Mergelyan approximation theorem for holomorphic Legendrian curves

Franc Forstnerič

Abstract In this paper we prove a Mergelyan type approximation theorem for immersed holomorphic Legendrian curves in an arbitrary complex contact manifold (X, ξ). Explicitly, we show that if S is a compact admissible set in a Riemann surface M and $f : S \to X$ is a ξ-Legendrian immersion of class $\mathscr{C}^r(S, X)$ for some $r \geq 2$ which is holomorphic in the interior of S, then f can be approximated in the $\mathscr{C}^r(S, X)$ topology by holomorphic Legendrian embeddings from open neighbourhoods of S into X. This result has numerous applications, some of which are indicated in the paper. In particular, by using Bryant's correspondence for the Penrose twistor map $\mathbb{CP}^3 \to S^4$ we show that the Mergelyan approximation theorem and the Calabi-Yau property hold for superminimal surfaces in the 4-sphere S^4.

Keywords complex contact manifold, Legendrian curve, Mergelyan theorem, superminimal surface

MSC (2020): Primary 53D35, 32E30, 34D10. Secondary 37J55, 53A10.

Date: May 10, 2020

1. Introduction

A *complex contact manifold* is a pair (X, ξ), where X is a complex manifold of necessarily odd dimension $2n + 1 \geq 3$ and ξ is a *complex contact subbundle* of the holomorphic tangent bundle TX, that is, a maximally nonintegrable holomorphic hyperplane subbundle of TX. More precisely, the *O'Neill tensor* $\xi \times \xi \to TX/\xi = L$, $(v, w) \mapsto [v, w] \mod \xi$, is nondegenerate. Consider the short exact sequence

$$0 \longrightarrow \xi \hookrightarrow TX \xrightarrow{\eta} L \longrightarrow 0, \tag{1.1}$$

where η is a holomorphic 1-form on X with values in the line bundle $L = TX/\xi$ realising the quotient projection, so $\xi = \ker \eta$. The contact condition is equivalent to $\eta \wedge (d\eta)^n$ having no zeros on X. A complex contact structure is locally in a neighbourhood of any point of X equivalent to the standard contact structure on $\mathbb{C}^{2n+1}$ given by the 1-form

$$\eta_{\rm std} = dz + \sum_{j=1}^{n} x_j dy_j. \tag{1.2}$$

See Darboux [16], Moser [41], and Geiges [29, p. 67] for the real case, and Alarcón et al. [7, Theorem A.2] for the holomorphic case. More complete introductions to complex contact manifolds are available in the papers by LeBrun [37], LeBrun and Salamon [38], Beauville [11], and the paper by Alarcón and Forstnerič [4].

A map $f : M \to X$ of class $\mathscr{C}^1$ from a real or complex manifold M is said to be *isotropic*, or an *integral submanifold* of the contact structure ξ on X, if

$$df_x(T_x M) \subset \xi_{f(x)} \ \text{ holds for all } \ x \in M. \tag{1.3}$$

Equivalently, $f^*\eta = 0$ for any 1-form η on X with $\ker \eta = \xi$. If f is an immersion at a generic point of M, then $\dim_{\mathbb{R}} M \le 2n$ where $\dim_{\mathbb{C}} X = 2n+1$. Isotropic submanifolds of maximal dimension $2n$ are necessarily complex submanifolds of X (see [4, Lemma 5.1]); they are called *Legendrian*. In this paper, isotropic holomorphic curves from open Riemann surfaces are called *holomorphic Legendrian curves* irrespectively of the dimension of X.

We shall be considering Legendrian immersions from sets of the following type.

Definition 1.1 (Admissible sets)**.** Let M be a connected Riemann surface. A proper compact subset $S \subsetneq M$ is *admissible* if it is of the form $S = K \cup E$, where K is a finite union of pairwise disjoint compact domains in M with piecewise $\mathscr{C}^1$ boundaries and $E = \overline{S \setminus K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only in their endpoints (if at all) and such that their intersections with the boundary bK of K are transverse.

Clearly, an admissible set has a well defined tangent space at every point $x \in S$ which equals $T_x M$ when $x \in K$, and equals the tangent space of the curve $E = \overline{S \setminus K}$ at any point $x \in S \setminus K$. Given a complex manifold X, we denote by $\mathscr{A}^r(S,X)$ the space of maps $S \to X$ of class $\mathscr{C}^r$ which are holomorphic in the interior $\mathring{S}$; we shall write $\mathscr{A}^r(S) = \mathscr{A}^r(S,\mathbb{C})$. Every $f \in \mathscr{A}^r(S)$ can be approximated in $\mathscr{C}^r(S)$ by meromorphic functions on M with poles off S, and every map $S \to X$ of class $\mathscr{A}^r(S,X)$ into an arbitrary complex manifold X can be approximated in the $\mathscr{C}^r(S,X)$ topology by holomorphic maps $U \to X$ from open neighbourhoods $U \subset M$ of S (see [18, Theorem 16 and Corollary 9]).

A map $f \in \mathscr{A}^1(S,X)$ from an admissibe set S into a contact manifold (X,ξ) is said to be *Legendrian* if condition (1.3) holds with M replaced by S.

Our main result is the following.

Theorem 1.2 (Mergelyan approximation theorem for Legendrian immersions)**.** *Let (X,ξ) be a complex contact manifold, and let S be an admissible set in a Riemann surface M. Every Legendrian immersion $f : S \to X$ of class $\mathscr{A}^r(S,X)$, $r \ge 2$, can be approximated in the $\mathscr{C}^r(S,X)$ topology by holomorphic Legendrian immersions $\tilde f : U \to X$ from open neighbourhoods of S in M. Furthermore, $\tilde f$ can be chosen to agree with f on any given finite subset A of S (to any given finite order at the points of $A \cap \mathring{S}$), and it can be chosen an embedding provided that f is injective on A.*

Theorem 1.2 is proved in Section 4; here is an outline. We have $\xi = \ker \eta$ where η is a holomorphic contact form (see (1.1)). The immersion $f : S \to X$ extends to an immersion $F : S \times \mathbb{B}^{2n} \to X$ of class $\mathscr{A}^r(S \times \mathbb{B}^{2n}, X)$ (see Lemma 2.3), and this gives a contact form $\beta = F^*\eta$ on $S \times \mathbb{B}^{2n}$ of class $\mathscr{A}^{r-1}(S \times \mathbb{B}^{2n})$. Next, we obtain a partial normal form of β along $S \times \{0\}^{2n}$ in the spirit of [4, Theorem 1.1], which amounts to a suitable change of the immersion F. From this point on, the main part of the proof is a careful study of the period map of solutions of the differential equation for β-Legendrian curves along closed real curves in S forming a homology basis. (This period map is also called the *monodromy map* when considered on the whole space of solutions, or the *Poincaré first return map* when considered on a neighbourhood of a given single valued solution.) The differential equation for Legendrian curves is underdetermined, and we can obtain a family of determined equations by choosing all but one independent variables as functions of the remaining independent variable and of some additional parameters. We show that a suitable choice of such family admits non-single valued solutions with arbitrarily given small periods around each closed curve in a homology basis of S. Next, we approximate F in the $\mathscr{C}^r(S\times\mathbb{B}^{2n},X)$ topology by a holomorphic immersion $\widetilde F: U\times\rho\mathbb{B}^{2n}\to X$, where $U\subset M$ is a neighbourhood of S and $0<\rho<1$, and consider the holomorphic contact form $\tilde\beta=\widetilde F^*\eta$ on $U\times\rho\mathbb{B}^{2n}$. If the approximation is close enough, the degree theory shows that there is a parameter value for which the associated differential equation in the perturbed family has a solution with vanishing periods on a neighbourhood of S. The $\widetilde F$-image of this single valued solution is an immersed holomorphic η-Legendrian curve $\tilde f: U\to X$ which approximates the given Legendrian curve $f:S\to X$. Finally, by [4, Theorem 1.2] we can approximate $\tilde f$ by embedded holomorphic Legendrian curves.

Remark 1.3. The problem of finding (solutions of) ordinary holomorphic differential equations with prescribed periods or holonomy is related to, and inspired by Hilbert's 21st problem. This problem asks about the existence of systems of ordinary linear holomorphic differential equations whose coefficients are rational functions with simple poles at a given finite set of points in the extended complex plane $\mathbb{C}\cup\{\infty\}=\mathbb{CP}^1$ (Fuchsian systems) and with a given monodromy. After major contributions by Hilbert (1905) and Plemelj (1908), the problem was completely solved in the negative by Bolibruch in 1989; see Anosov and Bolibruch [9]. Plemelj [44] gave an affirmative answer to Hilbert's problem in the bigger class of systems whose coefficients have simple poles and whose solutions grow at most polynomially at every pole. Here we are dealing with more general nonlinear Pfaffian holomorphic differential equations which are underdetermined and completely nonintegrable. This allows us to find a parametric family of determined equations which have solutions with arbitrary given small periods on curves in a homology basis. □

Admittedly, our proof is substantially more involved than the proof of the corresponding result in the seemingly only known special case when $X=\mathbb{C}^{2n+1}$ is a Euclidean space with the standard complex contact structure (1.2); see [7, Theorems 1.1 and 5.1]. Assuming that the admissible set $S\subset M$ is $\mathcal{O}(M)$-convex (equivalently, S has no holes in M), those results show that one can approximate a Legendrian map $S\to\mathbb{C}^{2n+1}$ of class $\mathscr{A}^1(S,\mathbb{C}^{2n+1})$ by proper holomorphic Legendrian embeddings $M\hookrightarrow\mathbb{C}^{2n+1}$. We do not see a comparably simple approach in an arbitrary complex contact manifold.

A basic example of an admissible set is the circle $S^1=\{z\in\mathbb{C}:|z|=1\}$ in $M=\mathbb{C}$. In this case, Theorem 1.2 gives the following corollary. Given $\rho>1$ we set $A_\rho=\{z\in\mathbb{C}:\rho^{-1}<|z|<\rho\}$.

Corollary 1.4. *Let X be a complex contact manifold. Every Legendrian loop $S^1\to X$ of class $\mathscr{C}^r(S^1,X)$ $(r\geq 2)$ can be approximated in $\mathscr{C}^r(S^1,X)$ by embedded holomorphic Legendrian annuli $A_\rho\hookrightarrow X$, where $\rho>1$ may depend on the map.*

Corollary 1.4 is clearly equivalent to the statement that every smooth Legendrian loop $S^1\to X$ in a complex contact manifold can be approximated by real analytic Legendrian loops, since the complexification of such are holomorphic Legendrian maps from surrounding annuli. This raises the following more general question.

Problem 1.5. Let X be a complex contact manifold and $f:M\to X$ be an isotropic immersion from a compact real analytic manifold M. Is it possible to approximate f by real analytic isotropic immersions $M\to X$?

The answer to this question is affirmative if (X, ξ) is a real analytic contact manifold, i.e., ξ is a real codimension one contact subundle of TX; see e.g. Cieliebak and Eliashberg [15, Sect. 6.7]. However, the method in the cited source does not apply in the holomorphic case, for two reasons. One is that holomorphic approximation theory is very different from real analytic approximation theory, the latter allowing arbitrarily close approximation (even in fine topologies) everywhere, and not only on the given subset. This is impossible in the holomorphic case where the approximants necessarily diverge outside the set of approximation, unless the original map extends holomorphically to a bigger domain in which case the problem is void. The second one is that the proof in [15] also relies on Gray's stability theorem for real contact structures which does not hold (not even locally) for real analytic deformations of complex contact structures. I wish to thank Roger Casals and Nicola Pia for having provided the following information (private communications, November 2019). The space of k-dimensional distributions on $\mathbb{R}^n$ (i.e., vector subbundles of rank k of the tangent bundle $T\mathbb{R}^n$) has functional dimension $k(n-k)$, the dimension of the Grassmanian manifold of k-planes in $\mathbb{R}^n$. The symmetry group of diffeomorphisms is given by n functions, so the functional dimension of k-distributions on $\mathbb{R}^n$ up to diffeomorphisms is $k(n-k)-n$. The only cases when this is non-positive (and there is a unique local normal form) are $(k, n) \in \{(1, n), (n-1, n), (2, 4)\}$, corresponding to vector fields, real contact and even-contact structures, and the Engel case. Any other pair (k, n) produces distributions which have functional moduli and thus are not be stable. More precise information can be found in Bryant et al. [14, Sect. II.5]; see in particular the examples on [14, pp. 48–49].

Theorem 1.2 reduces the problem of approximating Legendrian immersions $S \to X$ from an admissible set S in a Riemann surface M by holomorphic Legendrian immersions $M \to X$ to the following

Runge approximation problem for Legendrian immersions: Let S be a compact set in a Riemann surface M and $f : U \to X$ be a holomorphic immersion from an open neighbourhood of S to a complex contact manifold (X, ξ). When is it possible to approximate f on S by holomorphic Legendrian immersions $M \to X$?

The answer depends on the pair $S \subset M$ and on the complex structure and the contact structure on X. It is negative in general even if $X = \mathbb{C}^3$ and S is a closed disc in $\mathbb{C}$. Indeed, in [20] the author gave an example of a complex contact structure ξ on $\mathbb{C}^3$ which is Kobayashi hyperbolic; in particular, there are no nonconstant holomorphic Legendrian curves $\mathbb{C} \to (\mathbb{C}^3, \xi)$, and there are holomorphic Legendrian discs $\overline{\mathbb{D}} \to (\mathbb{C}^3, \xi)$ which are not approximable by holomorphic Legendrian discs $2\mathbb{D} \to (\mathbb{C}^3, \xi)$. Hence, Theorem 1.2 can not be improved in general even if X is an Oka manifold (see [21, Sect. 5.4] for this notion), for it is the properties of the contact structure on X that matter as well. Without the Legendrian condition, Runge approximation holds (in the absence of topological obstructions) for maps from all Stein manifolds, in particular from all open Riemann surfaces, to any Oka manifold; see [21, Theorem 5.4.4].

We now describe an application of Theorem 1.2 to Legendrian curves in the complex projective space $\mathbb{CP}^3$. Recall (see LeBrun and Salamon [37, 38]) that $\mathbb{CP}^3$ admits an essentially unique complex contact structure which is determined by the 1-form

$$\eta = z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2$$

on $\mathbb{C}^4$ via the standard projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$. On any affine chart $\mathbb{C}^3 \subset \mathbb{CP}^3$ this gives the standard contact structure on $\mathbb{C}^3$. The analogous statements hold on $\mathbb{CP}^{2n+1}$ for any $n \in \mathbb{N}$. It was shown by Alarcón et al. in 2019 (see [2, Theorems 3.2 and 3.4]) that holomorphic Legendrian curves and immersions from Riemann surfaces (both open and closed) into $\mathbb{CP}^3$ satisfy the Runge approximation theorem with interpolation, and every Legendrian immersion $M \to \mathbb{CP}^3$ from an open Riemann surface can be approximated uniformly on compacts in M by holomorphic Legendrian embeddings $M \hookrightarrow \mathbb{CP}^3$. This gives the following corollary to Theorem 1.2.

Corollary 1.6 (Mergelyan theorem for Legendrian immersions into $\mathbb{CP}^3$)**.** *Let S be an admissible set in a Riemann surface M. Every Legendrian immersion $f : S \to \mathbb{CP}^3$ of class $\mathscr{A}^r(S, \mathbb{CP}^3)$ $(r \geq 2)$ can be approximated in $\mathscr{C}^r(S, \mathbb{CP}^3)$ by holomorphic Legendrian immersions $F : M \to \mathbb{CP}^3$ (embeddings if M is an open Riemann surface and S has no holes). Furthermore, F can be chosen to agree with f on any given finite set of points in S.*

The fact that every compact Riemann surface embeds as a holomorphic Legendrian curve in $\mathbb{CP}^3$ was proved by Bryant in 1982, see [13, Theorem G]. As pointed out in [2], the Runge approximation theorem also holds for Legendrian curves in higher dimensional projective spaces of odd dimensions, thereby giving the corresponding generalization of Corollary 1.6.

Another positive result regarding Runge approximation of holomorphic Legendrian immersions was given by Forstnerič and Lárusson in [24, Corollaries 14 and 16]. It pertains to the projectivised cotangent bundle $X = \mathbb{P}(T^*Z)$ with its standard contact structure, where Z is an arbitrary Oka manifold (see [21, Sect. 5.4]) of dimension ≥ 2. The following is an immediate corollary.

Corollary 1.7. *Let S be an admissible set without holes in an open Riemann surface M. If Z is an Oka manifold of dimension at least 2, then every Legendrian immersion $S \to X = \mathbb{P}(T^*Z)$ of class $\mathscr{A}^r(S, X)$ $(r \geq 2)$ can be approximated in $\mathscr{C}^r(S, X)$ by holomorphic Legendrian embeddings $M \hookrightarrow X$.*

With the help of Theorem 1.2 we will also show that approximation by global holomorphic Legendrian immersions $M \to X$ is always possible if we allow deformations of the complex structure on M. The following result is proved in Sect. 5.

Theorem 1.8 (The soft Oka principle for Legendrian immersions)**.** *Assume that (M, J_0) is an open Riemann surface and K is a compact $\mathcal{O}(M)$-convex subset of M. Let X be a connected complex contact manifold endowed with a distance function* dist *compatible with its manifold topology. Given a continuous map $f : M \to X$ which is a J_0-holomorphic Legendrian immersion $U \to X$ on an open neighbourhood U of K, there exist for every $\epsilon > 0$ a complex structure J on M which agrees with J_0 on a smaller neighbourhood $U_1 \subset U$ of K and a J-holomorphic Legendrian embedding $F : (M, J) \hookrightarrow X$ such that*

$$\sup_{p \in K} \mathrm{dist}\,(F(p), f(p)) < \epsilon$$

and F is homotopic to f by a homotopy $f_t : M \to X$ $(t \in [0,1])$ such that f_t is a holomorphic Legendrian immersion on U_1 for every $t \in [0, 1]$.

Combining Theorems 1.2 and 1.8 we obtain the following corollary.

Corollary 1.9. *Let (M, J_0) be an open Riemann surface, and let $S \subset M$, X, and $f : S \to X$ satisfy the hypotheses of Theorem 1.2. Then there exist a complex structure J on M which agrees with J_0 on a neighbourhood of S and a J-holomorphic Legendrian embedding $F : (M, J) \hookrightarrow X$ which approximates f as closely as desired in the $\mathscr{C}^r(S, X)$ topology.*

The first systematic treatment of the existence and approximation problem for holomorphic maps $M \to X$ from a Stein manifold M (in particular, from an open Riemann surface) to an arbitrary complex manifold, allowing homotopic deformation of the Stein structure on the source manifold M, goes back to the 2007 papers by Forstnerič and Slapar [26, 25]. Results of this type are commonly referred to as (instances of) the *soft Oka principle*. For further recent examples, see the papers by Alarcón and López [8], Ritter [46], and the surveys [15, Theorem 8.43 and Remark 8.44], [21, Sects. 10.9–10.11], and [23].

In another direction, we obtain the following corollary by combining Theorem 1.2 with [4, Theorem 1.3] by Alarcón and Forstnerič. The assumption in the latter result is that a given Legendrian immersion $f : M \to X$ is holomorphic on a neighbourhood of M in an ambient open Riemann surface; this is now guaranteed by Theorem 1.2.

Corollary 1.10 (Calabi-Yau property for holomorphic Legendrian immersions)**.** *Assume that M is a compact bordered Riemann surface and (X, ξ) is a complex contact manifold. Every Legendrian immersion $f : M \to X$ of class $\mathscr{A}^2(M, X)$ can be approximated uniformly on M by topological embeddings $F : M \hookrightarrow X$ such that $F|_{\mathring{M}} : \mathring{M} \to X$ is a complete holomorphic Legendrian embedding.*

Using the terminology in [2, Definition 6.1], Corollary 1.10 asserts that holomorphic Legendrian curves from compact bordered Riemann surfaces to an arbitrary complex contact manifold enjoy the *Calabi-Yau property*. This terminology derives from the classical Calabi-Yau problem for immersed minimal surfaces in Euclidean spaces $\mathbb{R}^n$, $n \geq 3$; see the papers [5, 6] and the references therein for the latter. From the Calabi-Yau property for Legendrian curves in $\mathbb{CP}^3$ we can infer the analogous property for *superminimal surfaces* in the 4-sphere S^4 with the spherical metric. Among all minimal surfaces in S^4, superminimal surfaces form a natural and important subclass. This term was introduced by Bryant [13] in 1982, although they had been studied much earlier. They are characterized geometrically by the fact that the curvature ellipse in the normal plane to the surface at each of its points, which is determined by its second fundamental form in S^4, is a circle centred at the origin (see Friedrich [27, 28] and the brief survey in [19]).

The connection between holomorphic Legendrian curves in $\mathbb{CP}^3$ and superminimal surfaces in S^4 was discovered by Bryant [13, Sect. 2]; here is a brief description. Identifying $\mathbb{C}^4$ with the quaternionic 2-plane $\mathbb{H}^2$, the natural projection

$$h : \mathbb{H}^2_* = \mathbb{H}^2 \setminus \{0\} \to \mathbb{HP}^1 = S^4$$

onto the 1-dimensional quaternionic projective space splits as $h = \pi \circ \pi'$, where $\pi' : \mathbb{H}^2_* = \mathbb{C}^4_* \to \mathbb{CP}^3$ is the standard projection onto the complex projective 3-space and $\pi : \mathbb{CP}^3 \to S^4$ is the *Penrose's twistor map* (see Penrose [42, 43]), a real analytic (but nonholomorphic) fibre bundle projection whose fibres are projective lines. Bryant showed that the differential of π induces an isometry $d\pi : \xi \to TS^4$ from the contact subbundle $\xi \subset \mathbb{CP}^3$ in the Fubini-Study metric onto the tangent bundle of S^4 in the spherical metric. Furthermore, given a Riemann surface M, postcomposition by π defines a homeomorphism

$$\pi_* : \mathscr{L}(M, \mathbb{CP}^3) \longrightarrow \mathbb{S}_+(M, S^4) \tag{1.4}$$

from the space of holomorphic Legendrian immersions $M \to \mathbb{CP}^3$ to the space of superminimal immersions $M \to S^4$ of positive spin [13, Theorems B, B', D]. Furthermore, since $d\pi|_\xi : \xi \to TS^4$ is an isometry, π maps immersed holomorphic and antiholomorphic Legendrian curves in $\mathbb{CP}^3$ isometrically onto immersed superminimal surfaces in S^4. In particular, the projection of a complete Legendrian curve in $\mathbb{CP}^3$ is a complete superminimal surface in S^4, and vice versa. Postcomposition by an orientation reversing isometry $S^4 \to S^4$ (for example, by the antipodal map) reverses the spaces of superminimal surfaces of positive and negative spin in S^4, so the latter need not be considered separately.

The following two results are improvements of [2, Corollary 7.3 and Theorem 7.5], respectively. The difference is that we are now considering superminimal immersions defined on a compact domain, and not on an open neighbourhood of it. Both results use the Bryant correspondence (1.4) along with Theorem 1.2. If M is a compact bordered Riemann surface then every superminimal surface $M \to S^4$ of positive spin and of class $\mathscr{C}^r$ for some $r \geq 2$ lifts to a Legendrian curve $M \to \mathbb{CP}^3$ of class $\mathscr{C}^{r-1}$. Note however that the correspondence (1.4) essentially depends on complete nonintegrability of the contact subbundle $\xi \subset T\mathbb{CP}^3$, and it fails on admissible domains containing arcs. In fact, every smooth immersed arc in S^4 lifts isometrically to an immersed ξ-Legendrian arc in $\mathbb{CP}^3$.

Corollary 1.11 (Mergelyan approximation theorem for superminimal surfaces in S^4). *Let K be a compact domain with piecewise $\mathscr{C}^1$ boundary in a Riemann surface M. Every conformal superminimal immersion $X : K \to S^4$ of class $\mathscr{C}^3(K, S^4)$ can be approximated in $\mathscr{C}^2(K, S^4)$ by complete superminimal immersions $Y : M \to S^4$. We may choose Y to agree with X to a given finite order at finitely many given points in $\mathring{K}$.*

Corollary 1.12 (Calabi-Yau theorem for conformal superminimal surfaces in S^4). *If M is a compact bordered Riemann surface and $X : M \to S^4$ is a conformal superminimal immersion of class $\mathscr{C}^3(M, S^4)$, then X can be approximated as closely as desired uniformly on M by a continuous map $Y : M \to S^4$ whose restriction to the interior $\mathring{M}$ is a complete, generically injective conformal superminimal immersion, and whose restriction to the boundary bM is a topological embedding. In particular, $Y(bM) \subset S^4$ is a union of pairwise disjoint Jordan curves.*

The corresponding results for conformal minimal surfaces in flat Euclidean spaces $\mathbb{R}^n$, $n \geq 3$, were obtained in [1, 6].

2. Tubular neighbourhoods of immersions from admissible sets

In this section we prepare some necessary material concerning immersions from admissibe sets in Riemann surfaces into arbitrary complex manifolds. The main result that we shall need in the sequel is Lemma 2.3.

We begin by recalling the following result [21, Proposition 3.3.2]. The hypothesis that the manifold Z be Stein is accidentally missing in the cited source. For convenience of the reader we include a sketch of proof.

Proposition 2.1. *Assume that $\pi : Z \to M$ is a holomorphic submersion from a Stein manifold Z to a complex manifold M. Denote by $E = \ker d\pi \to Z$ the vertical tangent bundle of π, and let 0_z denotes the origin in the fibre E_z of E over $z \in Z$. Then there are an open Stein neighbourhood $\mathscr{E} \subset E$ of the zero section of E and a holomorphic map $\phi : \mathscr{E} \to Z$ such that for every $z \in Z$ we have $\phi(0_z) = z$ and ϕ maps the fibre $\mathscr{E}_z = \mathscr{E} \cap E_z$ biholomorphically onto a neighbourhood of z in the fibre $Z_{\pi(z)} = \pi^{-1}(\pi(z))$. We may choose $\mathscr{E}$ to be Runge in E and to have convex fibres $\mathscr{E}_z$, $z \in Z$.*

Proof. By Cartan's Theorem A there are finitely many holomorphic vector fields $V_1, \ldots, V_N$ on Z which are tangent to the fibres of π and span the vertical tangent space $E_z = \ker d\pi_z$ of Z at every point $z \in Z$. Let ϕ_t^i denote the holomorphic flow of V_i for the complex time t. There is an open neighbourhood Ω of the zero section $Z \times \{0\}^N$ in the trivial bundle $Z \times \mathbb{C}^N$ such that the holomorphic map $\Phi : \Omega \to Z$ given by

$$\Phi(z,t) = \phi^1_{t_1} \circ \cdots \circ \phi^N_{t_N}(z), \qquad z \in Z,\ t = (t_1, \ldots, t_N)$$

is well defined on Ω. Clearly, $\Phi(z,0) = z$ and $\pi \circ \Phi(z,t) = \pi(z)$ for all $(z,t) \in \Omega$. Since the vectors $V_i(z)$ span E_z at every point $z \in Z$, the map

$$\Theta(z) = \frac{\partial}{\partial t}\Phi(z,t)\Big|_{t=0} : \mathbb{C}^N \longrightarrow E_z$$

is surjective, and hence $\ker\Theta$ is a holomorphic vector subbundle of $Z \times \mathbb{C}^N$. By Cartan's Theorem B we have $Z \times \mathbb{C}^N = \ker\Theta \oplus E'$ for some holomorphic vector subbundle $E' \subset Z \times \mathbb{C}^N$. Clearly, the restriction $\Theta|_{E'} : E' \to E$ is a holomorphic vector bundle isomorphism, so we may identify E with the subbundle $E' \subset Z \times \mathbb{C}^N$. Let $\mathscr{E} = \Omega \cap E$. By shrinking Ω around the zero section if necessary, the implicit function theorem shows that the holomorphic map $\phi = \Phi|_{\mathscr{E}} : \mathscr{E} \to Z$ is fibrewise biholomorphic. □

The following lemma provides a coordinate Stein neighbourhood of the graph of a map of class $\mathscr{A}(S)$ over an admissible set S in a Riemann surface.

Lemma 2.2. *Assume that S is an admissible set in a Riemann surface M, X is a complex manifold, and $f : S \to X$ is a map of class $\mathscr{A}(S,X)$. Then, the graph*

$$G_f = \{(p, f(p)) : p \in S\} \subset M \times X$$

of f has a Stein neighbourhood $\widetilde\Omega \subset M \times X$ which is fibrewise bihomorphic to a Stein domain $\Omega \subset M \times \mathbb{C}^n$, $n = \dim X$. More precisely, there is a biholomorphic map $\Phi : \Omega \to \widetilde\Omega$ which commutes with the base projections $M \times \mathbb{C}^n \to M$ and $M \times X \to M$.

Proof. By Poletsky's theorem [45] the graph G_f has an open Stein neighbourhood Z in $M \times X$. (See also [18, Theorem 32] and the related discussion.) Let $\pi : Z \to M$ denote the restriction of the projection $\pi : M \times X \to M$, and let the domain $\mathscr{E} \subset E = \ker(d\pi)$ and the map $\phi : \mathscr{E} \to Z$ be as in Proposition 2.1. By [18, Corollary 9] we can approximate f as closely as desired uniformly on S by a holomorphic map $\tilde f : U \to X$ defined on an open neighbourhood $U \subset M$ of S. Denote by $\widetilde G \subset M \times X$ the graph of $\tilde f$ on U. If the approximation is close enough and after shrinking U around S if necessary, we have that $\widetilde G \subset Z$. Consider the restricted bundle $\widetilde E = E|_{\widetilde G} \to \widetilde G$. Since $\widetilde G$ is biholomorphic to the open Riemann surface $U \subset M$, the bundle $\widetilde E$ is holomorphically trivial by the Oka-Grauert principle (see [21, Theorem 5.3.1]), so we can identify it with $U \times \mathbb{C}^n$, $n = \dim X$. If the approximation of f by $\tilde f$ is close enough, then the restriction of the map $\phi : \mathscr{E} \to Z$ to the domain $\Omega := \mathscr{E} \cap \widetilde E \subset U \times \mathbb{C}^n$ provides a biholomorphic map

$$\Phi : \Omega \xrightarrow{\cong} \Phi(\Omega) = \widetilde\Omega \subset Z \subset M \times X \tag{2.1}$$

satisfying the lemma. Indeed, Ω is a Stein domain in $\widetilde E \cong U \times \mathbb{C}^n$, the map Φ is fibrewise biholomorphic and hence biholomorphic onto $\widetilde\Omega = \Phi(\Omega) \subset Z$, and $\widetilde\Omega$ contains the graph G_f of f provided that $\tilde f$ was chosen sufficiently close to f on S. □

We denote by $\mathbb{B}^n$ the unit ball of $\mathbb{C}^n$.

Lemma 2.3. *Assume that S is an admissible set in a Riemann surface M and X is a complex manifold of dimension n. Every immersion $f : S \to X$ of class $\mathscr{A}^r(S,X)$ $(r \geq 1)$ extends to an immersion $F : S \times \mathbb{B}^{n-1} \to X$ of class $\mathscr{A}^r(S \times \mathbb{B}^{n-1}, X)$.*

Proof. We may assume that M is an open Riemann surface. Let $pr_X : M \times X \to X$ denote the projection onto the second component. Choose a nowhere vanishing holomorphic vector field V on M; such exists by the Oka-Grauert principle (see [21, Theorem 5.3.1]) and it defines a trivialisation of the tangent bundle $TM \cong M \times \mathbb{C}$. For each $x \in S$ the vector $df_x(V_x) \in T_{f(x)}X$ is nonvanishing since $f : S \to X$ is an immersion.

Let $\Phi : \Omega \to \widetilde{\Omega} \subset U \times X$ be a fibre preserving biholomorphic map (2.1) furnished by Lemma 2.2, where Ω is a domain in $U \times \mathbb{C}^n$. In particular, the domain $\widetilde{\Omega}$ contains the graph of the immersion $f : S \to X$. Hence, there is a unique map $g : S \to \mathbb{C}^n$ of class $\mathscr{A}^r(S, \mathbb{C}^n)$ whose graph lies in Ω such that

$$pr_X \circ \Phi(x, g(x)) = f(x) \in X, \quad x \in S.$$

Choose $\rho > 0$ such that for every $x \in S$ and $z \in \rho\mathbb{B}^n \subset \mathbb{C}^n$ the point $(x, g(x) + z)$ belongs to Ω. Consider the map $\widetilde{F} : S \times \rho\mathbb{B}^n \to X$ given by

$$\widetilde{F}(x, z) = pr_X \circ \Phi(x, g(x) + z), \quad x \in S,\ z \in \rho\mathbb{B}^n. \tag{2.2}$$

Clearly, $\widetilde{F} \in \mathscr{A}^r(S \times \rho\mathbb{B}^n, X)$ and for all $x \in S$ we have that $\widetilde{F}(x, 0) = f(x)$ and $\widetilde{F}(x, \cdot\,) : \rho\mathbb{B}^n \to X$ maps the ball $\rho\mathbb{B}^n$ biholomorphically onto a neighbourhood of $f(x)$ in X. For each $x \in S$ let $W_x \in \mathbb{C}^n$ be the unique vector satisfying

$$\partial_z \widetilde{F}_{(x,0)}(W_x) = df_x(V_x) \in T_{f(x)}X,$$

where ∂_z denotes the partial differential with respect to the second component $z \in \mathbb{C}^n$. The map $W : S \to \mathbb{C}^n_*$ is of class $\mathscr{A}^{r-1}(S)$. By the Oka-Grauert principle and the approximation theorem for vector bundles of class $\mathscr{A}(S)$ (see Heunemann [32, 33]) we can split the trivial vector bundle $S \times \mathbb{C}^n$ into a direct sum

$$S \times \mathbb{C}^n = \mathrm{span}_{\mathbb{C}}(W) \oplus \nu,$$

where $\mathrm{span}_{\mathbb{C}}(W)$ is the complex line subbundle determined by W and ν is a complementary complex vector subbundle of rank $n - 1$ and of class $\mathscr{A}^r(S)$. (Heunemann's results are stated for compact strongly pseudoconvex domains in Stein manifolds, but they also hold on admissible sets in Riemann surfaces. In particular, over the arcs in $S \setminus K$ they are a consequence of the standard theory for smooth complex vector bundles. A simple proof of Heunemann's approximation theorem for complex vector subbundles of class $\mathscr{A}(S)$ in a trivial bundle $S \times \mathbb{C}^n$ by holomorphic vector subbundles over a neighbourhood of S (see [32, Theorem 1]) can be found in [17, Theorem A.1, pp. 248–249].) By the same results, the vector bundle $\nu \to S$ it trivial, isomorphic to $S \times \mathbb{C}^{n-1}$. Consider now the map

$$F = \widetilde{F}|_{\nu \cap (S \times \rho\mathbb{B}^n)} : \nu \cap (S \times \rho\mathbb{B}^n) \to X$$

of class $\mathscr{A}^r$. For each $x \in S$ we have $F(x, 0) = f(x)$ and the differential of F in the vertical direction maps the fibre ν_x onto a hyperplane in $T_{f(x)}X$ complementary to the vector $df_x(V_x)$. Decreasing $\rho > 0$ if necessary, it follows that F is an immersion of class $\mathscr{A}^r$ with the trivial normal bundle ν. After a change of coordinates mapping $\nu \cap (S \times \rho\mathbb{B}^n)$ onto $S \times \mathbb{B}^{n-1}$, we get an immersion $F : S \times \mathbb{B}^{n-1} \to X$ as in the lemma. □

3. Preliminaries

In this section we prepare the necessary background for the proof of Theorem 1.2. In Subsect. 3.1 we recall some basic facts concerning solutions of ordinary holomorphic differential equations, with emphasis on the case when the domain is an admissible set in a Riemann surface. In Subsect. 3.2 we recall the notion of the period map which plays an important role in the deformation theory for solutions of holomorphic differential equations. In Subsect. 3.3 we show that every admissible set S in a Riemann surface admits a homology basis consisting of finitely many closed curves whose union is Runge in S. In Subsect. 3.4 we recall a basic result on the topological degree of a map. Finally, in Subsect. 3.5 we recall a result of Arens concerning generators of the algebra $\mathscr{A}^r(S)$.

3.1. Holomorphic differential equations on admissible sets. Assume that M is an open Riemann surface. Fix a holomorphic immersion $z : M \to \mathbb{C}$ furnished by Gunning and Narasimhan [31]. Note that such an immersion provides a local holomorphic coordinate on a neighbourhood of any given point of M.

Let $S = K \cup E$ be an admissible set in M (see Def. 1.1). We shall need some basic results on the existence and behaviour of solutions of ordinary differential equations

$$dw = V(p, w, t)dz, \quad w(p_0, t) = w_0, \tag{3.1}$$

where the independent variable is $p \in S$, the dependent variable w belongs to some disc $\triangle \subset \mathbb{C}$ around the origin, the differentials dz and dw are taken with respect to p, $t = (t_1, \ldots, t_l)$ is a complex parameter in a ball $B \subset \mathbb{C}^l$ around the origin, and V is a function of class $\mathscr{C}^r$ on $S \times \triangle \times B \subset M \times \mathbb{C}^{l+1}$ for some $r \geq 1$ which is holomorphic on the interior $\mathring{S} \times \triangle \times B$. The function V may be thought of as a nonautonomous vector field $V\frac{\partial}{\partial w}$ of type $(1, 0)$ on the w-space, of class $\mathscr{C}^r$ in $(p, w) \in S \times \triangle$ and holomorphic over $\mathring{S}$, and also depending holomorphically on the parameter $t \in B$. On a neighbourhood of a point $p_0 \in S$, using the holomorphic immersion $z : M \to \mathbb{C}$ as a local coordinate near p_0 and setting $z_0 = z(p_0) \in \mathbb{C}$, the equation (3.1) assumes the more familiar form

$$\frac{dw}{dz} = V(z, w, t), \quad w(z_0, t) = w_0. \tag{3.2}$$

(The function V in (3.2) is obtained from the one in (3.1) by locally expressing $p = p(z)$.) This is an ordinary differential equation for $w = w(z, t)$ as a function of z, with t as a parameter. The precise local nature of this equation depends on the location of the point $p_0 \in S$ where we are considering it. If p_0 is an interior point of S, then (3.2) is a holomorphic differential equation near p_0 in the local coordinate z centred at $z_0 = z(p_0) \in \mathbb{C}$. Such equation admits a local holomorphic solution which is uniquely determined by an initial condition $w(z_0, t) = w_0$ and depends holomorphically on (z_0, w_0, t) (see E. Hille [34, Chapter 2]). One may find a local solution in terms of the power series expansion

$$w(z, t) = w_0 + \sum_{k=1}^{\infty} c_k (z - z_0)^k.$$

The coefficients $c_k = c_k(z_0, w_0, t)$ are uniquely determined by the equation (3.2). By the domination method of A. L. Cauchy or E. Lindelöf one can show that the power series converges in a disc around the point z_0, and it is possible to estimate its radius in terms of V. (See E. Hille [34, Sect. 2.6] or the book by E. Lindelöf [39] from 1905.)

This method does not apply at boundary points of the domain K in the admissible set S, or over the arcs $E \subset S$. Let us now explain an alternative approach which works up to the boundary of K. (We shall consider the equation on the arcs in $S \setminus K$ later.) Write the variables and the vector field in the form

$$z = x + \mathrm{i}y, \quad w = w_1 + \mathrm{i}w_2, \quad V = V_1 + \mathrm{i}V_2$$

with real components. The ordinary complex differential equation (3.2) is equivalent to the system of the following two real partial differential equations for $w = (w_1, w_2)$:

$$\frac{\partial w_1}{\partial x} = V_1, \qquad \frac{\partial w_2}{\partial x} = V_2, \tag{3.3}$$

$$\frac{\partial w_1}{\partial y} = -V_2, \qquad \frac{\partial w_2}{\partial y} = V_1. \tag{3.4}$$

A calculation shows that the vector fields

$$X = \frac{\partial}{\partial x} + V_1 \frac{\partial}{\partial w_1} + V_2 \frac{\partial}{\partial w_2}, \qquad Y = \frac{\partial}{\partial y} - V_2 \frac{\partial}{\partial w_1} + V_1 \frac{\partial}{\partial w_2}$$

commute when the function $V = V_1 + \mathrm{i}V_2$ is holomorphic in (z, w), and by continuity this persists up to the boundary of K. Hence, the flow ϕ_x of X commutes with the flow ψ_y of Y on their domains of definition. The local solution $w = w(z)$ of the initial value problem (3.2) is then the composition of these two flows, projected onto the w-space:

$$w(z_0 + x + \mathrm{i}y) = pr_w \circ \phi_x \circ \psi_y(z_0, w_0) = pr_w \circ \psi_y \circ \phi_x(z_0, w_0). \tag{3.5}$$

Indeed, differentiation of (3.5) on x and y gives the equations (3.3), (3.4) which are equivalent to (3.2). Clearly, a solution (3.5) also exists at the boundary points of K provided bK is piecewise $\mathscr{C}^1$. This gives local, and often also global holomorphic solutions of (3.1) in terms of flows of vector fields, an ostensibly simpler problem. The same method applies if the above vector fields depend holomorphically on an additional parameter t.

Finally, we can parameterise an arc or a closed curve Γ in $E = \overline{S \setminus K}$ by an injective immersion $p = h(s)$, $s \in [0, 1]$, except that $h(0) = h(1)$ if Γ is a closed Jordan curve. The differential equation (3.1) then takes the following form on Γ:

$$\frac{d}{ds} w(h(s), t) = V(h(s), w(h(s), t), t) \frac{d}{ds} z(h(s)), \qquad w(h(0), t) = w_0. \tag{3.6}$$

This is an ordinary differential equation for the function $s \mapsto w(h(s), t)$ on $s \in [0, 1]$.

It is classical that solutions of the initial value problem (3.2) depend holomorphically on the initial point (z_0, w_0) in the open set where V is holomorphic (see Hille [34, Theorem 2.8.2]). If V is of class $\mathscr{A}^r(K \times \Delta)$ then the flows in (3.5) are of class $\mathscr{C}^r$ up to the boundary of K, and hence so are the solutions. Furthermore, the solutions on $D = \{|z - z_0| < \rho\} \cap K$ corresponding to a pair of initial values w_0, w_1 at z_0 satisfy an estimate

$$|w(z; z_0, w_0) - w(z; z_0, w_1)| \le c_0 |w_0 - w_1| e^{c|z - z_0|} \tag{3.7}$$

as long as their graphs remain in $D \times \triangle$. Here, $c > 0$ is the Lipshitz constant for V with respect to the variable w:

$$|V(z, w) - V(z, w')| \le c|w - w'|, \quad z \in D,\ w, w' \in \triangle,$$

and the constant $c_0 \ge 1$ reflects the geometry of D (we may take $c_0 = 1$ if D is convex). This follows from Grönwall's inequality; see [34, Theorem 2.8.1] and [21, Lemma 1.9.3]. Covering S with finitely many discs such that the immersion $z : M \to \mathbb{C}$ gives a local coordinate of each of them, we get a similar estimate (3.7) globally on S.

Grönwall's inequality can also be used to estimate the difference between solutions of a perturbed equation and those of the original equation. Explicitly, if a function $\widetilde{V}$ of class $\mathscr{A}^r(S \times \triangle \times B)$ is uniformly close to V on $S \times \triangle \times B$ then for any $p_0 \in S$ the solution $w(p; p_0, w_0, t)$ of (3.1) is uniformly close to the solution $\widetilde{w}(p; p_0, w_0, t)$ of the same equation for $\widetilde{V}$, provided that both solutions exist and their graphs remain in the given domain. We refer to [21, Lemma 1.9.4] for a precise estimate in a similar context. From the equation

(3.1) we then infer that the solutions $w(p;p_0,w_0,t)$ and $\widetilde{w}(p;p_0,w_0,t)$ are also $\mathscr{C}^1$ close to each other. More generally, by a prolongation of the system we infer that if $\widetilde{V}$ is $\mathscr{C}^r$ close to V then their solutions for the same initial values are $\mathscr{C}^{r+1}$ close to each other.

3.2. The period map. So far we have been considering local solutions of the equation (3.1). If S is a compact simply connected domain with piecewise $\mathscr{C}^1$ boundary in a Riemann surface M, then by the uniqueness of local solutions they amalgamate into a global solution on S provided they remain in the domain of the vector field V. In particular, given a global solution $w(p;p_0,w_0)$ of (3.1) on S (neglecting the parameter t for the moment), any number $w_1 \in \mathbb{C}$ sufficiently close to w_0 determines a solution $w(p;p_0,w_1)$ of the same equation on all of S since, by (3.7), the solution $w(p;p_0,w_1)$ remains close to the original one.

The situation is rather different on an admissible set S with nontrivial fundamental group (equivalently, with nontrivial first holomology group $H_1(S,\mathbb{Z})$). In this case, an important role in the global existence and perturbation theory of solutions is played by the period map along closed homologically nontrivial curves in S; we now recall this notion.

Assume that C is a closed piecewise smooth Jordan curve in S, and choose a parameterisation $h:[0,1]\to C$ with $h(0)=h(1)=p_0\in C$. In the parameter $s\in[0,1]$, the equation (3.1) with the initial condition $w(p_0,t)=w_0$ takes the form (3.6). Assume that the solution $w(s;w_0,t)$ with $w(0;w_0,t)=w_0$ exists for all $s\in[0,1]$. The number

$$\mathscr{P}_C(p_0,w_0,t)=w(1;w_0,t)-w(0;w_0,t)=w(1;w_0,t)-w_0 \tag{3.8}$$

is called the *period* along C for the data (p_0,w_0,t). It is easily seen that this period is independent of the choice of parameterisation of C up to the sign; reversal of the orientation changes the sign of the period. A clear necessary condition for the existence of a single valued solution of the equation (3.1) along the curve C for the data (p_0,w_0,t) is that

$$\mathscr{P}_C(p_0,w_0,t)=0,$$

which simply means that the map $s\mapsto w(s;w_0,t)$ is 1-periodic. Conversely, if this holds then the equation (3.1) has a single valued solution on an annulus around C intersected with S. By varying the initial value at the point p_0 we obtain the map

$$\zeta \longmapsto \mathscr{P}_C(p_0,\zeta,t)\in\mathbb{C}, \qquad \zeta\in\mathbb{C} \text{ near } w_0,$$

called the *Poincaré first return map* of the closed orbit $s\mapsto w(s;w_0,t)$. This map describes the dynamics of orbits in a neighbourhood of the given periodic orbit. In particular, the return map vanishes identically if and only if all nearby solutions are periodic on C and their graphs form a foliation of the phase space near the graph of the initial solution.

Let us consider more closely the case when $S=K$ is a connected compact domain with piecewise $\mathscr{C}^1$ boundary and nontrivial homology group $H_1(K,\mathbb{Z})$. Then, $H_1(K,\mathbb{Z})=\mathbb{Z}^l$ is a free abelian group whose generators are represented by smooth closed Jordan curves $C_1,\ldots,C_l\subset \mathring{K}$. It is classical that the generating curves can be chosen to have a common base point $p_0\in\mathring{K}$, to satisfy $C_i\cap C_j=\{p_0\}$ for $1\le i\ne j\le l$, and such that their union $C=\bigcup_{i=1}^l C_i$ is a Runge set in K, i.e., $\mathring{K}\setminus C$ has no connected components with compact closure in $\mathring{K}$. (The Runge condition will be important in our proof since it implies that functions $C\to\mathbb{C}$ of class $\mathscr{C}^r(C)$ $(r\in\mathbb{Z}_+)$ can be approximated in $\mathscr{C}^r(C)$ by functions holomorphic on a neighbourhood of K in M; see [18, Theorem 16].) It follows that K admits a deformation retraction onto C, and the equation (3.1) has a single valued solution on K if and only if the solution remains in the domain of the equation and has vanishing period on each of the curves $C_1,\ldots,C_l$ in the homology basis of K.

3.3. Runge homology basis of an admissible set. When considering a holomorphic differential equation on a general admissible set S in a Riemann surface M, we must ensure vanishing of periods of solutions on closed curves in a basis for the homology group $H_1(S,\mathbb{Z})$, and also on the arcs in $S\setminus K$ connecting two connected components of K. To this end, we now construct a special Runge homology basis of an admissible set.

Given a Riemannian distance function dist on M and a number $\epsilon>0$, the set

$$S_\epsilon = \{p\in M : \mathrm{dist}(p,S)<\epsilon\} \tag{3.9}$$

is an open neighbourhood of S which admits a deformation retraction onto S provided $\epsilon>0$ is small enough; we shall call such S_ϵ a *regular neighbourhood* of S. Clearly, S has no holes and hence is Runge in any regular neighbourhood S_ϵ.

Lemma 3.1. *An admissible set S in a Riemann surface M has finitely generated first homology group $H_1(S,\mathbb{Z})\cong\mathbb{Z}^l$, and there is a homology basis $\mathcal{C}=\{C_1,\ldots,C_l\}$ consisting of closed piecewise smooth Jordan curves in S such that the compact set $C=\bigcup_{i=1}^l C_i$ is Runge in any regular neighbourhood S_ϵ (3.9) of S, and each curve $C_i\in\mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j\neq i}C_j$.*

One can also find a basis for $H_1(S,\mathbb{Z})$ consisting of smooth Jordan curves in S. Although the existence of such homology basis was used before (see in particular the papers [3, 5, 7], among others), a detailed construction has not been given; we shall compensate this here. The case when $S=K$ is a union of domains is classical.

Proof. Clearly we may assume that S is connected. Let $S=K\cup E$ with $K=\bigcup_{i=1}^m K_i$, where $K_1,\ldots,K_m$ are the connected components of K. In the special case when $K=\varnothing$, S is a single closed curve and the result is trivial. Assume now that $K\neq\varnothing$. Since S is connected, $E=\bigcup_{k=1}^n E_k$ is a union of finitely many smooth pairwise disjoint arcs E_k. In each component K_i we choose an interior point $q_i\in\mathring{K}_i$ which we shall call the *vertex* of K_i. The boundary $bK_i=\bigcup_{j=1}^{m_i}\Gamma_{i,j}$ consists of finitely many closed Jordan curves for some $m_i\geq 1$. The standard construction gives a basis for $H_1(K_i,\mathbb{Z})$ consisting of finitely many Jordan curves in $\mathring{K}_i$ passing through the vertex q_i and not intersecting elsewhere, together will all boundary components of bK_i except one, say $\Gamma_{i,1}$.

For every $j=1,\ldots,m_i$ we choose a smooth embedded arc $A_{i,j}\subset K_i$ connecting the vertex q_i to a boundary point $a_{i,j}\in\Gamma_{i,j}$ such that these arcs do not intersect each other except at q_i, and they intersect the chosen curves in the homology basis for K_i onoy at the endpoints. For every $j=2,\ldots,m_i$ we attach the arc $A_{i,j}$ to the boundary curve $\Gamma_{i,j}$, so the modified curve is also based at the vertex q_i. The closed curves obtained in this way form a homology basis of $K=\bigcup_{i=1}^m K_i$ and are placed into the family $\mathcal{C}$ under construction. Their union is Runge in S since every point in $\mathring{K}_i$ can be connected by a path in their complement to the boundary curve $\Gamma_{i,1}$.

On the component $\Gamma_{i,1}$ of bK_i (which is not a part of the homology basis of K_i) we choose a point $b_i\neq a_{i,1}$ which is not the endpoint of any of the arcs in E.

Recall that $E_1,\ldots,E_n$ are the connected components (arcs) of $\overline{S\setminus K}$. Set $S_k=K\cup\bigcup_{j=1}^k E_j$ for $k=0,\ldots,n$, so $S_0=K$ and $S_n=S$. Consider the change to the homology made by attaching E_k to S_{k-1}. If E_k is attached to K with only one of its endpoints, it does not contribute to the homology basis of S_k. If on the other hand E_k has both endpoints in bK, we have the following three distinct possibilities.

Case 1: Both endpoints of E_k lie in the same connected component $\Gamma_{i,j}$ of bK_i for some i. In this case, E_k forms a nontrivial new closed curve obtained by connecting the endpoints of E_k by an arc in $\Gamma_{i,j}$; if $j = 1$ we choose the one of both possible arcs not containing the point b_i. We add this new closed curve to the family $\mathcal{C}$.

Case 2: The endpoints of E_k lie in different connected component Γ_{i,j_1}, Γ_{i,j_2} of bK_i for some i. In this case, a new homologically nontrivial closed curve in S_k is obtained by connecting the endpoints of E_k inside K_i as follows. Start at the endpoint contained in Γ_{i,j_1} and go to the point a_{i,j_1} along an arc in Γ_{i,j_1}; if $j_1 = 1$, choose the one of both possible arcs which does not contain the point b_i. Next, go from a_{i,j_1} to the vertex q_i along the arc A_{i,j_1}, continue from q_i to a_{i,j_2} along A_{i,j_2}, and finally connect a_{i,j_2} to the other endpoint of E_k by an arc in Γ_{i,j_2} (if $j_2 = 1$, choose the one that does not contain the point b_i). Add the new closed curve to the family $\mathcal{C}$.

Case 3: The endpoints of E_k belong to different connected components of K. In this case, either no new homologically essential closed curve appears (this happens when E_k joins a pair of connected components of S_{k-1}), or else E_k is a part of a closed curve in S_k formed by a sequence of arcs $E_k = E_{k_1}, \dots, E_{k_s} \subset E \cap S_k$ and connected components $K_{i_1}, \dots, K_{i_s}$ of K. That is, the arc $E_{k_1} = E_k$ connects K_{i_1} to K_{i_2}, E_{k_2} connects K_{i_2} to K_{i_3}, etc., until the cycle closes with the arc E_{k_s} connecting K_{i_s} back to K_{i_1}. In this case we obtain a new closed curve in S_k by connecting the terminal point of each arc E_{k_j} in the above sequence within the domain K_{i_j} to the initial point of the next arc $E_{k_{j+1}}$, where $E_{k_{s+1}} = E_1$. These connecting curves in domains K_{i_j} are chosen as in Cases 1 and 2 above, depending on whether the points to be connected lie in the same or in different components of bK_{i_j}. The new curve obtained in this way is added to the family $\mathcal{C}$.

This process ends in n steps and yields a homology basis $\mathcal{C}$ of $S = S_n$. The union C of the curves in $\mathcal{C}$ is connected, and any point $p \in K_i \setminus C$ $(i = 1, \dots, m)$ can be connected by an arc in $K_i \setminus C$ to the point $b_i \in \Gamma_{i,1} \subset bK_i$. Hence, C has no holes in any regular neighbourhood S_ϵ of S, so it is Runge. It is evident from the construction that every curve $C_i \in \mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j \neq i} C_j$. $\square$

3.4. Topological degree of a map. We shall need the notion of the *topological degree* of a continuous map which was first defined and studied by L. E. J. Brouwer in 1911, [12]. For modern treatment, see e.g. M. Hirsch [35] or J. Milnor [40]. We recall a few basic facts.

Let $f : M \to N$ be a smooth map between closed (compact without boundary) connected oriented manifolds of the same dimension n. Then, every regular value $q \in N$ of f has finitely many preimages, and its degree $\deg(f)$ is the signed number of points in the fibre $f^{-1}(q)$ taking into account the local orientations. (For nonorientable manifolds one can introduce the notion of degree modulo 2.) It turns out that this number is independent of $q \in N$. Furthermore, a pair of homotopic maps $M \to N$ have the same degree, so the degree can also be defined for continuous maps between topological manifolds.

Let D be a compact domain in $\mathbb{R}^n$ which is a topological manifold with coherently oriented boundary bD. Given a continuous map $f : D \to \mathbb{R}^n$ and a point $q \in \mathbb{R}^n \setminus f(bD)$, we can define $\deg(f, q) \in \mathbb{Z}$ as the topological degree of the map $\pi \circ f : bD \to S^{n-1}$, where $\pi : \mathbb{R}^n \setminus \{q\} \to S^{n-1}$ is the retraction $\pi(x) = \frac{x-q}{|x-q|}$. If f is smooth and q is a regular value of f, then $\deg(f, q)$ is the signed number of points in the fibre $f^{-1}(q)$. This gives the following observation.

Proposition 3.2. *Assume that D is a compact domain in $\mathbb{R}^n$ which is a topological manifold with boundary bD. If $f : D \to \mathbb{R}^n$ is a continuous map such that $f(bD) \subset \mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$ and the map $\frac{f}{|f|} : bD \to S^{n-1}$ has nonzero degree, then there is a point $p \in D$ with $f(p) = 0$. In particular, if $0 \in \mathring{D}$ and $f_t : D \to \mathbb{R}^n$ $(t \in [0,1])$ is a homotopy with $f_0 = \mathrm{Id}_D$ such that $f_t(bD) \subset \mathbb{R}^n_*$ for all $t \in [0,1]$, then $0 \in f_t(D)$ for all $t \in [0,1]$.*

3.5. Generators of $\mathscr{A}^r(S)$. We shall need the following lemma.

Lemma 3.3. *Let S be an admissible set in a Riemann surface (see Definition 1.1). Given functions $f_1, \ldots, f_m \in \mathscr{A}^r(S)$ $(r \in \mathbb{Z}_+)$ without common zeros, there are functions $g_1, \ldots, g_m \in \mathscr{A}^r(S)$ such that*

$$f_1 g_1 + f_2 g_2 + \cdots + f_m g_m = 1. \tag{3.10}$$

Proof. For $r = 0$ this follows from the result of R. Arens [10] which states that every maximal ideal of the algebra $\mathscr{A}(S)$ is given by the evaluation at a point of S. (When $S = \overline{\mathbb{D}}$ is the closed unit disc in $\mathbb{C}$, this is a special case of results of W. Rudin [47] who described closed ideals of the disc algebra $\mathscr{A}(\overline{\mathbb{D}})$.) Hence, a collection of functions in $\mathscr{A}(S)$ without a common zero spans $\mathscr{A}(S)$, so (3.10) holds. (Arens's result applies in the more general situation when S is a compact set in a Riemann surface M, U is an open set in M contained in S, and $\mathscr{A}(S, U)$ is the algebra of continuous functions on S which are holomorphic on U. Here we are taking $U = \mathring{S}$.)

Suppose now that $r > 0$ and $f_1, \ldots, f_m \in \mathscr{A}^r(S)$ have no common zeros. Let $g_1, \ldots, g_m \in \mathscr{A}(S)$ satisfy (3.10). By Mergelyan's theorem we can approximate each g_j uniformly on S by a function $\tilde{g}_j \in \mathscr{O}(S)$. If the approximations are close enough then the function $h = \sum_{j=1}^m f_j \tilde{g}_j \in \mathscr{A}^r(S)$ has no zeros on S, and the functions $G_j = \tilde{g}_j / h \in \mathscr{A}^r(S)$ for $j = 1, \ldots, m$ then satisfy $\sum_{j=1}^m f_j G_j = 1$. □

We shall also need the following generalization of Lemma 3.3, analogous to [4, Lemma 2.1]. The proof given there applies verbatim if we replace the use of Cartan's Theorems A and B over an open Riemann surface by the corresponding results of Heunemann [32, 33] for complex vector bundles of class $\mathscr{A}(S)$ (see the proof of Lemma 2.3).

Lemma 3.4. *Let S be an admissible set in a Riemann surface M, and let A be a $m \times p$ matrix-valued function on S, $1 \leq m < p$, of class $\mathscr{A}^r(S)$ which has maximal rank m at every point of S. Then there exists a map $B : S \to GL_p(\mathbb{C})$ of class of class $\mathscr{A}^r(S)$ such that $A(p) \cdot B(p) = (I_m, 0)$ holds for all $p \in S$, where I_m is the $m \times m$ identity matrix.*

4. Proof of Theorem 1.2

Let (X, ξ) be a complex contact manifold with $\dim X = 2n + 1 \geq 3$, and let η be a holomorphic 1-form on X with values in the normal bundle TX/ξ such that $\xi = \ker \eta$ (see (1.1)). Assume that S is an admissible set in a Riemann surface M and $f \in \mathscr{A}^{r+1}(S, X)$ $(r \geq 1)$ is an immersed ξ-Legendrian curve. By Lemma 2.2, the graph

$$G_f = \{(\zeta, f(\zeta)) : \zeta \in S\} \subset M \times X$$

has a Stein neighbourhood $\widetilde{\Omega} \subset M \times X$. By Lemma 2.3, f extends to an immersion $F : S \times \mathbb{B}^{2n} \to X$ of class $\mathscr{A}^{r+1}(S \times \mathbb{B}^{2n}, X)$ whose graph is contained in $\widetilde{\Omega}$. By standard results (see e.g. [36, Lemma 4.3] or [15, Proposition 5.55]), F extends to an immersion

$F : U \times \mathbb{B}^{2n} \to X$ of class $\mathscr{C}^{r+1}$, where $U \subset M$ is an open neighbourhood of S, which is asymptotically holomorphic on $S \times \mathbb{B}^{2n}$ to order r, meaning that

$$D^r(\overline{\partial} F) = 0 \text{ on } S \times \mathbb{B}^{2n}. \tag{4.1}$$

Here, D^r stands for the total derivative of order r (the r-jet). Consider the 1-form

$$\beta = F^* \eta \tag{4.2}$$

of class $\mathscr{C}^r(U \times \mathbb{B}^{2n})$. In view of (4.1), β is a $(1,0)$-form at all points of $S \times \mathbb{B}^{2n}$ which is asymptotically holomorphic to order $r-1$ along S, and it is a holomorphic contact form on $\mathring{S} \times \mathbb{B}^{2n}$. For the same reason, its differential $d\beta = F^*(d\eta)$ is a 2-form of class $\mathscr{C}^r(U \times \mathbb{B}^{2n})$ which is a $(2,0)$-form at all points of $S \times \mathbb{B}^{2n}$, and

$$\beta \wedge (d\beta)^n = F^*(\eta \wedge (d\eta)^n) \neq 0$$

is a nowhere vanishing $(2n+1)$-form of class $\mathscr{C}^r(U \times \mathbb{B}^{2n})$ which is a $(2n+1,0)$-form at all points of $S \times \mathbb{B}^{2n}$. In this sense we shall understand β as a complex contact form on $S \times \mathbb{B}^{2n}$ of class $\mathscr{A}^r(S \times \mathbb{B}^{2n})$. Note also that $S \times \{0\}^{2n}$ is a β-Legendrian curve. (A more precise treatment of asymptotically holomorphic contact forms can be found in [22].)

We shall need the following partial normal form of β along $S \times \{0\}^{2n}$.

Lemma 4.1. *Let β be as in* (4.2)*, and let $z : M \to \mathbb{C}$ be a holomorphic immersion. There are fibre coordinates $(w, x_2, \ldots, x_n, y = y_1, y_2, \ldots, y_n)$ on $S \times \rho\mathbb{B}^{2n}$ and a nowhere vanishing function $h \in \mathscr{A}^r(S \times \rho\mathbb{B}^{2n})$ for some $0 < \rho < 1$ such that*

$$\frac{1}{h}\beta = dw - ydz - \sum_{i=2}^{n} y_i dx_i + \widetilde{\alpha} = \alpha + \widetilde{\alpha} \quad \text{on } S \times \rho\mathbb{B}^{2n}, \tag{4.3}$$

where

$$\alpha = dw - ydz - \sum_{i=2}^{n} y_i dx_i \tag{4.4}$$

and the remainder $\widetilde{\alpha}$ contains terms which do not contribute to $\beta \wedge (d\beta)^n$. The change of coordinates which brings β into this form is a shearlike transformation on $S \times \mathbb{C}^{2n}$ of class $\mathscr{A}^r(S \times \mathbb{C}^{2n})$ preserving the fibres $\{p\} \times \mathbb{C}^{2n}$ and keeping fixed the zero section $S \times \{0\}^{2n}$.

Proof. The proof is obtained by following the first part of the construction of Darboux charts around immersed holomorphic Legendrian curves given in [4, proof of Theorem 1.1]. However, the second step in the said proof, which uses Moser's flow method to get a complete normalisation without the remainder, does not preserve the tube over S, and we will not use it here.

Let p denote a variable point in M, and let $\zeta = (\zeta_1, \ldots, \zeta_{2n})$ be complex coordinates on $\mathbb{C}^{2n}$. Along the β-Legendrian curve $S \times \{0\}^{2n} = \{\zeta = 0\}$ we have that

$$\beta(p, 0) = \sum_{j=1}^{2n} a_j(p) d\zeta_j, \quad p \in S$$

for some functions $a_j \in \mathscr{A}^r(S)$ without common zeros. (The 1-form dz does not appear in the above expression since $S \times \{0\}^{2n}$ is a β-Legendrian curve.) Let $a = (a_1, \ldots, a_{2n}) : S \to \mathbb{C}^{2n} \setminus \{0\}$. Lemma 3.4 furnishes a map $B : S \to GL_{2n}(\mathbb{C})$ of class $\mathscr{A}^r(S)$ satisfying $a(p) \cdot B(p) = (1, 0, \ldots, 0)$ for all $p \in S$. We now introduce new fibre coordinates by

$$\zeta' = B(p)^{-1}\zeta, \quad p \in S,\ \zeta \in \mathbb{C}^{2n}.$$

Dropping the primes, this transforms β along $S \times \{0\}^{2n}$ into the constant 1-form $d\zeta_1$. Denoting the variable ζ_1 by w, we thus have

$$\beta = dw \ \text{ along } \ S \times \{0\}^{2n}.$$

We now consider those terms in the Taylor expansion of β along $S \times \{0\}^{2n}$ with respect to the fibre coordinates which give a nontrivial contribution to the coefficient function of the $(2n+1,0)$-form $\beta \wedge (d\beta)^n$. Since the coefficient of dw equals 1 on $S \times \{0\}^{2n}$, it is a nowhere vanishing function $h \in \mathscr{A}^r(S \times \rho\mathbb{B}^{2n})$ for some $\rho > 0$, and we have that

$$\tag{4.5} \frac{1}{h}\beta = dw + \left(\sum_{j=2}^{2n} b_j\zeta_j\right)dz + \sum_{j,k=2}^{2n} c_{j,k}\zeta_k\, d\zeta_j + \tilde\beta,$$

where $b_j, c_{j,k} \in \mathscr{A}^r(S)$. The remainder 1-form $\tilde\beta$ contains all terms $\zeta_j d\zeta_j$, terms whose coefficients are of order ≥ 2 in the variables $\zeta_2, \ldots, \zeta_{2n}$, and terms containing the w variable; such terms disappear in $\beta \wedge (d\beta)^n$ at all points of $S \times \{0\}^{2n}$.

We claim that the functions $b_2, \ldots, b_{2n}$ in (4.5) have no common zeros on S. Indeed, since $h = 1$ on $S \times \{0\}^{2n}$, at a common zero $p_0 \in S$ of these functions the form $d\beta$ at the point $(p_0, 0)$ does not contain the term dz and hence $\beta \wedge (d\beta)^n$ vanishes, a contradiction. Write $\zeta' = (\zeta_2, \ldots, \zeta_{2n})$. Lemma 3.4 applied to the map $(b_2, \ldots, b_{2n}) : S \to \mathbb{C}^{2n-1} \setminus \{0\}$ gives a change of coordinates of class $\mathscr{A}^r(S)$ and of the form

$$(p, w, \zeta') \mapsto (p, w, \widetilde{B}(p)\zeta'), \qquad \widetilde{B}(p) \in GL_{2n-1}(\mathbb{C})$$

such that the coefficient of dz becomes $-\zeta_2$, and hence

$$\tag{4.6} \frac{1}{h}\beta = dw - \zeta_2 dz + \sum_{j,k=2}^{2n} c_{j,k}\zeta_k\, d\zeta_j + \tilde\beta$$

for some new $c_{j,k} \in \mathscr{A}^r(S)$. Note that $(d\beta)^n$ must contain the factor $d(\zeta_2 dz) = d\zeta_2 \wedge dz$ since the differential dz in the S direction at the points of $S \times \{0\}^{2n}$ does not appear in any other way. Hence, the terms containing $d\zeta_2$ or $\zeta_2 d\zeta_j$ with $j > 2$ in (4.6) do not contribute to $(d\beta)^n$ and are placed into $\tilde\beta$. Renaming the variable ζ_2 to y we thus have

$$\tag{4.7} \frac{1}{h}\beta = dw - y dz + \sum_{j,k=3}^{2n} c_{j,k}\zeta_k\, d\zeta_j + \tilde\beta$$

with some new coefficients $c_{j,k} \in \mathscr{A}^r(S)$ and new remainder $\tilde\beta$.

If $n = 1$ (i.e., $\dim X = 3$), we are finished. Assume now that $n > 1$. We eliminate the variable ζ_3 from the coefficients of the differentials $d\zeta_4, \ldots, d\zeta_{2n}$ by the shear

$$w' = w + \sum_{j=4}^{2n} c_{j,3}\zeta_3\zeta_j.$$

This ensures that the functions $c_{3,k}$ in the coefficient of $d\zeta_3$ in (4.7) have no common zeros on S (since at such point $d\beta$ would not contain $d\zeta_3$ and hence $\beta \wedge (d\beta)^n$ would vanish). Applying Lemma 3.4 we change the coefficient of $d\zeta_3$ to $-\zeta_4$ by a linear change of the variables $\zeta_4, \ldots, \zeta_{2n}$ which is of class $\mathscr{A}^r(S)$ with respect to $p \in S$. Set $x_2 = \zeta_3$ and $y_2 = \zeta_4$. By the same argument as in the previous step, we can move the term with $dy_2 = d\zeta_4$, as well as all terms containing $y_2 = \zeta_4$ in the subsequent differentials $d\zeta_5, \ldots, d\zeta_{2n}$, to the remainder $\tilde\beta$. This gives

$$\frac{1}{h}\beta = dw - ydz - y_2 dx_2 + \sum_{j,k=5}^{2n} c_{j,k}\zeta_k\, d\zeta_j + \tilde\beta.$$

In finitely many steps of this kind we obtain the normal form (4.3), (4.4). $\square$

Precomposing the initial immersion $F : S \times \mathbb{B}^{2n} \to X$ by the change of coordinates furnished by Lemma 4.1 gives a new immersion $F : S \times \rho\mathbb{B}^{2n} \to X$ for some $0 < \rho < 1$ which agrees with f on $S \times \{0\}^{2n}$ such that the contact form $\frac{1}{h} F^*\eta$ on $S \times \rho\mathbb{B}^{2n}$ is of the form (4.3). With this assumption we now prove Theorem 1.2.

Proof of Theorem 1.2. We first prove the theorem in the special case when S is a compact connected domain in M (i.e., there are no attached arcs) and without the interpolation conditions. The general case will be considered afterwards.

Let $\mathcal{C} = \{C_1, \ldots, C_l\}$ be a homology basis of S furnished by Lemma 3.1, consisting of piecewise smooth oriented Jordan curves with the common base point $p_0 \in \mathring{S}$, such that $C = \bigcup_{i=1}^l C_i$ is $\mathscr{O}(S)$-convex and each curve $C_i \in \mathcal{C}$ contains a nontrivial arc I_i disjoint from $\bigcup_{j\neq i} C_j$. Let $z : M \to \mathbb{C}$ be the holomorphic immersion chosen at the beginning of the section such that (4.3) holds. As in [7, Sect. 4] we find a holomorphic spray of functions

$$y(p,t) = \sum_{i=1}^{l} t_i \xi_i(p), \quad p \in S,\ t = (t_1, \ldots, t_l) \in \mathbb{C}^l, \tag{4.8}$$

where $\xi_i \in \mathscr{O}(S)$ are holomorphic functions satisfying

$$\int_{C_i} \xi_j dz = \delta_{i,j}, \quad i, j = 1, \ldots, l. \tag{4.9}$$

(Here, $\delta_{i,j}$ is the Kronecker delta.) Inserting the values

$$y = y(p,t), \quad x_2 = \cdots = x_n = y_2 = \ldots = y_n = 0 \tag{4.10}$$

into the 1-form α (4.4) gives the equation

$$dw = y(p,t)dz, \quad p \in S \tag{4.11}$$

whose local solutions $w = w(p,t)$ are α-Legendrian curves. Since the variable w does not appear on the right hand side, the solutions are obtained by integration:

$$w(p,t) = w_0 + \int_{p_0}^{p} y(\cdot\,,t)dz = w_0 + \sum_{i=1}^{l} t_i \int_{p_0}^{p} \xi_i dz, \quad p \in S. \tag{4.12}$$

From (4.8) and (4.11) it follows that any solution satisfying the initial condition $w(p_0,t) = 0$ also satisfies

$$|w(p,t)| = O(|t|), \quad p \in S, \tag{4.13}$$

provided that in (4.12) we integrate along an approximately geodesic curve in S from p_0 to p. (The integral may of course depend on the choice of the curve.) Using the notation (3.8), we also see from (4.9) that the period map of the solution (4.12) along the curve C_i equals

$$\mathscr{P}_{C_i}(p_0, w_0, t) = t_i, \quad i = 1, \ldots, l.$$

Hence, the period map $\mathbb{C}^l \ni t \mapsto \mathscr{P}^\alpha(t) \in \mathbb{C}^l$ of (4.12) associated to the 1-form α and the spray (4.8) is the identity map

$$\mathscr{P}^\alpha(t) = \big(\mathscr{P}^\alpha_{C_1}(t), \ldots, \mathscr{P}^\alpha_{C_l}(t)\big) = t.$$

In particular, the only single valued α-Legendrian curve in this family satisfying the initial condition $w(p_0,t)=0$ is $w=0$ at the parameter value $t=0$.

Inserting the values (4.10) into the 1-form $\frac{1}{h}\beta = \alpha + \widetilde\alpha$ (4.3), the only nonvanishing terms in $\widetilde\alpha$ are those of the form wdy and ydy, possibly multiplied by other normal coordinates and by functions in $\mathscr{A}^r(S)$. This is seen from the proof of Lemma 4.1 where we stated explicitly at every step of the reduction which terms were placed into the remainder $\widetilde\alpha$. It follows from (4.8) and (4.13) that these terms disturb the period map of solutions of the resulting differential equation for β-Legendrian curves by a term of size $O(|t|^2)$. Hence, the period map of the β-Legendrian curve satisfying the initial condition $w(p_0,t)=0$ equals

$$\mathscr{P}^\beta(t) = t + O(|t|^2).$$

For every small $\delta>0$ the map $\mathscr{P}^\beta(t)$ is close enough to t on the closed polydisc

$$P_\delta = \{(t_1,\ldots,t_l) : |t_i| \le \delta,\ i=1,\ldots,l\} = \delta\mathbb{D}^l \subset \mathbb{C}^l \tag{4.14}$$

that it maps bP_δ to $\mathbb{C}^l_*$ and this map has degree one. (Since bP_δ is homotopic to the sphere S^{2l-1} in $\mathbb{C}^l_*$ and $\mathbb{C}^l_*$ retracts onto S^{2l-1}, the degree is well defined, cf. Subsect. 3.4.)

We now fix $\delta>0$; however, it's value will be determined only later.

Let $F : S\times\rho\mathbb{B}^{2n} \to X$ be the immersion described just before the beginning of the proof, so $F(\cdot\,,0) = f$ and $\frac{1}{h}F^*\eta = \frac{1}{h}\beta$ is of the form (4.3). After decreasing ρ slightly, we can approximate F as closely as desired in the $\mathscr{C}^{r+1}(S\times\rho\mathbb{B}^{2n},X)$ topology by a holomorphic immersion $\widetilde F : U\times\rho\mathbb{B}^{2n} \to X$, where $U\subset M$ is a neighbourhood of S (with U depending on $\widetilde F$). Let us explain this. By the construction, the graph of F is contained in a Stein domain $\widetilde\Omega \subset M\times X$. Hence, by using a holomorphic embedding of $\widetilde\Omega$ into a Euclidean space $\mathbb{C}^N$ and an ambient holomorphic retraction back to this embedded submanifold, the proof reduces to approximation of functions in $\mathscr{A}^{r+1}(S\times\rho\mathbb{B}^{2n})$ by holomorphic functions in a neighbourhood (decreasing $\rho>0$ slightly). For the details of this reduction, see [18, Sect. 7.2, Lemma 3]. In order to approximate a function $g\in\mathscr{A}^{r+1}(S\times\rho\mathbb{B}^{2n})$, we consider its Taylor series expansion in the fibre variable

$$g(p,z) = \sum_{I\in\mathbb{Z}^{2n}_+} a_I(p) z^I,$$

with coefficients $a_I \in \mathscr{A}^{r+1}(S)$. It remains to approximate the coefficients a_I in the $\mathscr{A}^{r+1}(S)$ by functions $\tilde a_I \in \mathscr{O}(S)$; this is accomplished by [18, Theorem 16].

Suppose now that $\widetilde F : U\times\rho\mathbb{B}^{2n} \to X$ is a holomorphic immersion approximating F in $\mathscr{C}^{r+1}(S\times\rho\mathbb{B}^{2n},X)$. Then, the pullback

$$\tilde\beta := \widetilde F^*\eta \tag{4.15}$$

is holomorphic contact form on $U\times\rho\mathbb{B}^{2n}$ which is $\mathscr{C}^r$-close to β on $S\times\rho\mathbb{B}^{2n}$. Furthermore, the coefficient $\tilde h\in\mathscr{O}(U\times\rho\mathbb{B}^{2n})$ of the differential dw in $\tilde\beta$ is close to the corresponding coefficient h of β on $S\times\rho\mathbb{B}^{2n}$ and hence is nonvanishing, perhaps after shrinking $U\supset S$ and decreasing $\rho>0$ slightly. The holomorphic contact form $\tilde h^{-1}\tilde\beta$ on $U\times\rho\mathbb{B}^{2n}$ is then $\mathscr{C}^r$ close to the form $h^{-1}\beta$ (4.3) on $S\times\rho\mathbb{B}^{2n}$.

We now insert the values (4.10) into $\tilde h^{-1}\tilde\beta$ and denote by $t \mapsto \mathscr{P}^{\tilde\beta}(t)$ the corresponding period map for solutions satisfying the initial condition $w(p_0,t) = 0$. Assuming that the approximations are close enough, the period map $\mathscr{P}^{\tilde\beta}(t)$ is so close to $\mathscr{P}^{\beta}(t) = t+O(|t|^2)$ on the polydisc $P_\delta \subset \mathbb{C}^l$ (see (4.14)) that it maps its boundary bP_δ to $\mathbb{C}^l_*$ and this map has degree one. By Proposition 3.2 there is a point $t^0 \in \mathring{P}_\delta$ such that

$$\mathscr{P}^{\tilde\beta}(t^0) = 0.$$

For $t = t^0$, the solution of the differential equation for $\tilde\beta$-Legendrian curves satisfying the initial condition $w(p_0,t^0) = 0$ has vanishing periods over the curves $C_1,\ldots,C_l$ in the homology basis of S. Assuming that $\delta > 0$ was chosen small enough and the approximations were close enough, we obtain an embedded holomorphic $\tilde\beta$-Legendrian curve on a neighbourhood of S in M (see Subsect. 3.1) which is $\mathscr{C}^{r+1}$-close to the initial β-Legendrian curve $S \times \{0\}^{2n}$. (Note that one gains one derivative when integrating the differential equation.) Its image by $\widetilde F$ is a holomorphic ξ-Legendrian immersion $\tilde f : U \to X$ which approximates the Legendrian immersion $f : S \to X$ in $\mathscr{C}^{r+1}(S,X)$.

This completes the proof of Theorem 1.2 in the special case.

Consider now the general case $S = K \cup E$ (see Definition 1.1). Let $\mathcal{C}$ denote the homology basis of S furnished by Lemma 3.1. The closed curves in each component K_i of K $(i = 1,\ldots,m)$ forming a basis of $H_1(K_i,\mathbb{Z})$ are based at a vertex $q_i \in \mathring{K}_i$. We enlarge the finite set $A \subset S$ (at which we shall interpolate) by adding to it the endpoints of all arcs $E_i \subset E$ (the connected components of E). We then form a family $\mathscr{C}$ of arcs and closed curves in S as follows.

(a) If a closed curve $C \in \mathcal{C}$ does not contain any points of A, we put it into $\mathscr{C}$. Otherwise, we split C into the union of finitely many arcs lying back to back, with the points of $A\cap C$ as the common endpoints of consecutive arcs, and we add all these arcs into $\mathscr{C}$.
(b) If E_j is a connected component of E which is not contained in any of the closed curves from the previous item, we split E_j into a union of subarcs with the points of $E_j \cap A$ being the common endpoints of consecutive arcs, and we add all these subarcs into the family $\mathscr{C}$. Furthermore, we connect each endpoint of E_j by an arc in the respective connected component K_i of K to the vertex q_i, choosing it disjoint from the curves already in $\mathscr{C}$ except at q_i. We also add these arcs into the family $\mathscr{C}$.
(c) Let A' denote the set of points $a \in A$ belonging to at least one curve in the family $\mathscr{C}$ constructed thus far. Any remaining point $a \in A\setminus A'$ lies in one of the components K_i of K. Choose an embedded arc $\Lambda_a \subset K_i$ connecting a to the vertex q_i such that Λ_a does not meet any of the arcs from $\mathscr{C}$ other than at q_i. We add the arcs Λ_a for $a \in A\setminus A'$ into the family $\mathscr{C}$ whose construction is now complete.

Note that the union $\widetilde C$ of all curves in the family $\mathscr{C}$ is a connected Runge set in S.

Let $y(p,t)$ be a spray (4.8) where the functions $\xi_i \in \mathscr{O}(S)$ satisfy conditions (4.9) on the curves in the family $\mathscr{C}$. As in the special case considered above, we approximate F in the $\mathscr{C}^{r+1}(S\times \rho\mathbb{B}^{2n}$ topology by a holomorphic immersion $\widetilde F : U\times\rho\mathbb{B}^{2n} \to X$ from a neighbourhood of $S \times \{0\}^{2n}$ into X which agrees with F at the finitely many points $A\times\{0\}^{2n}$, and let $\tilde\beta = \widetilde F^*\eta$ (4.15). As before, let $\tilde h \in \mathscr{O}(U\times\rho\mathbb{B}^{2n})$ denote the nonvanishing coefficient of dw in $\tilde\beta$. We then insert the values $y = y(p,t)$, $x_2 = \cdots = x_n = y_2 = \ldots = y_n = 0$ (see (4.10)) into the equation $\tilde h^{-1}\tilde\beta = 0$ for $\tilde\beta$-Legendrian curves. Pick a point $p_0 \in A$. For each $C \in \mathscr{C}$ we choose a parameterisation by a function $\gamma(s)$,

$s \in [0,1]$. For all $t \in \mathbb{C}^l$ sufficiently close to 0 we define

$$\mathscr{P}_C^{\tilde{\beta}}(t) = w(1,t) \in \mathbb{C},$$

where $[0,1] \ni s \mapsto w(s,t)$ is the unique solution on C of the differential equation for $\tilde{\beta}$-Legendrian curves with the initial value $w(0,t) = 0$. (Compare with (3.6) and (3.8).) This defines the period map $t \mapsto \mathscr{P}^{\tilde{\beta}}(t) \in \mathbb{C}^l$ for $t \in P_\delta \subset \mathbb{C}^l$ for a small enough $\delta > 0$ (see (4.14)). Assuming as we may that the approximations were close enough, the same argument as in the special case considered above gives a value $t^0 \in P_\delta$ such that $\mathscr{P}^{\tilde{\beta}}(t_0) = 0$. Since the union $\widetilde{C}$ of all curves in the family $\mathscr{C}$ is connected, it follows that the solution of the differential equation for $\tilde{\beta}$-Legendrian curves with $t = t^0$ and satisfying the initial condition $w(p_0,t^0) = 0$ is single valued on $\widetilde{C}$ and it vanishes at all points of the finite set A. Assuming as we may that $\delta > 0$ was chosen small enough and the approximation of β by $\tilde{\beta}$ was close enough, we obtain a single valued holomorphic solution on a neighbourhood of S in M which vanishes at all points of A. (See Subsect. 3.1.) Its image by $\widetilde{F}$ is a holomorphic ξ-Legendrian immersion $\tilde{f} : U \to X$ from a neighbourhood of S into X which approximates the initial Legendrian immersion $f : S \to X$ in $\mathscr{C}^{r+1}(S,X)$ and agrees with f at the points of A. It is clear that this method gives interpolation to any given finite order at the points in $A \cap \mathring{S}$ provided we choose $\widetilde{F}$ to match F to a suitable finite order at the points in $(A \cap \mathring{S}) \times \{0\}^{2n}$. The latter condition is a standard addition to the Mergelyan approximation theorem. $\square$

5. Proof of Theorem 1.8

By the general position theorem for Legendrian immersions (see [4, Theorem 1.2]) and shrinking the open set U around K if necessary, we may assume that $f : U \to X$ is a Legendrian embedding.

The proof of Theorem 1.8 is obtained by inductively applying the Mergelyan approximation theorem for Legendrian immersions, given by Theorem 1.2, and the procedure described (for Stein manifolds of arbitrary dimension $n \neq 2$) in [26, Proof of Theorem 1.2]. We provide the outline.

Choose a strongly subharmonic Morse exhaustion function $\rho : M \to \mathbb{R}_+$ and an increasing sequence $0 < c_0 < c_1 < c_2 \cdots$ of regular values of ρ with $\lim_{j\to\infty} c_j = +\infty$ such that, setting $M_j = \{\rho < c_j\}$ for $j \in \mathbb{Z}_+$, we have that $K \subset M_0 \subset \overline{M}_0 \subset U$ and for each $j > 0$ the function ρ has at most one critical point p_j in $M_j \setminus M_{j-1}$. Fix $\epsilon > 0$ and set $\epsilon_0 = \epsilon/2$, $W_0 = M_0$, and $h_0 = \mathrm{Id}_M$. We inductively construct an increasing sequence of smoothly bounded relatively compact domains $W_0 \Subset W_1 \Subset W_2 \Subset \cdots$ in M (not necessarily exhausting M), a sequence of continuous maps $f_j : M \to X$, a sequence of diffeomorphisms $h_j : M \to M$, and a decreasing sequence of numbers $\epsilon_j > 0$ such that the following conditions hold for all $j = 1, 2, \ldots$.

(i) The compact set $\overline{W}_{j-1}$ is $\mathscr{O}(W_j)$-convex.
(ii) The map f_j is a holomorphic Legendrian embedding on a neighborhood of $\overline{W}_j$, and it is homotopic to f_{j-1} by a homotopy $f_{j,t} : M \to X$ ($t \in [0,1]$) such that each $f_{j,t}$ is a holomorphic Legendrian embedding on $\overline{W}_{j-1}$ satisfying

$$\sup_{p \in \overline{W}_{j-1}} \mathrm{dist}\big(f_{j,t}(p), f_{j-1}(p)\big) < \epsilon_{j-1}.$$

(iii) $h_j(M_j) = W_j$, and $h_j = h_{j-1} \circ g_j$ where $g_j : M \to M$ is a diffeomorphism which is diffeotopic to Id_M by a diffeotopy that is fixed on a neighborhood of $\overline{M}_{j-1}$.

(iv) We have $\epsilon_j < \frac{1}{2} \min\{\epsilon_{j-1}, \delta_j\}$, where $\delta_j > 0$ is chosen such that any holomorphic map $g : W_j \to X$ satisfying $\sup_{p \in W_j} \big(f_j(p), g(p)\big) < \delta_j$ is an embedding on W_{j-1}.

Granted such sequences, it is easily verified that there exists the limit map

$$\tilde{f} = \lim_{j \to \infty} f_j : \Omega = \bigcup_{j=1}^{\infty} W_j \to X$$

which is a holomorphic Legendrian embedding, and there exists the limit diffeomorphism $h = \lim_{j \to \infty} h_j : M \to \Omega$ onto Ω. The composition $F = \tilde{f} \circ h : M \to X$ and the complex structure $J = h^*(J_0)$ on M then satisfy Theorem 1.8, i.e., F is a J-holomorphic Legendrian embedding approximating f on K and J agrees with the original complex structure J_0 on a neighborhood of K (since h is the identity there).

The induction begins with $f_0 = f$, $W_0 = M_0$, $h_0 = \mathrm{Id}_M$, and $\epsilon_0 = \epsilon/2$. We now explain the inductive step. Fix $j \in \mathbb{N}$. Assume that ρ has a (unique) critical point $p_j \in M_j \setminus \overline{M}_{j-1}$. If p_j is a local minimum of ρ, we let $E_j = \{p_j\}$. Otherwise, the Morse index of p_j equals 1 and the change of topology of the sublevel set $\{\rho < c\}$ at p_j is described by attaching to the compact domain $\overline{M}_{j-1}$ a smooth embedded arc $E_j \subset M_j$ intersecting $\overline{M}_{j-1}$ transversely at both endpoints and nowhere else. Finally, if ρ has no critical point in $M_j \setminus \overline{M}_{j-1}$, we let $E_j = \varnothing$. In all three cases, the compact set

$$\Gamma_j := \overline{M}_{j-1} \cup E_j \subset M_j \tag{5.1}$$

has arbitrarily small smoothly bounded neighbourhoods $M'_j \Subset M_j$ which are diffeotopic to M_j by a diffeotopy of M that is fixed on a neighbourhood of Γ_j.

Let $h_{j-1} : M \to M$ be the diffeomorphism from the previous step, so we have that $W_{j-1} = h_{j-1}(M_{j-1})$. Set $E'_j = h_{j-1}(E_j)$. Then,

$$S_j := \overline{W}_{j-1} \cup E'_j = h_{j-1}(\Gamma_j)$$

is an admissible subset of M (see Definition 1.1). By the induction hypothesis, the map $f_{j-1} : M \to X$ is a Legendrian embedding on a neighbourhood of $\overline{W}_{j-1}$. Our goal is to find a homotopic deformation of f_{j-1} to continuous map $f_j : M \to X$ which is a holomorphic Legendrian embedding on a neighbourhood of S_j.

If $E'_j = \varnothing$, there is nothing to do. If E'_j is a point, we let f_j agree with f_{j-1} near $\overline{W}_{j-1}$ and let it be an arbitrary holomorphic Legendrian embedding on a small neighbourhood of E'_j. Thus, the only nontrivial case is when E'_j is a smooth arc attached transversely to $\overline{W}_{j-1}$ at its endpoints. In this case we first homotopically deform f_{j-1}, keeping it fixed near $\overline{W}_{j-1}$, so that $f_{j-1} : E'_j \to X$ becomes a smoothly immersed Legendrian curve. (This is possible by the Chow-Rashevskiĭ theorem; see M. Gromov [30, 1.1, p. 113 and 1.2.B, p. 120].) This makes $f_{j-1} : S_j \to X$ a smooth Legendrian immersion which is holomorphic on a neighbourhood of $\overline{W}_{j-1}$. By Theorem 1.2 and the general position theorem (see [4, Theorem 1.2]) we can approximate it as closely as desired in the $\mathscr{C}^2$ topology on S_j by a holomorphic Legendrian embedding $f_j : U_j \to X$ from a neighborhood of S_j into X. After shrinking U_j around S_j there exists a homotopy $f_{j,t} : U_j \to X$ $(t \in [0, 1])$ between $f_{j-1}|_{U_j} = f_{j,0}$ and f_j consisting of maps which are Legendrian embeddings on a neighbourhood $V_j \subset U_j$ of $\overline{W}_{j-1}$ (see [4, Remark 3.2]). Assuming that the approximations were close enough we get condition (ii).

By what was said above, there is a smoothly bounded neighbourhood $W_j \Subset U_j$ of S_j of the form $W_j = h_{j-1}(M'_j)$, where $M'_j \subset M_j$ is a neighbourhood of Γ_j (5.1) diffeotopic to M_j by a diffeotopy which is fixed on a neighbourhood of Γ_j. Let $g_{j,t} : M \to M$ $(t \in [0,1])$ be such a diffeotopy with $g_{j,0} = \mathrm{Id}_M$ and $g_{j,1}(M_j) = M'_j$. Then,

$$h_{j,t} := h_{j-1} \circ g_{j,t} : M \to M, \quad t \in [0,1],$$

is a diffeotopy connecting $h_{j,0} = h_{j-1}$ and $h_{j,1} = h_j$ such that $h_j(M_j) = W_j$. This gives condition (iii). By using a cutoff function in the parameter of the homotopy we can extend f_j and the homotopy $f_{j,t}$ (keeping it fixed on a neighbourhood of S_j) to a continuous map $M \to X$ which agrees with f_{j-1} on $M \setminus U_j$. Finally, we choose the next number $\epsilon_j > 0$ sufficiently small so that condition (iv) holds. This concludes the proof.

Acknowledgements. This research is supported by research program P1-0291 and grant J1-9104 from ARRS, Republic of Slovenia, and by the Stefan Bergman Prize 2019 awarded by the American Mathematical Society.

I wish to thank Roger Casals and Nicola Pia for their helpful explanation and references concerning local stability and functional dimension of Pfaffian systems, described in the introduction just after Problem 1.5 (personal communications, November 2019). I also thanks Antonio Alarcón for helpful remarks which helped me to improve the presentation.

References

[1] A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, and F. J. López. Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. *Proc. Lond. Math. Soc. (3)*, 111(4):851–886, 2015.

[2] A. Alarcón, F. Forstnerič, and F. Lárusson. Holomorphic Legendrian curves in $\mathbb{CP}^3$ and superminimal surfaces in $\mathbb{S}^4$. *arXiv e-prints*, Oct 2019.

[3] A. Alarcón and F. Forstnerič. Null curves and directed immersions of open Riemann surfaces. *Invent. Math.*, 196(3):733–771, 2014.

[4] A. Alarcón and F. Forstnerič. Darboux charts around holomorphic Legendrian curves and applications. *Int. Math. Res. Not. IMRN*, (3):893–922, 2019.

[5] A. Alarcón and F. Forstnerič. New complex analytic methods in the theory of minimal surfaces: a survey. *J. Aust. Math. Soc.*, 106(3):287–341, 2019.

[6] A. Alarcón and F. Forstnerič. The Calabi-Yau problem for Riemann surfaces with finite genus and countably many ends. *arXiv e-prints*, Apr. 2019. `https://arXiv:1904.08015`. *Rev. Mat. Iberoam.*, to appear.

[7] A. Alarcón, F. Forstnerič, and F. J. López. Holomorphic Legendrian curves. *Compos. Math.*, 153(9):1945–1986, 2017.

[8] A. Alarcón and F. J. López. Proper holomorphic embeddings of Riemann surfaces with arbitrary topology into $\mathbb{C}^2$. *J. Geom. Anal.*, 23(4):1794–1805, 2013.

[9] D. V. Anosov and A. A. Bolibruch. *The Riemann-Hilbert problem*. Aspects of Mathematics, E22. Friedr. Vieweg & Sohn, Braunschweig, 1994.

[10] R. Arens. The closed maximal ideals of algebras of functions holomorphic on a Riemann surface. *Rend. Circ. Mat. Palermo (2)*, 7:245–260, 1958.

[11] A. Beauville. Holomorphic symplectic geometry: a problem list. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 49–63. Springer, Heidelberg, 2011.

[12] L. E. J. Brouwer. Über Abbildung von Mannigfaltigkeiten. *Math. Ann.*, 71(1):97–115, 1911.

[13] R. L. Bryant. Conformal and minimal immersions of compact surfaces into the 4-sphere. *J. Differential Geom.*, 17(3):455–473, 1982.

[14] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior differential systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1991.

[15] K. Cieliebak and Y. Eliashberg. *From Stein to Weinstein and back. Symplectic geometry of affine complex manifolds*, volume 59 of *Amer. Math. Soc. Colloquium Publications*. Amer. Math. Soc., Providence, RI, 2012.
[16] G. Darboux. Sur le problème de Pfaff. *C. R. Acad. Sci., Paris*, 94:835–837, 1882.
[17] B. Drinovec Drnovšek and F. Forstnerič. Holomorphic curves in complex spaces. *Duke Math. J.*, 139(2):203–253, 2007.
[18] J. E. Fornæss, F. Forstnerič, and E. F. Wold. Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. *arXiv e-prints*, Feb. 2018. `https://arxiv.org/abs/1802.03924`. To appear in the volume *Advancements in Complex Analysis* by Springer-Verlag.
[19] F. Forstneric. The Calabi-Yau property of superminimal surfaces in self-dual Einstein four-manifolds. *arXiv e-prints*, 2020. `https://arxiv.org/abs/2004.03536`.
[20] F. Forstnerič. Hyperbolic complex contact structures on $\mathbb{C}^{2n+1}$. *J. Geom. Anal.*, 27(4):3166–3175, 2017.
[21] F. Forstnerič. *Stein manifolds and holomorphic mappings (The homotopy principle in complex analysis)*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Cham, second edition, 2017.
[22] F. Forstnerič. H-principle for complex contact structures on Stein manifolds. *arXiv e-prints*, Oct 2018. `https://arxiv.org/abs/1810.12943`. *J. Symplectic Geom.*, to appear.
[23] F. Forstnerič. Holomorphic embeddings and immersions of Stein manifolds: a survey. In *Geometric complex analysis*, volume 246 of *Springer Proc. Math. Stat.*, pages 145–169. Springer, Singapore, 2018.
[24] F. Forstnerič and F. Lárusson. Holomorphic Legendrian curves in projectivised cotangent bundles. *arXiv e-prints*, page arXiv:1809.09391, Sep. 2018.
[25] F. Forstnerič and M. Slapar. Deformations of Stein structures and extensions of holomorphic mappings. *Math. Res. Lett.*, 14(2):343–357, 2007.
[26] F. Forstnerič and M. Slapar. Stein structures and holomorphic mappings. *Math. Z.*, 256(3):615–646, 2007.
[27] T. Friedrich. On surfaces in four-spaces. *Ann. Global Anal. Geom.*, 2(3):257–287, 1984.
[28] T. Friedrich. On superminimal surfaces. *Arch. Math. (Brno)*, 33(1-2):41–56, 1997.
[29] H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.
[30] M. Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.
[31] R. C. Gunning and R. Narasimhan. Immersion of open Riemann surfaces. *Math. Ann.*, 174:103–108, 1967.
[32] D. Heunemann. An approximation theorem and Oka's principle for holomorphic vector bundles which are continuous on the boundary of strictly pseudoconvex domains. *Math. Nachr.*, 127:275–280, 1986.
[33] D. Heunemann. Theorem B for Stein manifolds with strictly pseudoconvex boundary. *Math. Nachr.*, 128:87–101, 1986.
[34] E. Hille. *Ordinary differential equations in the complex domain*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976. Pure and Applied Mathematics.
[35] M. W. Hirsch. *Differential topology*. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.
[36] L. Hörmander and J. Wermer. Uniform approximation on compact sets in $\mathbb{C}^n$. *Math. Scand.*, 23:5–21 (1969), 1968.
[37] C. LeBrun. Fano manifolds, contact structures, and quaternionic geometry. *Internat. J. Math.*, 6(3):419–437, 1995.
[38] C. LeBrun and S. Salamon. Strong rigidity of positive quaternion-Kähler manifolds. *Invent. Math.*, 118(1):109–132, 1994.
[39] E. Lindelöf. Le calcul des résidus et ses applications à la théorie des fonctions. Paris: Gauthier-Villars. vi, 141 S. 8° (1905), 1905.
[40] J. W. Milnor. *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.
[41] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.*, 120:286–294, 1965.
[42] R. Penrose. Twistor algebra. *J. Mathematical Phys.*, 8:345–366, 1967.
[43] R. Penrose. Twistor theory as an approach to fundamental physics. In *Foundations of mathematics and physics one century after Hilbert*, pages 253–285. Springer, Cham, 2018.
[44] J. Plemelj. Riemannsche Funktionenscharen mit gegebener Monodromiegruppe. *Monatsh. Math. Phys.*, 19(1):211–245, 1908.
[45] E. A. Poletsky. Stein neighborhoods of graphs of holomorphic mappings. *J. Reine Angew. Math.*, 684:187–198, 2013.

[46] T. Ritter. A soft Oka principle for proper holomorphic embeddings of open Riemann surfaces into $(\mathbb{C}^*)^2$. *J. Reine Angew. Math.*, 745:59–82, 2018.

[47] W. Rudin. The closed ideals in an algebra of analytic functions. *Canadian J. Math.*, 9:426–434, 1957.

Franc Forstnerič

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI–1000 Ljubljana, Slovenia

Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI–1000 Ljubljana, Slovenia

e-mail: franc.forstneric@fmf.uni-lj.si