group of M equals $H_1(M, \mathbb{Z}) = \mathbb{Z}^k$ for some $k \in \mathbb{Z}_+ \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$, and M is homotopy equivalent to a bouquet of k circles. It follows that the space $\mathscr{C}(M, E)$ has 2^k path components, each determined by the winding numbers modulo two of a map on a collection of loops forming a basis of $H_1(M, \mathbb{Z})$ (see Corollary 2.2). Together with Theorem 1.1 we obtain the following result.

Corollary 1.2. *For any open Riemann surface, M, the space of holomorphic immersions $M \to \mathbb{CP}^1$ has 2^k path components where $H_1(M, \mathbb{Z}) = \mathbb{Z}^k$. A path component is determined by the winding numbers modulo two of the derivative of an immersion $M \to \mathbb{CP}^1$ on a basis of the homology group $H_1(M, \mathbb{Z})$.*

More precise information is obtained from the commutative diagram

$$\begin{tikzcd} & \mathscr{I}(M,\mathbb{C}) \arrow[r, hook, "\iota"] \arrow[d, "\Phi"'] & \mathscr{I}(M,\mathbb{CP}^1) \arrow[d, "\Phi"] \\ \mathscr{C}(M,S^1) \arrow[r, "\simeq"] & \mathscr{C}(M,E|_{\mathbb{C}}) \arrow[r, hook, "\iota"] & \mathscr{C}(M,E) \end{tikzcd}$$

induced by the inclusion $\mathbb{C} \hookrightarrow \mathbb{CP}^1$. Note that $E|_{\mathbb{C}} \cong \mathbb{C} \times \mathbb{C}^* \simeq S^1$, the inclusion of the circle S^1 into $\mathbb{C}^*$ being a homotopy equivalence. Hence, the spaces $\mathscr{C}(M, E|_{\mathbb{C}})$ and $\mathscr{C}(M, S^1)$ are homotopy equivalent, so we may take $\mathscr{C}(M, S^1)$ as the space of formal immersions $M \to \mathbb{C}$.

The first construction of holomorphic immersions from an arbitrary open Riemann surface, M, into $\mathbb{C}$ was given by R. C. Gunning and R. Narasimhan in 1967, [15]. Much more recently, it was proved by F. Forstnerič and F. Lárusson in 2018 (see [9, Theorem 1.5]) that for any such M, holomorphic immersions $M \to \mathbb{C}^n$ for any $n \geq 1$ satisfy the parametric h-principle. More precisely, given a nowhere vanishing holomorphic vector field V on M, the map $\Phi : \mathscr{I}(M, \mathbb{C}^n) \to \mathscr{C}(M, S^{2n-1})$ defined by

$$\Phi(f)(x) = \frac{df_x(V_x)}{\|df_x(V_x)\|} \in S^{2n-1} \subset \mathbb{C}^n, \quad x \in M,$$

satisfies the parametric h-principle, and hence is a weak homotopy equivalence. (Here, S^{2n-1} denotes the unit sphere of $\mathbb{C}^n = \mathbb{R}^{2n}$.) It is also a genuine homotopy equivalence if M is of finite topological type; see [9, Remark 6.3]. For $n = 1$, this result shows that the vertical map in the left column of the above diagram is a weak homotopy equivalence, and is a homotopy equivalence if M is of finite topological type. By Theorem 1.1, the vertical map in the right column is also a weak homotopy equivalence. Hence, Theorem 1.1 and Corollary 2.2 imply the following.

Corollary 1.3. *The natural inclusion $\mathscr{I}(M, \mathbb{C}) \hookrightarrow \mathscr{I}(M, \mathbb{CP}^1)$ induces a surjective map $\pi_0(\mathscr{I}(M, \mathbb{C})) \to \pi_0(\mathscr{I}(M, \mathbb{CP}^1))$ of the respective spaces of path components. This map is determined by sending the winding numbers of the derivative of an immersion $f \in \mathscr{I}(M, \mathbb{C})$ on a basis of the homology group $H_1(M, \mathbb{Z})$ to their reductions modulo 2. In particular, every holomorphic immersion $M \to \mathbb{CP}^1$ can be deformed through a path of holomorphic immersions to a holomorphic immersion $M \to \mathbb{C}$.*

Example 1.4. Let M be a domain in $\mathbb{C}$ with the coordinate z. Fix the standard trivialisation of $T\mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ given by the vector field $\partial/\partial z$. The fibre component of the map Φ (1.2) then takes an immersion $f : M \to \mathbb{C}$ to its complex derivative $f' : M \to \mathbb{C}^*$.

Consider the simplest nontrivial case when M is an annulus in $\mathbb{C}$, and assume for simplicity that M contains the unit circle $\mathbb{T} = \{|z| = 1\}$. The path components of $\mathscr{I}(M, \mathbb{C})$ are then represented by the immersions $z \mapsto z^d$ for $d \in \mathbb{Z}\setminus\{0\}$, and by the *figure eight immersion* for $d = 0$. Indeed, the derivative $(z^d)' = dz^{d-1}$ has winding number $d - 1 \neq -1$ on $\mathbb{T}$, which covers all integers except -1. Let $f : \mathbb{T} \to \mathbb{C}$ be a real analytic figure eight immersion whose tangent vector map $e^{\mathrm{i}t} \mapsto \frac{d}{dt} f(e^{\mathrm{i}t}) = \mathrm{i} f'(e^{\mathrm{i}t})e^{\mathrm{i}t}$ has winding number zero. Then, f complexifies to a holomorphic immersion of a surrounding annulus, and the winding number of $z \mapsto f'(z)$ along the circle $z = e^{\mathrm{i}t}$ equals -1. Immersions $M \to \mathbb{C}$ with winding numbers d_1, d_2 are isotopic as immersions into $\mathbb{CP}^1$ if and only if $d_1 - d_2$ is even, and the two path components of the space $\mathscr{I}(M, \mathbb{CP}^1)$ are represented by any pair immersions $M \to \mathbb{C}$ with $d_1 - d_2$ odd. $\square$

We wish to place Theorem 1.1 in the context of known results.

We have already mentioned that immersions from open Riemann surfaces into $\mathbb{C}^n$ satisfy the parametric h-principle (see [9, Theorem 1.5]). Much earlier, Y. Eliashberg and M. Gromov established the basic h-principle for holomorphic immersions of Stein manifolds of any dimension n to Euclidean spaces $\mathbb{C}^N$ with $N > n$ (see [14], [12, Sect. 2.1.5], and the survey in [7, Sect. 9.6]). A parametric h-principle was obtained in this context by D. Kolarič [17]; however, since it does not pertain to pairs of parameter spaces, his result does not suffice to infer the weak homotopy equivalence, and not even bijectivity between path components of genuine and formal immersions. Recall that a one dimensional Stein manifold is the same thing as an open Riemann surface (see H. Behnke and K. Stein [2]).

A major open problem is whether the h-principle holds for immersions $M^n \to \mathbb{C}^n$ from Stein manifolds of dimension $n > 1$. A formal immersion is given by a trivialisation of the tangent bundle TM, but it is not known whether triviality of TM implies the existence of a holomorphic immersion $M \to \mathbb{C}^n$ (see [7, Problem 9.13.3]). However, there is a Stein structure J' on M, homotopic to the original Stein structure J through a path of Stein structures, such that (M, J') admits a holomorphic immersion into $\mathbb{C}^n$ (see F. Forstnerič and M. Slapar [10] and K. Cieliebak and Y. Eliashberg [4, Theorem 8.43 and Remark 8.44].) The basic h-principle for holomorphic submersions $M \to \mathbb{C}^q$ from Stein manifolds with $\dim M > q \geq 1$ was proved by the author in [6]. Parametric h-principle also holds for directed holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$, provided the directional subvariety $A \subset \mathbb{C}^n$ is a complex cone and $A \setminus \{0\}$ is an Oka manifold (see F. Forstnerič and F. Lárusson [9] and note that immersions into $\mathbb{C}^n$ for any $n \geq 1$ are a special case). The basic case was obtained by A. Alarcón and F. Forstnerič in [1].

The author is not aware of other results in the literature concerning the validity of the h-principle for holomorphic immersions from Stein manifolds to complex manifolds. The h-principle typically fails for maps from non-Stein manifolds, in particular, from compact complex manifolds. It also fails in general for immersions into non-Oka manifolds, for example, into Kobayashi hyperbolic manifolds, due to holomorphic rigidity obstructions.

In the smooth world, the h-principle for immersions $M \to N$ between a pair of smooth or real analytic manifolds holds whenever $\dim M < \dim N$, or $\dim M = \dim N$ and M is an open manifold (see S. Smale [21], M. Hirsch [16], and M. Gromov [12]). However, methods used in the smooth case do not suffice to treat the holomorphic case, and often there are genuine obstructions coming from holomorphic rigidity properties of complex manifolds.

2. Topological preliminaries

Recall that $E = T\mathbb{CP}^1 \setminus \{0\} \xrightarrow{\pi} \mathbb{CP}^1$ denotes the tangent bundle of $\mathbb{CP}^1$ with the zero section removed. Note that $E|_{\mathbb{C}} = \mathbb{C} \times \mathbb{C}^*$. The line bundle $T\mathbb{CP}^1 \to \mathbb{CP}^1$ has degree (Euler number) 2. Indeed, the coordinate vector field $\frac{\partial}{\partial z}$ has no zeros on $\mathbb{C}$, while in the coordinate $w = 1/z$ centred at $\infty = \mathbb{CP}^1 \setminus \mathbb{C}$ it equals $-w^2 \frac{\partial}{\partial w}$, so it has a second order zero at ∞.

Lemma 2.1. *The fundamental group of E equals $\pi_1(E) \cong \mathbb{Z}_2$. Furthermore, the homomorphism*

$$\pi_1(E|_{\mathbb{C}}) = \pi_1(\mathbb{C} \times \mathbb{C}^*) = \mathbb{Z} \longmapsto \mathbb{Z}_2 = \pi_1(E),$$

induced by the inclusion $E|_{\mathbb{C}} \hookrightarrow E$, is $\mathbb{Z} \ni m \mapsto (m \bmod 2) \in \mathbb{Z}_2$.

Proof. We have the comutative diagram

$$\begin{tikzcd} E|_{\mathbb{C}} \arrow[r, hook] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{C} \arrow[r, hook] & \mathbb{CP}^1 \end{tikzcd}$$

The exact sequence of homotopy groups associated to these fibrations is

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2(\mathbb{C}) \arrow[r] \arrow[d] & \pi_1(\mathbb{C}^*) \arrow[r, "\alpha"] \arrow[d] & \pi_1(E|_{\mathbb{C}}) \arrow[r] \arrow[d, "\gamma"] & \pi_1(\mathbb{C}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_2(\mathbb{CP}^1) \arrow[r, "\delta"] & \pi_1(\mathbb{C}^*) \arrow[r, "\beta"] & \pi_1(E) \arrow[r] & \pi_1(\mathbb{CP}^1) \arrow[r] & \cdots, \end{tikzcd}$$

the vertical maps being induced by the horizontal inclusions in the above diagram. In the top line we have that $\pi_2(\mathbb{C}) = 0 = \pi_1(\mathbb{C})$, and α is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. In the bottom line we have

$$\cdots \longrightarrow \pi_2(\mathbb{CP}^1) = \mathbb{Z} \xrightarrow{\delta} \pi_1(\mathbb{C}^*) = \mathbb{Z} \xrightarrow{\beta} \pi_1(E) \longrightarrow \pi_1(\mathbb{CP}^1) = 0 \longrightarrow \cdots,$$

so $\pi_1(E)$ is the cokernel of the boundary map $\pi_2(\mathbb{CP}^1) = \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} = \pi_1(\mathbb{C}^*)$. Take a generator of $\pi_2(\mathbb{CP}^1)$ in the form of a continuous map from the closed disc D onto $\mathbb{CP}^1$, collapsing the boundary of D to a point p in $\mathbb{CP}^1$. Lift this map to E using a nowhere vanishing holomorphic vector field V on $\mathbb{CP}^1 \setminus \{p\}$ with a double zero at p. (For $p = \infty$ we may take $V = \frac{\partial}{\partial z}$ as seen above.) The boundary of D then lifts to a loop in E_p that winds twice around zero. Thus, the map $\delta : \mathbb{Z} \to \mathbb{Z}$ equals $m \mapsto 2m$, so $\pi_1(E) \cong \mathbb{Z}_2$ and $\beta : \mathbb{Z} \to \mathbb{Z}_2$ is the map $m \mapsto m \bmod 2$. The diagram also implies that $\gamma : \mathbb{Z} \to \mathbb{Z}_2$ is surjective, so it equals $m \mapsto m \bmod 2$. ☐

Corollary 2.2. *For every open Riemann surface, M, the space of continuous maps $M \to E$ has 2^k path components where $H_1(M,\mathbb{Z}) = \mathbb{Z}^k$, $k \in \{0,1,2,\ldots,\infty\}$. The map $\mathscr{C}(M,S^1) \hookrightarrow \mathscr{C}(M,E)$, induced by the inclusion of the circle S^1 into a fibre $E_z \cong \mathbb{C}^*$, determines a surjective map*

$$\mathbb{Z}^k = H^1(M,\mathbb{Z}) = \pi_0(\mathscr{C}(M,S^1)) \longrightarrow \pi_0(\mathscr{C}(M,E)) = H^1(M,\mathbb{Z}_2) = \mathbb{Z}_2^k$$

given on every generator by $\mathbb{Z} \ni m \mapsto (m \bmod 2) \in \mathbb{Z}_2$.

Proof. This follows immediately from Lemma 2.1 and the fact that M has the homotopy type of a bouquet of k circles, where $H_1(M,\mathbb{Z}) = \mathbb{Z}^k$. Note that $\pi_0(\mathscr{C}(M,S^1)) = [M,S^1] = H^1(M,\mathbb{Z}) = \mathbb{Z}^k$, and similarly for $\pi_0(\mathscr{C}(M,E))$. ☐

3. A parametric approximation theorem for immersions from discs to $\mathbb{CP}^1$

In this section we prove a homotopy approximation theorem for holomorphic immersions from a pair of discs in $\mathbb{C}$ into $\mathbb{CP}^1$; see Proposition 3.1. This is one of the main ingredients in the proof of Theorem 1.1.

Let $Q \subset P$ be compact Hausdorff spaces which will be used as parameter spaces. (To establish weak homotopy equivalence in Theorem 1.1, it suffices to consider two special cases: $P = S^k$ (the k-dimensional sphere) for any $k \in \mathbb{N}$ and $Q = \varnothing$, and $P = \mathbb{B}^k$ (the closed ball in $\mathbb{R}^k$) for any $k \in \mathbb{Z}_+$ and $Q = bP = S^{k-1}$.) The following conventions will be used in the sequel.

(1) A holomorphic map on a compact set K in a complex manifold M is one that is holomorphic on an unspecified open neighbourhood of K.
(2) A homotopy of maps is holomorphic on K if all maps in the family are holomorphic on the same open neighbourhood of K in M.

(3) A holomorphic map f is said to enjoy a certain property on K if it enjoys that property on a neighbourhood of K.
(4) When performing standard procedures such (uniform) approximation of a family of holomorphic maps on a compact set K, their domain is allowed to shrink around K.

Proposition 3.1. *Let $Q \subset P$ be as above, and let $\Delta_0 \subset \Delta_1$ be a pair of compact smoothly bounded discs in $\mathbb{C}$ (diffeomorphic images of the closed unit disc). Assume that $f_p : \Delta_0 \to \mathbb{CP}^1$ is a family of holomorphic immersions depending continously on the parameter $p \in P$ such that for all $p \in Q$ the map f_p extends to a holomorphic immersion $f_p : \Delta_1 \to \mathbb{CP}^1$. Let* dist *denote the spherical distance function on $\mathbb{CP}^1$. Given $\epsilon > 0$ there exists a continuous family of holomorphic immersions $\tilde{f}_p : \Delta_1 \to \mathbb{CP}^1$ $(p \in P)$ such that*

(a) $\mathrm{dist}(f_p(z), \tilde{f}_p(z)) < \epsilon$ *for all $z \in \Delta_0$ and $p \in P$, and*
(b) $\tilde{f}_p = f_p$ *for all $p \in Q$.*

Proof. A holomorphic immersion $U \to \mathbb{CP}^1$ from an open set $U \subset \mathbb{C}$ is effected by a meromorphic function f on U with only simple poles such that $f'(z) \neq 0$ for any point $z \in U$ which is not a pole of f. At a pole $a \in U$ of f we have

$$f(z) = \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \cdots, \quad f'(z) = \frac{-c_{-1}}{(z-a)^2} + c_1 + \cdots. \tag{3.1}$$

Thus, f' has a second order pole at a and its residue equals zero. Conversely, a meromorphic function on a simply connected domain U which has no zeros, and whose poles (if any) are precisely of the second order with vanishing residues, is the derivative of a holomorphic immersion $U \to \mathbb{CP}^1$.

Consider first the special case when the functions f_p have no poles on their domains. Pick a point $z_0 \in \Delta_0$. Since the derivatives f'_p are novanishing holomorphic functions, there is a continuous family of holomorphic logarithms

$$\xi_p = \log(f'_p/f'_p(z_0)) \in \mathscr{O}(\Delta_0), \quad p \in P$$

(and $\xi_p \in \mathscr{O}(\Delta_1)$ if $p \in Q$) such that $\xi_p(z_0) = 0$ for all $p \in P$. By the parametric Oka-Weil theorem [7, Theorem 2.8.4] we can approximate this family uniformly on $(P \times \Delta_0) \cup (Q \times \Delta_1)$ by a continuous family of holomorphic functions $\{\tilde{\xi}_p \in \mathscr{O}(\Delta_1)\}_{p\in P}$ such that $\tilde{\xi}_p(z_0) = 0$ for all $p \in P$ and $\tilde{\xi}_p = \xi_p$ for all $p \in Q$. The family of holomorphic functions given by

$$\tilde{f}_p(z) = f_p(z_0) + f'_p(z_0) \int_{z_0}^{z} e^{\tilde{\xi}_p(\zeta)}\, d\zeta, \quad z \in \Delta_1,\ p \in P,$$

then clearly satisfies the conclusion of the proposition.

The proof is more involved in the presence of poles. We shall need the following lemma.

Lemma 3.2. *(Assumptions as in Proposition 3.1.) Write $P_0 = P$. There are an integer $k \in \mathbb{N}$, a neighbourhood $P_1 \subset P_0$ of Q, and for every $p \in P$ a family of k not necessarily distinct points $A(p) = \{a_1(p), \dots, a_k(p)\}$ in $\mathbb{C}$, depending continuously on $p \in P$ and satisfying the following conditions.*

(a) *For every $p \in P$, the points in $A(p) \cap \Delta_1$ are pairwise distinct.*
(b) *For every $p \in P_j$ $(j \in \{0, 1\})$, $A(p) \cap \Delta_j$ is the set of poles of f_p in Δ_j.*

More precisely, we consider A as a map $A : P \to \mathrm{Sym}^k(\mathbb{C})$ into the kth symmetric power of $\mathbb{C}$, and its continuity is understood in this sense. A point $a_i(p) \in A(p)$ such that $a_i(p) = a_j(p)$ for some $i \neq j$ is called a *multiple point* of $A(p)$, and the remaining points are called *simple points*.

Proof. By the parametric Oka principle for maps into the complex homogeneous manifold $\mathbb{CP}^1$ (see [7, Theorem 5.4.4 and Proposition 5.6.1]) we can approximate the family of immersions $\{f_p\}_{p \in P}$ uniformly on a neighbourhood of $(P \times \Delta_0) \cup (Q \times \Delta_1)$ in $P \times \mathbb{C}$ by a continuous family of rational functions $\{\tilde{f}_p\}_{p \in P}$. Replacing $\tilde{f}_p$ by $\tilde{f}_p(z) + cz^N$ for some small $c > 0$ and big $N \in \mathbb{N}$, we may ensure that for each $p \in P$ the function $\tilde{f}_p$ has a pole of order N at $\infty = \mathbb{CP}^1 \setminus \mathbb{C}$. For each $p \in P$ we denote by $B(p) = \{b_1(p), \dots, b_k(p)\}$ the family of poles of $\tilde{f}_p$ lying in $\mathbb{C}$ (which is all except the one at ∞), where each point is listed with multiplicity equal to the order of the pole. Since ∞ is an isolated pole of each $\tilde{f}_p$, there is a disc in $\mathbb{C}$ containing $B(p)$ for all $p \in P$. Assuming as we may that the approximation of f_p by $\tilde{f}_p$ is close enough for each p, there are open neighbourhoods $P_1 \subset P_0 = P$ of Q, and $\Delta'_j \subset \mathbb{C}$ of Δ_j for $j = 0, 1$, such that $B(p)$ has only simple points in Δ'_j for all $p \in P_j$ $(j \in \{0, 1\})$. This means that $\tilde{f}_p$, considered as a map into $\mathbb{CP}^1$, is an immersion on Δ'_j for all $p \in P_j$, $j \in \{0, 1\}$. The remaining poles of $\tilde{f}_p$ may be of higher order.

Assuming that the approximation of the immersion f_p by $\tilde{f}_p$ is close enough for each $p \in P$ on the respective domain, there is a continuous family of injective holomorphic maps ϕ_p $(p \in P = P_0)$, defined and close to the identity map on a neighbourhood of Δ_j if $p \in P_j$ $(j \in \{0, 1\})$, such that

$$f_p = \tilde{f}_p \circ \phi_p, \quad p \in P. \tag{3.2}$$

This holds by the parametric version of [7, Lemma 9.12.6] or [6, Lemma 5.1], which is easily seen by the same proof. Thus, for $p \in P_j$ $(j \in \{0, 1\})$, ϕ_p maps the set of poles of $\tilde{f}_p$ near the disc Δ_j bijectively onto the set of poles of f_p near Δ_j. We now extend ϕ_p to a continuous family of smooth diffeomorphisms $\phi_p : \mathbb{CP}^1 \to \mathbb{CP}^1$ $(p \in P)$ which are fixed near ∞ such that the families of points $A(p) := \phi_p(B(p)) = \{\phi_p(b_j(p))\}_{j=1}^k$ for $p \in P$ satisfy the conclusion of the lemma. This is accomplished by choosing ϕ_p for $p \in P \setminus Q$ such that it expels all multiple points of $B(p)$ out of the big disc Δ_1. $\square$

We continue with the proof of Proposition 3.1. For any $p \in P$ let $A(p)$ be given by Lemma 3.2. Consider the following family of holomorphic polynomials on $\mathbb{C}$ depending continuously on the parameter $p \in P$:

$$\Theta_p(z) = \prod_{j=1}^{k} (z - a_j(p))^2, \quad z \in \mathbb{C}.$$

The function

$$z \mapsto h_p(z) = f'_p(z)\Theta_p(z), \quad p \in P, \tag{3.3}$$

is then nonvanishing holomorphic on Δ_0 for every $p \in P$, and it is nonvanishing holomorphic on Δ_1 if p lies in a small neighbourhood $P_1 \subset P$ of Q.

Fix $p \in P$ and a point $a \in A(p) \cap \Delta_0$ (resp. $a \in A(p) \cap \Delta_1$ if $p \in P_1$). Let

$$g_{p,a}(z) := \Theta_p(z)/(z-a)^2 = \prod_{b \in A(p) \setminus \{a\}} (z-b)^2, \quad z \in \mathbb{C}.$$

A calculation shows that for any holomorphic function $h(z)$ near $z = a$,

$$\mathrm{Res}_{z=a} \frac{h(z)}{\Theta_p(z)} = \lim_{z \to a} \left(\frac{h(z)}{g_{p,a}(z)} \right)' = \frac{g_{p,a}(a)h'(a) - g'_{p,a}(a)h(a)}{g_{p,a}(a)^2}.$$

The function h/Θ_p admits a meromorphic primitive at a if and only if this residue vanishes, which is equivalent to the condition

$$\frac{h'(a)}{h(a)} = \frac{g'_{p,a}(a)}{g_{p,a}(a)} =: c_{p,a}. \tag{3.4}$$

This holds for $h_p/\Theta_p = f'_p$ whose primitive is f_p. Thus, when approximating the function h_p (3.3) on Δ_0 by a function $\tilde{h}_p \in \mathscr{O}(\Delta_1)$, we must ensure that

$$\frac{\tilde{h}'_p(a)}{\tilde{h}_p(a)} = c_{p,a}, \quad a \in A(p) \cap \Delta_1. \tag{3.5}$$

We now explain how to do this. Fix a point $z_0 \in \Delta_0$. The family of logarithms

$$\xi_p(z) = \log(h_p(z)/h_p(z_0)), \quad \xi_p(z_0) = 0, \tag{3.6}$$

is well defined and holomorphic on Δ_j for $p \in P_j$ ($j = 0, 1$). Note that

$$\eta_p := \xi'_p = \frac{h'_p}{h_p}, \tag{3.7}$$

so the conditions (3.4) for the functions $h = h_p$ are equivalent to

$$\eta_p(a) = c_{p,a} \ \text{ for all } a \in A(p) \cap \Delta_j, \ p \in P_j, \ j = 0, 1. \tag{3.8}$$

(Recall that $P_0 = P$.) To complete the proof, we must find a continuous family of holomorphic functions $\tilde\eta_p \in \mathscr{O}(\Delta_1)$, $p \in P$, such that

(i) $\tilde\eta_p$ approximates η_p uniformly on Δ_0 for all $p \in P$,
(ii) $\tilde\eta_p = \eta_p$ for all $p \in Q$, and
(iii) $\tilde\eta_p(a) = c_{p,a}$ for all $a \in A(p) \cap \Delta_1$ and $p \in P$.

Indeed, having such function $\tilde\eta_p$, we retrace our path back by setting for all z in a neighbourhood of Δ_1 and all $p \in P$:

$$\tilde\xi_p(z) = \xi_p(z_0) + \int_{z_0}^{z} \tilde\eta_p(\zeta)\,d\zeta, \qquad \tilde h_p(z) = h_p(z_0)\exp\tilde\xi_p(z),$$

$$\tilde f_p(z) = f_p(z_0) + \int_{z_0}^{z} \frac{\tilde h_p(\zeta)}{\Theta_p(\zeta)}\,d\zeta$$

(see (3.3), (3.6), and (3.7)). The integrals for $\tilde f_p$ are well defined and independent of the choice of a path in the disc Δ_1 since, by the construction, the function $\tilde h_p/\Theta_p$ has vanishing residue at every point in $A(p) \cap \Delta_1$.

It remains to construct a family of functions $\tilde\eta_p$ satisfying conditions (i)–(iii) above. This is a linear interpolation problem at finitely many points depending continuously on $p \in P$. Since a convex combination of solutions is again one, we may use partitions of unity on the parameter space P. We proceed as follows. Fix a point $p_0 \in P$. If p_0 belongs to the neighbourhood P_1 of Q, there is nothing to do since h_p is already holomorphic on the big disc Δ_1 for $p \in P_1$ and we shall use this family in the sequel. Assume now that $p_0 \in P \setminus P_1$. For $j = 0, 1$ we choose open neighbourhoods $D_j \subset \mathbb{C}$ of Δ_j such that $\Delta_j \subset D_j \subset \Delta'_j$ and $A(p_0) \cap bD_j = \varnothing$. By continuity of the map $p \mapsto A(p)$, there is an open neighbourhood $U = U_{p_0} \subset P \setminus Q$ of p_0 such that $A(p) \cap bD_j = \varnothing$ for $j = 0, 1$ and $p \in U$. It follows that for $j = 0, 1$ the number of points in the set $A(p) \cap D_j$ is independent of $p \in U$. Let

$$A(p) \cap D_1 = \{a_1(p), \ldots, a_m(p)\}, \qquad p \in U,$$

where the points $a_j(p)$ are distinct and depend continuously on $p \in U$. Consider the polynomials

$$\phi_{p,i}(z) = \frac{\prod_{j\neq i}(z - a_j)}{\prod_{j\neq i}(a_i - a_j)}, \qquad p \in U,\ i = 1, \ldots, m.$$

Then, $\phi_{p,i}(a_j) = \delta_{i,j}$. For $p \in U$, any function η_p satisfying condition (3.8) is of the form

$$\eta_p(z) = \sum_{i=1}^{m} c_{p,a_i}\phi_{p,i}(z) + \sigma_p(z)\prod_{i=1}^{m}(z - a_i)$$

for some $\sigma_p \in \mathscr{O}(\Delta_0)$. Approximating σ_p uniformly on Δ_0 by a function $\tilde\sigma_p \in \mathscr{O}(\Delta_1)$ depending continuously on $p \in U$, we get functions $\tilde\eta_p \in \mathscr{O}(\Delta_1)$ given by

$$\tilde\eta_p(z) = \sum_{i=1}^m c_{p,a_i}\phi_{p,i}(z) + \tilde\sigma_p(z)\prod_{i=1}^m (z-a_i)$$

depending continuously on $p \in U$ and satisfying conditions (i) and (iii). On the other hand, condition (ii) is vacuous since $U \cap Q = \varnothing$.

To complete the proof, we cover the compact set $P \setminus P_1$ by finitely many open sets $U_1, \ldots, U_l \subset P \setminus Q$ of this type, add the set $U_0 = P_1$ into the collection, choose a partition of unity $\{\chi_i\}_{i=0}^l$ on P subordinate to the open cover $\{U_0, \ldots, U_l\}$ of P, and use it to combine the resulting families of solutions $\tilde\eta_{p,i}$ for $p \in U_i$ $(i = 0, \ldots, l)$ into a global family of solutions

$$\tilde\eta_p = \sum_{i=0}^l \chi_i(p)\tilde\eta_{p,i}, \quad p \in P.$$

Since none of sets $U_1, \ldots, U_l$ intersects Q, the resulting family $\tilde\eta_p$ also satisfies condition (ii) for $p \in Q$. This completes the proof. □

4. Parametric Mergelyan theorem for manifold valued maps

The main result of this section, Theorem 4.3, provides a parametric version of Mergelyan's approximation theorem for maps from certain compact sets in Riemann surfaces to arbitrary complex manifolds. Although this is a relatively straightforward extension of the nonparametric case (see [8, Theorem 1.4] and [5, Theorem 16]), we could not find it in the literature, so we take this opportunity to prove it. Our proof also applies to families of maps from certain compact subsets in higher dimensional complex manifolds; see Remark 4.4.

Given complex manifolds M and X and a compact subset S of M, we denote by $\mathscr{A}(S,X)$ the space of continuous maps $S \to X$ which are holomorphic on the interior of S. We write $\mathscr{A}(S,\mathbb{C}) = \mathscr{A}(S)$.

Definition 4.1. A compact set S in a Riemann surface has the *Mergelyan property* (or the *Vitushkin property*) if every function in $\mathscr{A}(S)$ can be approximated uniformly on S by functions holomorphic on neighbourhoods of S.

Denote by $\overline{\mathscr{O}}(S)$ the uniform closure in $\mathscr{C}(S)$ of the set $\{f|_S : f \in \mathscr{O}(S)\}$, so S has the Mergelyan property if and ony if $\mathscr{A}(S) = \overline{\mathscr{O}}(S)$. If S is a plane compact then by Runge's theorem [20] the set $\overline{\mathscr{O}}(S)$ equals the rational algebra $\mathscr{R}(S)$, i.e., the uniform closure in $\mathscr{C}(S)$ of the space of rational functions on $\mathbb{C}$ with poles off S. A characterization of this class of plane compacts in terms of the continuous analytic capacity was given by A. G. Vitushkin in 1966 [22, 23]. See also the exposition in T. W. Gamelin's book [11].

We shall also consider compact sets of the following special kind.

Definition 4.2 (Admissible sets in Riemann surfaces)**.** A compact set S in a Riemann surface M is *admissible* if it is of the form $S = K \cup \Lambda$, where K is the union of finitely many pairwise disjoint compact domains in M with piecewise $\mathscr{C}^1$ boundaries and $\Lambda = \overline{S \setminus K}$ is the union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only in their endpoints (or not at all) and such that their intersections with the boundary bK of K are transverse.

It was shown in [8, Theorem 1.4] that if a compact set S in a Riemann surface has the Mergelyan property for functions, then it also has the Mergelyan property for maps into an arbitrary complex manifold X. Furthermore, if S is admissible then the Mergelyan approximation theorem in the $\mathscr{C}^r(S,X)$ topology holds for maps in $\mathscr{A}^r(S,X) = \mathscr{A}(S,X) \cap \mathscr{C}^r(S,X)$ (see [5, Theorem 16]). We now prove the following parametric version of this result.

Theorem 4.3. *If M is a Riemann surface and S is a compact set in M with the Mergelyan property, then S has the parametric Mergelyan property for maps to an arbitrary complex manifold X.*

More precisely, given a family of maps $f_p \in \mathscr{A}(S,X)$ depending continuously on a parameter p in a compact Hausdorff space P, a Riemannian distance function dist *on X, and a number $\epsilon > 0$, there are a neighbourhood $U \subset M$ of S and a family of holomorphic maps $\tilde{f}_p : U \to X$, depending continuously on $p \in P$, such that $\mathrm{dist}(\tilde{f}_p(x), f_p(x)) < \epsilon$ holds for all $x \in S$ and $p \in P$.*

If $S = K \cup \Lambda$ is an admissible set in M and $f_p \in \mathscr{A}^r(S,X)$ for some $r \in \mathbb{N}$ with a continuous dependence on $p \in P$, then the family f_p can be approximated in the $\mathscr{C}^r(S,X)$ topology by a family of holomorphic maps $\tilde{f}_p \in \mathscr{O}(S,X)$ in an open neighbourhood of S, depending continuously on $p \in P$.

If in addition there is a compact subset Q of P such that $f_p \in \mathscr{O}(S)$ for all $p \in Q$, then the family $\tilde{f}_p$ can be chosen such that $\tilde{f}_p = f_p$ for all $p \in Q$.

If M is an open Riemann surface, the set S has no holes in$M$, X is an Oka manifold, and $f_p \in \mathscr{O}(M,X)$ for all $p \in Q$, then the approximating family of maps $\tilde{f}_p$ $(p \in P)$ in these results can be chosen holomorphic on all of M.

The last statement is a direct consequence of the previous ones and the parametric Oka principle for maps from Stein manifolds (in particular, from open Riemann surfaces) to Oka manifolds; see [7, Theorem 5.4.4].

Proof. Recall that a compact set S in a complex manifold M is a *Stein compact* if S admits a basis of open Stein neighbourhoods. Every proper compact subset of a connected Riemann surface is obviously a Stein compact.

For the sake of motivation, we first recall the proof of [8, Theorem 1.4] in the nonparametric case. Assume that S has the Mergelyan property, i.e., $\mathscr{A}(S) = \overline{\mathscr{O}}(S)$. Let $f \in \mathscr{A}(S, X)$. Pick a point $s_0 \in S$ and choose a closed disc $D \subset M$ around s_0. By the theorem of Boivin and Jiang [3, Theorem 1], the assumption $\mathscr{A}(S) = \overline{\mathscr{O}}(S)$ implies $\mathscr{A}(S \cap D) = \overline{\mathscr{O}}(S \cap D)$. By choosing D small enough, $f(S \cap D)$ lies in a coordinate chart of X, and hence (by the Mergelyan property for functions) the map f can be approximated uniformly on $S \cap D$ by maps into X that are holomorphic on neighbourhoods of $S \cap D$. Thus, f can be approximated locally on S by holomorphic maps. It follows from a theorem of E. Poletsky [19] (see also [5, Theorem 32]) that its graph $G_f = \{(s, f(s)) : s \in S\}$ is a Stein compact in $M \times X$. From this, we easily infer that S enjoys the Mergelyan property for maps $S \to X$ (see [5, Lemma 3]); here is an outline of proof.

Let $V \subset M \times X$ be a Stein neighbourhood of the graph G_f. By the Remmert-Bishop-Narasimhan theorem (see [7, Theorem 2.4.1]) there is a proper holomorphic embedding $\phi : V \hookrightarrow \mathbb{C}^N$ into a complex Euclidean space. By the Docquier-Grauert theorem (see [7, Theorem 3.3.3]) there is a neighbourhood $\Omega \subset \mathbb{C}^N$ of $\phi(V)$ and a holomorphic retraction $\rho : \Omega \to \phi(V)$. Assuming that $\overline{\mathscr{O}}(S) = \mathscr{A}(S)$, we can approximate the map $\phi \circ f : S \to \mathbb{C}^N$ as closely as desired uniformly on S by a holomorphic map $G : U \to \Omega \subset \mathbb{C}^N$ from an open neighbourhood $U \subset M$ of S. The holomorphic map

$$g = pr_X \circ \phi^{-1} \circ \rho \circ G : U \to X$$

then approximates f uniformly on S.

We now consider the parametric case. When $X = \mathbb{C}$, the proof is a simple application of the nonparametric case, using a continuous partition of unity on P. Indeed, there is a finite set $\{p_1, \ldots, p_k\} \subset P$ and for each $j = 1, \ldots, k$ an open set $P_j \subset P$, with $p_j \in P_j$, such that

$$\|f_p - f_{p_j}\|_S := \max_{s \in S} |f_p(s) - f_{p_j}(s)| < \frac{\epsilon}{4} \quad \text{for every } p \in P_j,\ j = 1, \ldots, k. \tag{4.1}$$

Since S has the Mergelyan property, there are functions $g_j \in \mathscr{O}(S)$ such that

$$\|g_j - f_{p_j}\|_S < \frac{\epsilon}{4} \quad \text{for } j = 1, \ldots, k. \tag{4.2}$$

Let $\{\chi_j\}_{j=1}^k$ be a partition of unity on P subordinate to the cover $\{P_j\}_{j=1}^k$. Set

$$\tilde{f}_p = \sum_{j=1}^{k} \chi_j(p) g_j \in \mathscr{O}(S), \quad p \in P.$$

For every $p \in P$ we then have $\tilde{f}_p - f_p = \sum_{j=1}^k \chi_j(p)(g_j - f_p)$. If $p \in P_j$ then

$$\|g_j - f_p\|_S \le \|g_j - f_{p_j}\|_S + \|f_{p_j} - f_p\|_S < \frac{\epsilon}{2}$$

by (4.1) and (4.2). If on the other hand $p \notin P_j$ then $\chi_j(p) = 0$, so this term does not appear in the above sum for $\tilde{f}_p$. It follows that

$$\|\tilde{f}_p - f_p\|_S \le \sum_{j=1}^k \chi_j(p)\|g_j - f_p\|_S < \frac{\epsilon}{2} \quad \text{for every } p \in P.$$

Finally, to satisfy the last condition in the theorem (i.e., fixing the maps $f_p \in \mathscr{O}(S)$ for the parameter values $p \in Q$), we proceed as follows. Choose a compact neighbourhood $K \subset M$ of S such that $f_p \in \mathscr{A}(K)$ for all $p \in Q$. As $\mathscr{A}(K)$ is a Banach space, Michael's extension theorem [18] (see also [7, Theorem 2.8.2]) yields a continuous extension of the family $\{f_p \in \mathscr{A}(K)\}_{p\in Q}$ to a continuous family $\{\xi_p \in \mathscr{A}(K)\}_{p\in P}$ such that $\xi_p = f_p$ for $p \in Q$. Let $\{\tilde{f}_p\}_{p\in P}$ be the family constructed above. Choose a small neighbourhood $P_0 \subset P$ of Q such that $\|\xi_p - f_p\|_S < \epsilon/2$ for all $p \in P_0$. Pick a continuous function $\chi : P \to [0,1]$ supported in P_0 such that $\chi = 1$ on Q and replace $\tilde{f}_p$ by $\chi(p)\xi_p + (1 - \chi(p)\tilde{f}_p$. This family enjoys all required conditions.

Consider now the general case of maps to a complex manifold X. Given a continuous family $\{f_p\}_{p\in P} \in \mathscr{A}(S, X)$, the proof of the basic case and the compactness of P yield an open cover $\{P_j\}_{j=1}^k$ of P and Stein domains V_j in $M \times X$ for $j = 1, \ldots, k$ such that

$$\overline{\bigcup_{p\in P_j} G_{f_p}} \subset V_j, \qquad j = 1, \ldots, k.$$

Embedding V_j into a Euclidean space $\mathbb{C}^N$, the proof of the special case and the parametric approximation theorem for functions (hence for maps to $\mathbb{C}^N$) allow us to approximate each family $\{f_p\}_{p\in P_j}$ as closely as desired uniformly on S by a continuous family $\{g_{p,j}\}_{p\in P_j} \in \mathscr{O}(U, X)$, where $U \subset M$ is an open neighbourhood of S. Furthermore, we can ensure that $g_{p,j} = f_p$ for $p \in P_j \cap Q$. Assuming that the approximations are close enough and shrinking U around S if necessary, we can patch the families $\{g_{p,j}\}_{p\in P_j}$ into a single family $\{\tilde{f}_p\}_{p\in P} \in \mathscr{O}(U)$ satisfying the conclusion of the theorem by applying the *method of successive patching*; see [7, p. 78]. This means that we patch a pair of families at a time, using the embedding $V_j \hookrightarrow \mathbb{C}^N$ of the Stein domain containing maps from both families on the set of patching.

If S is an admissible set then the same proof applies to continuous families of maps in $\mathscr{A}^r(S, X)$ for any $r \in \mathbb{N}$. We use [5, Theorem 16] to approximate single maps, and the rest of the procedure is exactly as above. $\square$

Remark 4.4. Theorem 4.3 and its proof generalise to the case when M is a manifold of higher dimension and S is a *strongly admissible set* in M in the sense of [5, Definition 5]. This means that S is a Stein compact of the form $S = K \cup \Lambda$, where $K = \overline{D}$ is the closure of a strongly pseudoconvex Stein domain and $\Lambda = \overline{S \setminus K}$ is a totally real submanifold of M. A discussion of this subject can be found in [5, Sect. 7.2]; see in particular [5, Corollary 9] which gives the basic (nonparametric) case of the Mergelyan approximation theorem in this situation. A slightly less precise result in this direction (with some loss of derivatives), but applying to a more general geometric situation concerning sections of holomorphic submersion onto Stein manifolds, is [7, Theorem 3.8.1]. Its parametric generalisations are used in the cited book with ad hoc proofs, similar to the one given in the proof of Theorem 4.3 above.

5. Proof of Theorem 1.1

Recall that E stands for the tangent bundle of $\mathbb{CP}^1$ with the zero section removed, and continuous maps $M \to E$ from an open Riemann surface M are called *formal immersions* from M to $\mathbb{CP}^1$. Let V be a nowhere vanishing holomorphic vector field on M. Such V serves to trivialise the tangent bundle TM; the precise choice will not be important. Every genuine holomorphic immersion $f : M \to \mathbb{CP}^1$ determines the formal immersion $\Phi(f) = df(V) : M \to E$ (see (1.1)). Now, the weak homotopy equivalence asserted in Theorem 1.1 follows from the following parametric h-principle which basically says that a continuous family of formal immersions $M \to E$ can be deformed to a continuous family of genuine holomorphic immersions $M \to \mathbb{CP}^1$, and the homotopy may be kept fixed on a compact subset of the parameter space where the given family already consists of genuine immersions.

Theorem 5.1 (The parametric h-principle for immersions $M \to \mathbb{CP}^1$)**.** *Let M be an open Riemann surface, V be a nowhere vanishing holomorphic vector field on M, and $Q \subset P$ be compact Hausdorff spaces. Assume that $f_p : M \to \mathbb{CP}^1$, $p \in Q$, is a continuous family of holomorphic immersions and $\sigma_p : M \to E$, $p \in P$, is a continuous family of maps (formal immersions) such that for all $p \in Q$ we have $\sigma_p = \Phi(f_p) := df_p(V)$. Then, the family $\{f_p\}_{p\in Q}$ extends to a continuous family of holomorphic immersions $f_p : M \to \mathbb{CP}^1$, $p \in P$, such that there is a homotopy $\sigma_p^t : M \to E$ ($p \in P$, $t \in [0,1]$) which is fixed for $p \in Q$ and satisfies $\sigma_p^0 = \sigma_p$ and $\sigma_p^1 = \Phi(f_p)$ for all $p \in P$.*

Indeed, Theorem 1.1 follows from Theorem 5.1 applied with the pairs of parameter spaces $P = S^k$ (the k-dimensional sphere) and $Q = \varnothing$, and $P = \mathbb{B}^k$ (the closed ball in $\mathbb{R}^k$) and $Q = bP = S^{k-1}$ (cf. [7, proof of Corollary 5.5.6]).

Proof. The main ingredients have already been established: the parametric approximation theorem for holomorphic immersions from a pair of discs into $\mathbb{CP}^1$ (see Proposition 3.1), the parametric Mergelyan approximation theorem on admissible sets (see Theorem 4.3), and the parametric h-principle for smooth immersions due to Smale [21] and Hirsch [16] (see also Gromov [13, 12]). The proof of the theorem amounts to an induction in which these ingredients are combined. Although this construction is rather standard and is similar to those given in [7, proof of Theorem 5.4.4] and in [1, 9], among many others. we include the details by the request of the referee and for the benefit of the readers who may not be familiar with the h-principle.

We exhaust M by an increasing sequence

$$\varnothing = D_0 \subset D_1 \subset D_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} D_j = M$$

of compact, smoothly bounded, not necessarily connected domains without holes (i.e., such that $M \setminus D_j$ has no relatively compact connected components for any j), where D_1 is a disc and every pair $D_j \subset \mathring{D}_{j+1}$ is of one of the following two types:

(i) *The noncritical case:* D_{j+1} is the disjoint union $D'_{j+1} \cup D''_{j+1}$ of smoothly bounded domains such that $\overline{D'_{j+1} \setminus D_j}$ is diffeomorphic to $bD_j \times [0,1]$ (so it is the union of finitely many pairwise disjoint compact annuli) and D''_{j+1} is either a disc or the empty set.

(ii) *The critical case:* D_{j+1} admits a deformation retraction onto an admissible set $S = D_j \cup \Lambda \subset \mathring{D}_{j+1}$ (see Definition 4.2), where Λ is a smooth arc in $\mathring{D}_{j+1} \setminus D_j$ attached with its endpoints to bD_j. (This handle attachment occurs in only one connected component of D_{j+1}, while the other components are noncritical extensions of the corresponding components of D_j as in case (i).)

The critical case (ii) has the following three topologically distinct subcases.

(ii$_1$) The endpoints of the arc Λ are attached to the same connected component of bD_j. In this case, the topological genus satisfies $g(D_{j+1}) = g(D_j)$ and bD_{j+1} has one more connected component than bD_j.

(ii$_2$) The endpoints of Λ are attached to different connected components of the boundary of the same connected component of D_j. In this case, $g(D_{j+1}) = g(D_j) + 1$ and the number of boundary curves remains the same. The domain $\overline{D_{j+1} \setminus D_j}$ consists of a *pair of pants* (i.e., a compact surface with genus one and three boundary components) together with finitely many pairwise disjoint compact annuli.

(ii$_3$) The endpoints of Λ are attached to different connected components of D_j. In this case, $g(D_{j+1}) = g(D_j)$ and the number of boundary curves decreases by one.

In all three cases (ii$_1$)–(ii$_3$) the Euler numbers satisfy $\chi(D_{j+1}) = \chi(D_j) - 1$.

An exhaustion of this type is obtained by taking regular sublevel sets of a strongly subharmonic Morse exhaustion function $\rho : M \to \mathbb{R}_+$ such that ρ has at most one critical point in $\mathring{D}_{j+1} \setminus D_j$ for every $j = 0, 1, 2, \ldots$. The case (i) with $D''_{j+1} \neq \varnothing$ is usually included in the critical case since it corresponds to passing a local minimum of ρ at which a new connected component of the sublevel set $\{\rho < c\}$ appears, however, the procedure that will be required in this case is similar to the one in the noncritical case.

Given an open subset U of M, we denote by

$$\mathscr{I}(P \times U, \mathbb{CP}^1)$$

the space of continuous maps $f : P \times U \to \mathbb{CP}^1$ such that for every $p \in P$ the map $f_p = f(p, \cdot\,) : U \to \mathbb{CP}^1$ is a holomorphic immersion. Let dist denote the spherical distance function on $\mathbb{CP}^1$. Let $f^0 = f \in \mathscr{I}(Q \times M, \mathbb{CP}^1)$ be as in the theorem. Pick a number $\epsilon > 0$ and set $\epsilon_0 = \epsilon$.

We shall inductively construct sequences of maps $f^j \in \mathscr{I}(P \times U_j, \mathbb{CP}^1)$, where $U_j \subset M$ is a small open neighbourhood of D_j, numbers $\epsilon_j > 0$, and homotopies $\sigma^{j,t} : P \times M \to E$ ($t \in [0, 1]$) such that $\sigma^{1,0} = \sigma : P \times M \to E$ is the map in the statement of the theorem and the following conditions hold for every $j \in \mathbb{N}$, where conditions (ii) and (v)–(vii) are void for $j = 1$.

(i) $f^j_p = f_p|_{U_j}$ for all $p \in Q$.

(ii) $\mathrm{dist}(f^j, f^{j-1}) < \epsilon_{j-1}$ on $P \times D_{j-1}$.

(iii) $\epsilon_j < \epsilon_{j-1}/2$, and if a map $h : P \times D_j \to \mathbb{CP}^1$ satisfies $\mathrm{dist}(h, f^j) < 2\epsilon_j$ on $P \times D_j$ and $h_p = h(p, \cdot\,)$ is holomorphic on $\mathring{D}_j$ for every $p \in P$, then $h_p : D_{j-1} \to \mathbb{CP}^1$ is an immersion for every $p \in P$.

(iv) $\sigma^{j,t}_p = \sigma_p$ for all $p \in Q$ and $t \in [0, 1]$.

(v) $\sigma^{j,0} = \sigma^{j-1,1}$.

(vi) $\sigma^{j,0}_p = \Phi(f^{j-1}_p)$ on D_{j-1} and $\sigma^{j,1}_p = \Phi(f^j_p)$ on D_j for all $p \in P$.

(vii) $\sigma^{j,t}_p = \Phi(f^{j-1,t}_p)$ on D_{j-1} for all $p \in P$, where the homotopy $f^{j-1,t} \in \mathscr{I}(P \times D_{j-1}, \mathbb{CP}^1)$ ($t \in [0, 1]$) satisfies $f^{j-1,0} = f^{j-1}$, $f^{j-1,1} = f^j|_{P \times D_{j-1}}$, and $\mathrm{dist}(f^{j-1,t}, f^{j-1}) < \epsilon_{j-1}$ on $P \times D_{j-1}$.

Assume for a moment that such sequences exist. Conditions (i)–(iii) ensure that the sequence f^j converges uniformly on compacts in $P \times M$ to a map $f \in \mathscr{I}(P \times M, \mathbb{CP}^1)$ which extends the given map $f \in \mathscr{I}(Q \times M, \mathbb{CP}^1)$ in the theorem. Define the homotopy $\sigma^t : P \times M \to E$ for $0 \le t < 1$ by

$$\sigma^t = \sigma^{j,\tau_j(t)} \quad \text{on } t \in [1-2^{-j+1}, 1-2^{-j}], \quad j \in \mathbb{N},$$

where $\tau_j(t) = 2^j(t-1+2^{-j+1})$. (Note that τ_j maps the interval $[1-2^{-j+1}, 1-2^j]$ linearly onto $[0,1]$.) Condition (v) ensures compatibility of the definition at the points $t = 1-2^{-j}$ and hence σ^t is continuous in $t \in [0,1)$, while (iv) shows that $\sigma^t = \sigma$ on $Q \times M$ for all $t \in [0,1)$. Conditions (vi) and (vii) ensure the existence of the limit map $\lim_{t\to 1} \sigma^t = \sigma^1 : P \times M \to E$ such that $\sigma^1_p = \Phi(f_p)$ on M for all $p \in P$. This completes the proof of the theorem, granted that we have sequences with the stated properties.

Let us now show how one obtains such sequences. It is instructive to look at the initial step of the induction, in particular since this argument will also be used in some of the subsequent steps.

0) The initial step: the domain D_1 is a closed disc, and our goal is to extend the given family of holomorphic immersions $f_p|_{D_1} : D_1 \to \mathbb{CP}^1$, $p \in Q$, to a continuous family of holomorphic immersions $f^1_p : D_1 \to \mathbb{CP}^1$, $p \in P$, such that the family $\Phi(f_p) : D_1 \to E$, $p \in P$, is homotopic to the given family of formal immersions σ_p on D_1. (We adopt the convention from Sect. 3 concerning the notion of holomorphic families of maps from compact subsets of M.)

Fix a point $x_1 \in \mathring{D}_1$. There is a holomorphic coordinate $z : U_1 \to \mathbb{C}$ on a neighborhood of D_1 in M such that $z(D_1) = \overline{\mathbb{D}} \subset \mathbb{C}$ is the closed unit disc and $z(x_1) = 0$. Let $\pi : E \to \mathbb{CP}^1$ denote the base projection. The formal immersions $\sigma_p : M \to E$ determine the map $a(p) = \pi \circ \sigma_p(x_1) \in \mathbb{CP}^1$, $p \in P$, and we shall choose our discs f^1_p such that $f^1_p(x_1) = a(p)$ for all p. The formal immersions also determine the derivative of f^1_p at x_1 as follows. Write $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and define the sets

$$P_0 = \{p \in P : a(p) \in \mathbb{CP}^1 \setminus \{\infty\}\}, \quad P_1 = \{p \in P : a(p) \in \mathbb{CP}^1 \setminus \{0\}\}.$$

Clearly these sets form an open cover of P. Choose a complex coordinate w on $\mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$ and a trivialisation $E|_{\mathbb{C}} \cong \mathbb{C} \times \mathbb{C}^*$ of the $\mathbb{C}^*$-bundle $E \to \mathbb{CP}^1$ over $\mathbb{C}$. The fibre component of the point $\sigma_p(x_1) \in E$ for any $p \in P_0$ is then a number $v(p) \in \mathbb{C}^*$ depending continuously on $p \in P_0$. Let V be the nowhere vanishing holomorphic vector field on M as in the theorem. In the coordinate z on D_1 we have $V(x_1) = c\frac{\partial}{\partial z}\big|_{x_1}$ for some $c \neq 0$. The embedded holomorphic discs $g_p : D_1 \to \mathbb{CP}^1$, $p \in P_0$, given in the pair of coordinates z and w by

$$w = a(p) + c^{-1} v(p) z \in \mathbb{C} \subset \mathbb{CP}^1$$

then satisfy

$$d(g_p)_{x_1}(V(x_1)) = (a(p), v(p)) = \sigma_p(x_1), \quad p \in P_0.$$

We repeat the construction for $p \in P_1$ with respect to the holomorphic coordinate $\zeta = 1/w$ on $\mathbb{CP}^1 \setminus \{0\}$ and the respective trivialisation of $E \to \mathbb{CP}^1 \setminus \{0\}$ to get another family of embedded discs $h_p : D_1 \to \mathbb{CP}^1 \setminus \{0\}$ satisfying

$$d(h_p)_{x_1}(V(x_1)) = \sigma_p(x_1), \quad p \in P_1.$$

Pick a continuous function $\chi : P \to [0, 1]$ with support in P_0 and consider the family of holomorphic discs

$$f_p^1 = \chi(p) g_p + (1 - \chi(p)) h_p : D_1 \to \mathbb{CP}^1, \quad p \in P.$$

(The convex combination is nontrivial only on the set $\{0 < \chi < 1\} \subset P_0 \cap P_1$, and for p in this set the centre $g_p(x_1) = h_p(x_1)$ lies in $\mathbb{C}^* = \mathbb{CP}^1 \setminus \{0, 1\}$.) Clearly, $d(f_p^1)_{x_1}(V(x_1)) = \sigma_p(x_1)$ for all $p \in P$. Hence, there is a smaller disc $D_1' \subset D_1$ around x_1 such that $f_p^1 : D_1' \to \mathbb{CP}^1$ $(p \in P)$ is a continuous family of embedded holomorphic discs. It is trivial to find a homotopy of formal immersions $\sigma^{1,t} : P \times M \to E$ from the initial one, $\sigma^{1,0} = \sigma$, to $\sigma^{1,1}$ such that $\sigma_p^{1,1} = d(f_p^1)(V) = \Phi(f_p^1)$ holds on D_1' for every $p \in P$. The problem of extending these immersions and homotopies (by approximation) from D_1' to D_1 is a part of the next step where we deal with the noncritical case.

We now explain the induction step $j \to j + 1$ in each of the two cases.

1) The noncritical case. In this case, $\overline{D_{j+1} \setminus D_j}$ is a finite union of compact pairwise disjoint annuli and perhaps an additional disc D_{j+1}'' not intersecting D_j. We extend the family of immersions $f_p^j : U_j \to \mathbb{CP}^1$ to a small disc in D_{j+1}'' just as in the initial case explained above, thereby reducing the problem to the case when $\overline{D_{j+1} \setminus D_j}$ consists only of annuli. Hence, D_{j+1} is obtained from D_j by successively attaching finitely many discs so that we have a Cartan pair at every step; this is a special case of [7, Lemma 5.10.3] which pertains to the more general case of noncritical strongly pseudoconvex cobordisms. In fact, we can recover a given cylinder by attaching two well chosen discs.

The induction step is obtained by applying Proposition 3.1 finitely many times, once for each disc attachment. Let us explain the procedure at each step. Thus, we have attached a compact smoothly bounded disc $B \subset M$ to a compact smoothly bounded domain $A \subset M$ such that $C = A \cap B$ is also a disc, $A \cup B$ is smoothly bounded, and (A, B) is a Cartan pair: $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$. In a coordinate chart $z : U \to U' \subset \mathbb{C}$ on a neighbourhood $U \subset M$ of B, the pair $C \subset B$ corresponds to a pair of compact discs $\Delta_0 \subset \Delta_1$ in $\mathbb{C}$.

By Proposition 3.1 we can approximate a continuous family of immersions $f_p : A \to \mathbb{CP}^1$ $(p \in P)$ as closely as desired on a neighbourhood of C by a continuous family of immersions $g_p : B \to \mathbb{CP}^1$ $(p \in P)$, keeping fixed those for $p \in Q$ which are already defined on M. Then, there is a smaller open neighbourhood U of C such that

$$f_p = g_p \circ \gamma_p \ \text{ on } U \text{ for all } p \in P, \tag{5.1}$$

where $\gamma_p : U \to M$ $(p \in P)$ is a continuous family of injective holomorphic maps close to the identity, with γ_p being the identity for $p \in Q$. As has already been mentioned in connection to (5.1), such transition maps γ_p are given by the parametric version of [7, Lemma 9.12.6] or [6, Lemma 5.1].

By the splitting lemma for biholomorphic maps close to the identity on a Cartan pair (see [6, Theorem 4.1] or [7, Theorem 9.7.1]), we have that

$$\gamma_p = \beta_p \circ \alpha_p^{-1}, \qquad p \in P,$$

where $\alpha_p : A \to M$ and $\beta_p : B \to M$ are injective holomorphic maps close to the identity on a pair of open neighbourhoods $\widetilde{A} \supset A$, $\widetilde{B} \supset B$ of the respective domains, depending continuously on $p \in P$ and agreeing with the identity map for $p \in Q$. It follows that for all $p \in P$,

$$f_p \circ \alpha_p = g_p \circ \beta_p \ \text{ holds on a neighbourhood of } C.$$

Hence, the two sides amalgamate into a continuous family of holomorphic immersions $\tilde{f}_p : A \cup B \to \mathbb{CP}^1$, $p \in P$, such that $\tilde{f}_p = f_p$ for $p \in Q$.

Applying this procedure to $f^j \in \mathscr{I}(U_j, \mathbb{CP}^1)$ furnishes in finitely many steps a map $f^{j+1} \in \mathscr{I}(U_{j+1}, \mathbb{CP}^1)$, where U_{j+1} is a neighborhood of D_{j+1}, which approximates f^j to any given precision on a fixed neighbourhood of D_j and such that $f_p^{j+1} = f_p^j$ holds for all $p \in Q$.

Assuming as we may that the approximations are close enough, there is a homotopy of holomorphic immersions $f_p^{j,t} : U_j \to \mathbb{CP}^1$ $(p \in P,\ t \in [0,1])$ on a neighbourhood U_j of D_j satisfying condition (vii). This can be seen by writing $f_p^{j+1} = f_p^j \circ \gamma_p$, where $\gamma_p : U_j' \to M$ $(p \in P)$ is a continuous family of injective holomorphic maps which are defined and close to the identity map on a neighbourhood of D_j, and they equal the identity for $p \in Q$ (see (5.1) and the references given there.) By [7, Proposition 3.3.1] there are a neighbourhood $\Omega \subset TM = M \times \mathbb{C}$ of $D_j \subset M$ with convex fibres in the (trivial) tangent bundle of M and a holomorphic map $s : \Omega \to M$ which takes the fibre of Ω over any point $x \in \Omega \cap M$ biholomorphically onto a neighbourhood of x in M, with $s(x,0) = x$. Assuming as we may that γ_p is close enough to the identity, it follows that $\gamma_p = s \circ \lambda_p$ where λ_p is a holomorphic section of Ω over a neighbourhood of D_j. By radially deforming λ_p to the zero section (recall that Ω has convex fibres), we obtain for each $p \in P$ a homotopy γ_p^t $(t \in [0,1])$ from $\gamma_p^1 = \gamma_p$ to $\gamma_p^0 = \mathrm{Id}$. Hence,

$$f_p^{j,t} := f_p^j \circ \gamma_p^t, \qquad p \in P,\ t \in [0,1],$$

is a homotopy of immersions satisfying condition (vii).

Since there is no change of topology, it is a trivial matter to deform the family of formal immersions accordingly to satisfy conditions (iv)–(vii). Condition (iii) holds for any sufficiently small number $\epsilon_{j+1} > 0$ which we choose at this point. This completes the induction step in the noncritical case.

2) The critical case. It suffices to explain the procedure in the unique connected component of D_{j+1} containing a component of D_j (or a pair of components of D_j in subcase (ii$_3$)) as a topologically nontrivial extension. The remaining pairs of components form noncritical extensions and hence the method in the previous case applies to them.

To simplify the presentation, we therefore assume without loss of generality that D_{j+1} is connected. We begin with a map $f^j \in \mathscr{I}(U_j, \mathbb{CP}^1)$, where U_j is an open neighbourhood of D_j. Choose a compact smoothly bounded domain $D'_j \subset U_j$ containing D_j in its interior and diffeotopic to D_j. Then, D_{j+1} admits a deformation retraction onto an admissible set $S = D'_j \cup \Lambda \subset \mathring{D}_{j+1}$, where Λ is a smooth arc in $\mathring{D}_{j+1} \setminus D'_j$ attached with its boundary points to bD'_j.

In the first step, we extend the immersions $f^j_p : D'_j \to \mathbb{CP}^1$, $p \in P$, across the arc Λ to obtain a continuous family of smooth immersions $f^j_p : S \to \mathbb{CP}^1$ which are holomorphic in the interior of S, keeping fixed the maps f^j_p for $p \in Q$, such that the family of maps $\Phi(f^j_p) : S \to E$, $p \in P$, is homotopic on S to the given family $\sigma_p : M \to E$ of formal immersions and the homotopy is fixed for $p \in Q$. Such extensions exist by the Smale-Hirsch-Gromov parametric h-principle for smooth immersions (see [21, 16, 13, 12]); in the case at hand we are considering immersions from an arc Λ into the Riemann sphere $\mathbb{CP}^1$, with fixed values and derivatives at the endpoints of Λ.

In the second step, we apply the parametric Mergelyan theorem (see Theorem 4.3) to approximate the new family of immersions $S \to \mathbb{CP}^1$ in the $\mathscr{C}^1(S)$ topology by a continuous family of holomorphic immersions from a neighbourhood $B \subset M$ of S into $\mathbb{CP}^1$. Since $D'_j \cup \Lambda$ is a deformation retract of D_{j+1}, we can choose B to be a smoothly bounded domain such that D_{j+1} is a noncritical extension of B, i.e., $\overline{D_{j+1} \setminus B}$ is a union of annuli. By applying the noncritical case established above, we can therefore extend the family of immersions (by approximation on B) to D_{j+1}.

Choose a number ϵ_{j+1} satisfying condition (iii). The remaining steps, finding a homotopy $f^{j+1,t}$ satisfying condition (vii) and adjusting the homotopy of formal immersions such that conditions (iv)–(vi) hold, are done as in the noncritical case. This completes the induction step. $\square$

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