Developments in Oka theory since 2017

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Abstract This paper is a survey of main developments in Oka theory since the publication of my book *Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis)*, Second Edition, Springer, Cham, 2017. The paper is self-contained to the extent possible and is accessible also to readers who are new to the field. It will be updated periodically and available at `https://www.springer.com/gp/book/9783319610573`.

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CONTENTS

1. Introduction

Central notions of Oka theory are Oka manifolds and Oka maps[1]. These are complex manifolds and holomorphic maps which enjoy natural flexibility properties for holomorphic maps from Stein manifolds, akin to those which hold for maps from Stein manifolds to Euclidean spaces but taking into account possible topological obstructions. In particular, Oka manifolds are the most natural targets of holomorphic maps from Stein spaces, and many complex analytic problems which can be formulated in terms of such maps have only

[1]MSC 2020 classification introduced the new subfield 32Q56 Oka principle and Oka manifolds.

topological obstructions. Oka theory is primarily an existence theory, while holomorphic rigidity (in particular, Kobayashi hyperbolicity) theory is essentially a holomorphic obstruction theory. Many of the long-standing complex analytic problems lie in the grey area between these two theories where there are no obvious obstructions for the existence of solutions and no methods for solving them either.

The state of the art of Oka theory up to the first half of the year 2017 is summarized in [28, Chapters 5–7] and the older surveys [27, 34]). A brief historical account is included in Section 2 of the present paper for the benefit of readers new to the field. In the remainder we survey the main new developments since 2017. The paper will be updated periodically, so that together with [28] it will maintain presenting the current state of the theory.

The most significant new contributions are due to Yuta Kusakabe [52, 53, 54, 56, 55, 57] and are described in Sections 3–6. His results provide further conceptual unification of the theory and give several new constructions and examples of Oka manifolds and Oka maps. Although the general theory has been fairly well developed by 2010, the subject suffered from lack of examples. This is no longer the case, at least for noncompact manifolds. Indeed, Kusakabe showed that the complement $\mathbb{C}^n\setminus K$ of any compact polynomially convex subset $K \subset \mathbb{C}^n$ for $n > 1$ is an Oka manifold (see Theorem 5.1); the same holds for complements of closed rectifiable curves in $\mathbb{C}^n$ $(n > 2)$ and of several closed noncompact sets. Analogous results hold for any Stein manifold having Varolin's density property in place of $\mathbb{C}^n$; such manifolds share many complex analytic properties with Euclidean spaces (see [5, 31, 41, 59]).

Progress was made in the study of Oka theory for regular algebraic maps from affine algebraic varieties into algebraic manifolds, mainly by Lárusson and Truong [66] and Kusakabe [55, 58]; see Section 6.

Kutzschebauch, Lárusson and Schwarz [60] took steps in the development of an equivariant version of modern Oka theory, and they introduced the notion of a G-Oka manifold where G is to a reductive complex Lie group. On the same topic, Kusakabe (see [55, Appendix]) provided a characterization of G-Oka manifolds by a G-equivariant version of his new characterization of Oka manifolds by condition Ell_1; Definition 3.1 (b).

In another direction, Luca Studer extended modern Oka theory to certain Oka pairs of sheaves [74], thereby generalizing the classical work of Forster and Ramspott [17]. He also developed an abstract homotopy theorem based on Oka theory [73]. See Section 7.

Finally, new approximation theorems of Carleman and Arakelian type for maps to Oka manifolds were proved by Brett Chenoweth [12] and the author [30]; see Section 8.

There were important developments in other fields of analysis and geometry closely intertwined with Oka theory. One of them is Andersén-Lempert theory which concerns Stein manifolds with big groups of holomorphic automorphisms. This subject, which is treated in [28, Chapter 4], has a major impact on Oka theory, both in the proofs of fundamental results and by way of providing examples. A recent survey from 2019 is due to Frank Kutzschebauch [59]. Important results on parametric jet-interpolation by automorphisms were proved by Ugolini [75] and Ramos-Peon and Ugolini [70]. New notions of tame sets in complex manifolds were introduced and studied by Andrist and Ugolini [7] and Winkelmann [78, 77]. A detailed review of this topic is not included here.

Results and methods of Oka theory have lately influenced several other fields of analysis and geometry. Foremost among them are applications of Oka-theoretic methods in the study of minimal surfaces in Euclidean spaces; see the survey by Alarcón and the author [2]. Another one is the study of holomorphic Legendrian curves in complex contact manifolds; see [3, 1, 4, 36]. Recently these results were applied to constructions of superminimal surfaces in self-dual four-dimensional Einstein manifolds [32, 33] by exploring the connection, provided by the theory of twistor spaces, between this class of Riemannian four-manifolds and three-dimensional complex contact manifolds.

These applications might indicate the beginning of a new development of Oka theory towards the study of the Oka principle for *holomorphic partial differential relations*. The analogy with the title of Gromov's monograph [47] is intentional, for I have in mind a range of complex analytic problems where not only values of maps, but also their jets must satisfy certain conditions. This includes holomorphic differential equations along with a variety of open differential relations. Both applications mentioned above, to minimal surfaces and directed holomorphic curves (such as Legendrian curves), are of this type. The study of regular holomorphic maps, including immersions, submersions and locally biholomorphic maps, also fits in this framework. In particular, the structure of the semigroup of locally biholomorphic self-maps of Euclidean spaces $\mathbb{C}^n$ remains a complete mystery.

2. A brief history of Oka theory up to 2017

Oka theory evolved from classical works of Kiyoshi Oka [69] (1939), Hans Grauert [45] (1958), and Mikhail Gromov [47, 48], the main contributors up to 1989 when Gromov's paper [48] appeared. The principal motivation behind the works of Oka and Grauert was to understand the classification of holomorphic principal bundles and their associated bundles (in particular, vector bundles) on Stein spaces, and their main results were that the holomorphic classification of such bundles agrees with the topological one. This early period in Oka-Grauert theory is summarized by the following heuristic formulation of the Oka-Grauert principle found on p. 145 of the monograph [46] by Grauert and Remmert:

Analytic problems on Stein manifolds which can be cohomologically formulated have only topological obstructions.

Problems of this type often reduce to properties of maps to classifying spaces, and hence it became of interest to understand the class of complex manifolds having the property that every continuous map from a Stein manifold or a Stein space to such a manifold can be deformed to a holomorphic map, which some natural additions. It also became clear that for many interesting geometric applications, such as the problem of the minimal embedding dimension for Stein manifolds into Euclidean spaces considered in the early 1970's by Forster [16] and Eliashberg and Gromov [49], classical methods did not suffice. This eventually led to the broader perspective initiated by Gromov in his 1986 monograph [47] and the 1989 paper [48]. He replaced the cohomological interpretation of the problem by a homotopy theoretic one and proposed several sufficient conditions in terms of the existence of dominating holomorphic sprays. His 1989 paper remains a source of new ideas even after three decades. One of the first major applications of these new methods was a solution of the optimal embedding problem for Stein manifolds by Eliashberg and Gromov in 1992 [13], with an improvement due to Schürmann [72]; see the exposition in [28, Secs. 9.3–9.4]. Numerous other applications are described in [28, Chaps. 8–10].

The first steps to understand, explain and develop Gromov's ideas outlined in [48] were taken by Jasna Prezelj and myself in the papers [37, 38, 39, 18] published during 2000-2002. These paper in particular contain detailed proofs and some extensions of the main results from [48]. The study of the Oka principle for sections of branched holomorphic maps was initiated in [19]. At the same time, Finnur Lárusson started developing an abstract homotopy-theoretic approach which culminated in his construction of a model category for Oka theory; see [61, 62, 63], [27, Appendix], and [28, Sect. 7.5].

Subsequent developments focused on finding analytic sufficient conditions for the Oka principle which would also be necessary, or at least close to necessary. An important step was the author's paper [22] from 2006, showing that many natural Oka properties of a complex manifold Y are implied by a simple Runge approximation property for holomorphic maps from convex sets in Euclidean spaces $\mathbb{C}^n$ to Y by entire maps $\mathbb{C}^n \to Y$; see Definition 2.1. In [22] and the subsequent papers [20, 23, 25] the theory was developed to a stage when it became clear that most Oka-type properties considered in the literature, including their parametric versions, are pairwise equivalent. This motivated the following definition of Oka manifold[2] in [23] which was later adopted in [28, Definition 5.4.1].

Definition 2.1. A complex manifold Y is an Oka manifold if every holomorphic map from a neigbourhood of a compact convex set K in a Euclidean space $\mathbb{C}^n$ (for any $n \in \mathbb{N}$) into Y is a uniform limit on K of entire maps $\mathbb{C}^n \to Y$.

This Runge approximation condition was introduced in [22] as the *convex approximation property* (CAP); it suffices to test it on compact convex polyhedra (see Lemma 3.5). An Oka manifold Y enjoys all Oka properties previously considered in the literature: every continuous map $f_0 : X \to Y$ from a reduced Stein space X to Y is homotopic to a holomorphic map $f_1 : X \to Y$; if f_0 is already holomorphic on a neighbourhood of a compact $\mathscr{O}(X)$-convex subset K of X then a homotopy $f_t : X \to Y$ $(t \in [0,1])$ from f_0 to f_1 can be chosen holomorphic and uniformly close to f_0 on K; if in addition f_0 is holomorphic on a closed complex subvariety $A \subset X$ then the homotopy f_t can be chosen fixed on A (and fixed to any given order along A if f_0 is holomorphic on a neighbourhood of A); finally, the corresponding properties hold for continuous families of maps $X \to Y$. See [28, Theorem 5.4.4 and Corollary 5.4.5] for precise statements. Furthermore, these Oka-type properties are pairwise equivalent as shown in [28, Sect. 5.15]. (For recent developments and additional characterizations of the class of Oka manifolds, see Section 3.) Grauert's results [43, 44] say in particular that every complex homogeneous manifold is an Oka manifold. Modern Oka theory may be summarized as follows:

Analytic problems on Stein manifolds which can be formulated in terms of maps to Oka manifolds have only topological obstructions.

Like any heuristic principle, this must be taken with a grain of salt. For instance, to construct proper holomorphic maps one needs stronger geometric assumptions.

The concept of *Oka map* generalizes that of Oka manifold. A holomorphic map $h : Z \to Y$ between complex manifolds is said to be an Oka map if it is a topological fibration (i.e, a Serre fibration or a Hurewicz fibration, these conditions being equivalent for maps between manifolds and refer to the homotopy lifting property) which enjoys the above mentioned Oka properties for lifts of holomorphic maps $f : X \to Y$ from reduced Stein spaces X to holomorphic maps $X \to Z$. For example, every continuous lift $X \to Z$ of

[2] The terms Oka manifold and Oka map were proposed by Lárusson [62] in 2004. The notion of Oka manifold was formally introduced by the author in [23, Definition 1.2] (2009) when the above mentioned equivalences were established. The term became widely used after the publication of the monograph [26] in 2011. Oka manifolds coincide with Ell_∞ spaces introduced by Gromov in [47, p. 73] and [48, Definition 3.1.A].

f is homotopic (through lifts of f) to a holomorphic lift, with similar additions concerning approximation, interpolation, and parametric versions that hold for maps to Oka manifolds. For a precise definition, see Lárusson [62] or [28, Definition 7.4.7]. In particular, a complex manifold Y is an Oka manifold if and only if the map $Y \to \mathrm{point}$ is an Oka map. A new characterization of Oka maps due to Kusakabe [54] is presented in Section 4.

The main lines of questions in Oka theory are the following:

(a) Find examples of Oka manifolds and Oka maps, as well as new sufficient conditions, characterizations, and operations preserving these classes of manifolds and maps.
(b) Develop the Oka theory for algebraic manifolds.
(c) Develop the Oka theory for maps to singular targets, i.e., complex spaces, and for sections of branched holomorphic maps.
(d) Find new applications of Oka theory.

Oka manifolds are the very opposite of Kobayashi hyperbolic manifolds, the former being strongly dominable by Euclidean spaces while the latter not admitting any nonconstant holomorphic maps from $\mathbb{C}$. A big majority of complex manifolds display at least some holomorphic rigidity; this holds in particular for all compact complex manifolds of general type since these are not dominable by Euclidean spaces according to Kobayashi and Ochiai [51], and hence no such manifold is Oka.

For a long time it had seemed that Oka manifolds are few and very special. However, in May 2020 it became clear through the work of Kusakabe [57] that they are much more plentiful than previously thought, especially among noncompact complex manifolds; see Section 5. These results open entire new vistas of possibilities.

3. Elliptic characterization of Oka manifolds

In this section we present a new conceptual unification of Oka theory, due to Yuta Kusakabe [53] (2018), which establishes the equivalence between the Oka property of a complex manifold and Gromov's ellipticity type condition Ell_1; see Definition 3.1 (b) and Theorem 3.3. This provides an affirmative answer to a question of Gromov [47, p. 72]. A more recent result of Kusakabe [54] from 2020 gives the analogous characterization of Oka maps by convex ellipticity; see Theorem 4.5. An important consequence is a localization theorem for Oka manifolds (see Theorem 3.6) which has already led to many new examples. A fascinating application of this new characterization is the fact that the complement of any compact polynomially convex set in $\mathbb{C}^n$ for $n > 1$ is Oka (see Section 5).

3.1. Ellipticity conditions. In [47, 48] Gromov introduced several *ellipticity conditions* for complex manifolds and holomorphic maps which provide geometric sufficient conditions for Oka properties. These conditions are based on the notion of a *dominating spray*, a prime example being the exponential map on a complex Lie group.

Let X and Y be complex manifolds. A *holomorphic spray of maps* $X \to Y$ is a holomorphic map $F : X \times \mathbb{C}^N \to Y$ for some $N \in \mathbb{N}$. The map $f = F(\cdot\,, 0) : X \to Y$ is called the *core* of F, and F is a *spray over* f. The spray F is said to be *dominating* if

$$\frac{\partial}{\partial w}\Big|_{w=0} F(x,w) : \mathbb{C}^N \to T_{f(x)}Y \quad \text{is surjective for every} \quad x \in X.$$

More generally, F is dominating on a subset $U \subset X$ if the above condition holds for every $x \in U$. A more general type of a spray is a holomorphic map $F : E \to Y$ from the total space of a holomorphic vector bundle $\pi : E \to X$; its core is the restriction of F to the zero section of E (which we identify with X), and domination is defined in the same way.

Definition 3.1. Let Y be a complex manifold.

(a) (Gromov [48, 0.5, p. 8.5.5]; see also [28, Definition 5.6.13].) Y is *elliptic* if it admits a dominating holomorphic spray $F : E \to Y$, where $\pi : E \to Y$ is a holomorphic vector bundle and $F(0_y) = y$ for all $y \in Y$. The manifold Y is *special elliptic* if it admits a dominating holomorphic spray as above from a trivial bundle $E = Y \times \mathbb{C}^N$.
(b) (Gromov [47, p. 72].) Y enjoys condition Ell_1 if every holomorphic map $X \to Y$ from a Stein manifold is the core of a dominating holomorphic spray $X \times \mathbb{C}^N \to Y$.
(c) Y enjoys condition C-Ell_1 (convex Ell_1) if for any compact convex set $K \subset \mathbb{C}^n$, open set $U \subset \mathbb{C}^n$ containing K and map $f \in \mathscr{O}(U, Y)$ there are an open set V with $K \subset V \subset U$ and a dominating holomorphic spray $F : V \times \mathbb{C}^N \to Y$ over $f|_V$.

Every elliptic Stein manifold Y is also special elliptic. Indeed, by (a small extension of) Cartan's Theorem A, any holomorphic vector bundle $\pi : E \to Y$ over a Stein manifold admits finitely many (say N) holomorphic sections which span the fibre $E_y = \pi^{-1}(y)$ over each point $y \in Y$. This gives a surjective holomorphic vector bundle map $\phi : Y \times \mathbb{C}^N \to E$, and precomposing a dominating holomorphic spray $F : E \to Y$ by ϕ gives a dominating spray $Y \times \mathbb{C}^N \to Y$. This fails on non-Stein manifolds. In fact, every compact special elliptic manifold is complex homogeneous (see [30, Proposition 6.2]).

Condition Ell_1 can be interpreted as saying that the space $\mathscr{O}(X, Y)$ is dominable by some $\mathbb{C}^N$ at every point $f \in \mathscr{O}(X, Y)$. Obviously Ell_1 implies C-Ell_1, the latter being a restricted version of Ell_1 applying to compact convex sets in Euclidean spaces, and we ask that a dominating spray exists over a smaller neighbourhood of the set. (This comes very handy in proofs.)

One of Gromov's main results in [48] is that every elliptic manifold is Oka (see also [28, Corollary 5.6.14]). In fact, ellipticity implies CAP (see Definition 2.1); this is a special case of [28, Theorem 6.6.1]). Conversely, it is easily seen that every Stein Oka manifold is elliptic [28, Proposition 5.6.15]. Kusakabe constructed in [56] the first known examples of non-Stein Oka manifolds which fail to be elliptic or even just (weakly) subelliptic. His more recent results described in Section 5 provide a plethora of such examples. However, the following problem seems to remain open.

Problem 3.2. Is there a compact Oka manifold which fails to be (sub-) elliptic?

3.2. Characterization of Oka manifolds by condition Ell_1.

Every Oka manifold satisfies condition Ell_1 (see [28, Corollary 8.8.7]). However, it came as a surprise that the converse holds as well. The following result to this effect is due to Kusakabe [53, Theorem 1.3].

Theorem 3.3. *A complex manifold which satisfies condition* C-Ell_1 *is an Oka manifold. In particular, a complex manifold is Oka if and only if it satisfies condition* Ell_1 *(or* C-Ell_1*).*

It follows that conditions Ell_1, Ell_2 and Ell_∞ introduced by Gromov in [48] are pairwise equivalent and characterize the class of Oka manifolds. See also [53, Conjecture 4.6 and Corollary 4.7] for a more precise description of Gromov's conjectures.

Theorem 3.3 enables the construction of many new examples of Oka manifolds; see in particular Theorem 3.6 and the examples in Section 5. The main point is that it is easier to construct sprays whose domain is a Stein manifold, or even just a convex domain in a Euclidean space, rather than a general complex manifold.

We begin with some preparations. Given a compact set K in a complex manifold X and a complex manifold Y, we denote by $\mathscr{O}(K,Y)$ the space of germs on K of holomorphic maps from open neighbourhoods $U \subset X$ of K to Y. Thus, $\mathscr{O}(K,Y)$ is the colimit (also called the direct limit) of the system $\mathscr{O}(U,Y)$ over open sets $U \subset X$ containing K, with the natural restrictions maps $r_{U,V} : \mathscr{O}(V,Y) \to \mathscr{O}(U,Y)$ given for any pair $U \subset V$ by $r_{U,V}(f) = f|_U$. The space $\mathscr{O}(K,Y)$ carries the colimit topology defined as follows. Fix a distance function dist on Y inducing the manifold topology. A basic open neighbourhood of an element of $\mathscr{O}(K,Y)$, represented by a map $f \in \mathscr{O}(U,Y)$, is a set of the form

$$\mathscr{V}(f,U',K',\epsilon) = \big\{ g \in \mathscr{O}(U',Y) : \sup_{z\in K'} \mathrm{dist}(f(z),g(z)) < \epsilon \big\} \tag{3.1}$$

where K' is a compact set containing K in its interior, U' is an open set with $K' \subset U' \Subset U$, and $\epsilon > 0$. Equivalently, let $(U_k)_{k\ge 1}$ be a decreasing basis of open neighbourhoods of K such that U_{k+1} is relatively compact in U_k for all $k \ge 1$. The colimit topology on $\mathscr{O}(K,Y)$ is the finest topology that makes all maps $j_k : \mathscr{O}(U_k,Y) \to \mathscr{O}(K,Y)$ continuous. By saying that a map $K \to Y$ is holomorphic, we mean that it belongs to $\mathscr{O}(K,Y)$.

A *(convex) polyhedron* in $\mathbb{R}^N$ is a compact set which is the intersection of finitely many closed affine half-spaces. Recall the following definition (cf. [28, Definition 5.15.3]).

Definition 3.4. A pair $K \subset L$ of compact convex sets in $\mathbb{R}^N$ is a *special polyhedral pair* if L is a polyhedron and $K = \{z \in L : \lambda(z) \le 0\}$ for some affine linear function $\lambda \colon \mathbb{R}^N \to \mathbb{R}$.

The following observation is due to Kusakabe [52] (see [28, Lemma 5.15.4]).

Lemma 3.5. *Suppose that Y is a complex manifold such that for each special polyhedral pair $K \subset L$ in $\mathbb{C}^n$, $n \in \mathbb{N}$, every holomorphic map $K \to Y$ can be approximated uniformly on K by holomorphic maps $L \to Y$. Then Y enjoys* CAP *and hence is an Oka manifold.*

Proof of Theorem 3.3. Let $K \subset L$ be special polyhedral pair in $\mathbb{C}^n$. Denote by $A(K,Y)$ the set of all $f \in \mathscr{O}(K,Y)$ which can be approximated uniformly on K by maps $g \in \mathscr{O}(L,Y)$. Then $A(K,Y)$ is a nonempty closed subset of $\mathscr{O}(K,Y)$. Since K is convex and Y is connected, the space $\mathscr{O}(K,Y)$ is clearly connected. Hence, to prove the theorem it suffices to show that the set $A(K,Y)$ is also open in $\mathscr{O}(K,Y)$.

Fix $f \in A(K,Y)$ and represent it by a map $f \in \mathscr{O}(U,Y)$ from an open set $U \subset \mathbb{C}^n$ containing K. Condition C-Ell$_1$ gives a convex open set V with $K \subset V \subset U$ and a dominating holomorphic spray $F : V \times \mathbb{C}^N \to Y$ with $F(\cdot\,,0) = f|_V$. By factoring out the kernel of $\partial F(z,w)/\partial w|_{w=0} : \mathbb{C}^N \to T_{f(z)}Y$ (which is a trivial holomorphic subbundle of $V \times \mathbb{C}^N$ with trivial quotient) we may assume that $N = \dim Y$ and the above derivative is an isomorphism for every $z \in V$. Hence, up to shrinking V around K if necessary there is an open ball $0 \in W \subset \mathbb{C}^N$ such that the map $\widetilde{F} = (\mathrm{Id}, F) : V \times \mathbb{C}^N \to V \times Y$ given by

$$\widetilde{F}(z,w) = (z, F(z,w)), \quad z \in V,\ w \in \mathbb{C}^N \tag{3.2}$$

maps $V \times W$ biholomorphically onto its image in $V \times Y$. Since $f \in A(K,Y)$, there are a neighbourhood $\Omega \subset \mathbb{C}^n$ of L and a map $g \in \mathscr{O}(\Omega,Y)$ whose graph $\{(z,g(z)) : z \in K\}$ belongs to $\widetilde{F}(V \times W)$. Up to shrinking Ω around L, [28, Lemma 5.10.4] provides a local dominating holomorphic spray $G : \Omega \times W \to Y$ over $G(\cdot\,,0) = g$. Replacing $G(z,w)$ by $G(z,tw)$ for a small $t > 0$ we may assume that the map $\widetilde{G}(z,w) = (z, G(z,w))$

satisfies $\widetilde G(K \times W) \Subset \widetilde F(V \times W)$. Hence, there is an open convex set $U_1 \subset \mathbb{C}^n$ with $K \subset U_1 \Subset V \cap \Omega$ such that $\widetilde G(U_1 \times W) \Subset \widetilde F(V \times W)$. Since the map $\widetilde F$ (3.2) is biholomorphic on $V \times W$, there is a unique holomorphic map $H : U_1 \times W \to W$ such that

$$F(z, H(z,w)) = G(z,w) \quad \text{for all } (z,w) \in U_1 \times W. \tag{3.3}$$

Pick a slightly larger polyhedron L' containing L in its interior and a small $\epsilon > 0$ and set

$$A = \{z \in L' : \lambda(z) \le 2\epsilon\} \subset U_1, \qquad B = \{z \in L' : \lambda(z) \ge \epsilon\} \subset \Omega.$$

The polyhedra A and B form a Cartan pair (see [28, Definition 5.7.1]) with $A \cup B = L'$ and $C := A \cap B = \{z \in L' : \epsilon \le \lambda(z) \le 2\epsilon\}$. Let

$$K' = \{z \in L' : \lambda(z) \le \epsilon/2\}.$$

Pick a convex open set $U_0 \subset \mathbb{C}^n$ such that $K' \subset U_0 \subset U_1$ and $\overline U_0 \cap C = \varnothing$. Let $\phi : U_0 \to \mathbb{C}^N$ be any holomorphic map. Since K' and C are disjoint compact convex sets in $\mathbb{C}^n$ and $W \subset \mathbb{C}^N$ is a ball, $(K' \cup C) \times \overline W$ is a polynomially convex subset of $\mathbb{C}^n \times \mathbb{C}^N$. The Oka-Weil theorem furnishes a holomorphic map $\tilde\phi : A \times W \to \mathbb{C}^N$ which approximates $\phi(z)$ on $(z,w) \in K' \times W$ and approximates H on $C \times W$. In view of (3.3), the local holomorphic spray $\Phi : A \times W \to Y$ defined by

$$\Phi(z,w) = F(z, \tilde\phi(z,w)), \qquad z \in A,\ w \in W$$

then approximates the spray G on $C \times W$, while on $K' \times W$ it is close to the map

$$(z,w) \mapsto f_\phi(z) := F(z, \phi(z)), \qquad z \in K'. \tag{3.4}$$

Provided the approximations are close enough and noting that the spray G is dominating over C, we can apply [28, Proposition 5.9.2] on the Cartan pair (A,B) to glue Φ and G into a holomorphic spray $\Theta : L' \times W' \to Y$ for a smaller parameter ball $0 \in W' \subset W$. By the construction, its core map $\tilde f := \Theta(\cdot\,, 0) : L' \to Y$ then approximates the map f_ϕ (3.4) on K'. Since the map $\widetilde F$ (3.2) is biholomorphic on $V \times W$, every holomorphic map $K' \to Y$ sufficiently uniformly close to f on K' is of the form f_ϕ (3.4), and hence it belongs to the set $A(K,Y)$ of approximable maps. This shows that $A(K,Y)$ is open as claimed. □

3.3. A localization theorem for Oka manifolds. A domain U in a complex manifold Y is said to be *Zariski open* if its complement $A = Y \setminus U$ is a closed complex subvariety of Y. An important application of Theorem 3.3 is the following localization criterion.

Theorem 3.6. (Kusakabe, [53, Theorem 1.4].) *If Y is a complex manifold which is the union of Zariski open Oka domains, then Y is an Oka manifold.*

Previously, a localization theorem has only been known for algebraically subelliptic manifolds (see [28, Proposition 6.4.2]).

The proof of Theorem 3.6 uses the following corollary to [28, Theorem 7.2.1] or its more general version, [28, Theorem 8.6.1].

Proposition 3.7 (Proposition 3.1 in [53])**.** *Let Y be complex manifold and $\Omega \subset Y$ be a Zariski open Oka domain. Given a Stein manifold X and a holomorphic map $f : X \to Y$, there is a holomorphic spray $F : X \times \mathbb{C}^N \to Y$ over f which is dominating on $f^{-1}(\Omega)$.*

Proof of Theorem 3.6. By Theorem 3.3 it suffices to show that Y enjoys condition C-Ell$_1$. Let K be a compact convex set in $\mathbb{C}^n$ and $f \in \mathscr{O}(U,Y)$ be a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^n$ of K. Let $\Omega_i \subset Y$ be a collection of Zariski open domains with $\bigcup_i \Omega_i = Y$. Since K is compact, $f(K)$ is contained in the union of finitely many

Ω_i's; call them $\Omega_1, \ldots, \Omega_m$. Proposition 3.7 furnishes a spray $F_1 : U \times \mathbb{C}^{N_1} \to Y$ with core f which is dominating on $f^{-1}(\Omega_1)$. Applying Proposition 3.7 to F_1 furnishes a spray $F_2 : U \times \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \to Y$ with core F_1 which is dominating on $F_1^{-1}(\Omega_2)$. Considering F_2 as a spray over f, it is dominating on $f^{-1}(\Omega_1 \cup \Omega_2)$. After m steps of this kind we obtain a spray $F : U \times \mathbb{C}^N \to Y$ over f which is dominating on a neighbourhood of K. □

3.4. Sprays generating the tangent space. Theorem 3.3 implies several other criteria for a manifold Y to be Oka in terms of sprays over maps from Stein manifolds into Y. The following result combines Corollaries 4.1 and 4.2 in Kusakabe's paper [53].

Corollary 3.8. *For a complex manifold Y the following are equivalent.*

(a) *Y is Oka.*
(b) *For every Stein manifold X, holomorphic map $f : X \to Y$, and holomorphic section V of f^*TY (such V may be thought of as a holomorphic vector field on Y along the map f) there is a holomorphic spray $F : X \times \mathbb{C} \to Y$ over f such that*
$$\partial_t|_{t=0} F(x,t) = V(x) \in T_{f(x)}Y \quad \textit{for all } x \in X.$$
(c) *For every Stein manifold X, holomorphic map $f : X \to Y$, and point $x \in X$ there are finitely many holomorphic sprays $F_j : X \times \mathbb{C}^{N_j} \to Y$ $(j = 1, \ldots, k)$ such that $\sum_{j=1}^k \partial_t|_{t=0} F_j(x,t)(\mathbb{C}^{N_j}) = T_{f(x)}Y$.*
(d) *Condition* (c) *holds for every convex domain $X \subset \mathbb{C}^n$, $n \in \mathbb{N}$.*

Condition (c), when applied to sprays over the identity map on the manifold Y (which need not be Stein), coincides with *weak subellipticity* of Y (see [28, Definition 5.6.13 (f)]), and this condition implies that Y is Oka (see [28, Corollary 5.6.14]).

Proof. (a)⇒(b): If Y is Oka then by [28, Corollary 8.8.7] there is a dominating holomorphic spray $G : X \times \mathbb{C}^N \to Y$ over $f = G(\cdot\,, 0)$ for some $N \in \mathbb{N}$. This means that
$$\Theta := \partial_t|_{t=0} G(\cdot\,, t) : X \times \mathbb{C}^N \to f^*TY$$
is a surjective holomorphic map of holomorphic vector bundles. Hence there is a holomorphic section W of the trivial bundle $X \times \mathbb{C}^N \to X$ such that $\Theta(W) = V$. (This follows from Cartan's theory and is a special case of [28, Corollary 2.6.5].) The spray $F : X \times \mathbb{C} \to Y$ defined by $F(x,t) = \Theta(tW(x))$ then clearly satisfies condition (b).

The implications (b)⇒(c)⇒(d) are obvious.

(d)⇒(a): Let $K \subset \mathbb{C}^n$ be a compact convex set, $X \subset \mathbb{C}^n$ be an open convex set containing K, and $f \in \mathscr{O}(X,Y)$. Fix $x \in K$. By condition (d) there is a spray $F_1 : X \times \mathbb{C} \to Y$ over f such that the vector $V_1 := \partial_t|_{t=0} F_1(x,t) \in T_{f(x)}Y$ is nonzero. Applying condition (d) to F_1 gives a spray $F_2 : X \times \mathbb{C} \times \mathbb{C} \to Y$ over F_1 such that the vector $V_2 := \partial_t|_{t=0} F_2(x,0,t) \in T_{f(x)}Y$ is linearly independent from V_1. Continuing this way we obtain after $d = \dim Y$ steps a spray $F : X \times \mathbb{C}^d \to Y$ over f which dominates at x, and hence on a Zariski neighbourhood of x. A repetition of this process over other points of K gives a holomorphic spray over f which is dominating on an open neighbourhood $\Omega \subset X$ of K. Thus, Y enjoys condition C-Ell$_1$ and hence is Oka by Theorem 3.3. □

3.5. An implication of $\mathbb{C}$-connectedness. In his first paper [52] on Oka theory, Kusakabe characterized the class of Oka manifolds by the following $\mathbb{C}$-connectedness property of the space of holomorphic maps from Stein manifolds.

Theorem 3.9. ([52, Theorem 3.2].) *For a complex manifold Y the following are equivalent.*

(a) *Y is an Oka manifold.*
(b) *For every Stein manifold X and homotopic holomorphic maps $f_0, f_1 : X \to Y$ there is a holomorphic map $F : X \times \mathbb{C} \to Y$ such that $F(\cdot\,, t) = f_t$ for $t = 0, 1$.*
(c) *Condition* (b) *holds for any bounded convex domain $X \subset \mathbb{C}^n$, $n \in \mathbb{C}^n$.*

This result and its proof are also presented in [28, Theorem 5.15.2]. The proof of the main implication (c)⇒(a) uses the same geometric scheme as the proof of Theorem 3.3.

The following is an outstanding open problem in Oka theory; see [28, Problem 7.6.4].

Problem 3.10 (The union problem for Oka manifolds)**.** Let Y be a complex manifold and $Y' \subset Y$ be a closed complex submanifold. If Y' and $U := Y \setminus Y'$ are Oka, is Y Oka?

In an apparent attempt to approach this problem, Kusakabe combined Theorem 3.9 and [28, Theorem 7.2.1] to show the following.

Theorem 3.11 (Theorem 4.4 in [53])**.** *For a complex manifold Y with a Zariski open Oka domain $U \subset Y$, the following are equivalent.*

(a) *Y is an Oka manifold.*
(b) *For every Stein manifold X and map $f \in \mathscr{O}(X, Y)$ which is homotopic to a continuous map $X \to U$ there exists $F \in \mathscr{O}(X \times \mathbb{C}, Y)$ with $F(\cdot\,, 0) = f$ and $F(\cdot\,, 1) \in \mathscr{O}(X, U)$.*
(c) *For any bounded convex domain $X \subset \mathbb{C}^n$ $(n \in \mathbb{C}^n)$ and $f \in \mathscr{O}(X, Y)$ there is a holomorphic map $F : X \times \mathbb{C} \to Y$ such that $F(\cdot\,, 0) = f$ and $F(\cdot\,, 1) \in \mathscr{O}(X, U)$.*

It is not clear how to find a spray F over f satisfying condition (c) if the image $f(X)$ intersects both U and the subvariety $Y' = Y \setminus U$. If $f(X) \subset Y'$ and X is convex, then such F exists by [35, proof of Theorem 2] (see also [28, proof of Theorem 7.1.8]).

4. Elliptic characterization of Oka maps

In this section we present Kusakabe's recent characterization from [54] of the Oka property of holomorphic submersions by a new condition called *convex ellipticity* which he introduced; see Theorem 4.5. Before we get to that, we briefly survey the extant theory.

A holomorphic map $h : Y \to Z$ between reduced complex spaces is said to enjoy the *parameteric Oka property with approximation and interpolation* (POPAI) if for every Stein space X and holomorphic map $f : X \to Z$, each continuous lift $F_0 : X \to Y$ is homotopic (through lifts of f) to a holomorphic lift $F = F_1 : X \to Y$ as in the following diagram,

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & Z \end{tikzcd}$$

with natural additions concerning the approximation on compact $\mathscr{O}(X)$-convex subsets of X and interpolation on closed complex subvarieties of X on which F_0 happens to be holomorphic; the analogous conditions must hold for families of maps $f_p : X \to Z$ depending continuously on a parameter p in a compact Hausdorff space. See [28, Definitions 7.4.1 and 7.4.7] for the details and note that these conditions correspond to those listed on p. 4 in the special case when Z is a singleton.

Definition 4.1. A holomorphic map $h : Y \to Z$ of reduced complex spaces is an *Oka map* if it enjoys POPAI and is a Serre fibration (see [28, Definition 7.4.7] and [62]).

For a holomorphic submersion $h : Y \to Z$, POPAI is a local condition in the sense that it holds if every point $z_0 \in Z$ has an open neighbourhood $U \subset Z$ such that the restricted submersion $h : h^{-1}(U) \to U$ enjoys POPAI (see [25, Theorem 4.7] and also [28, Definition 6.6.5 and Theorem 6.6.6]). Furthermore, for such h the basic Oka property (BOPAI, referring to lifts of a single map $f : X \to Z$) implies POPAI; see [24].

One of the simplest examples of Oka maps is a holomorphic fibre bundle with Oka fibre. Furthermore, every stratified subelliptic holomorphic submersion satisfies the Oka property, so it is an Oka map provided it is a Serre fibration (see [28, Corollary 7.8.4]).

As mentioned in [28, p. 335], a holomorphic map is an Oka map if and only if it is the intermediate fibration in Lárusson's model category [62]. In particular, we have the following result (see [27, Corollary 2.51] and [28, Theorem 5.6.5]).

Theorem 4.2. *If $h : Y \to Z$ is an Oka map of complex manifolds, then Y is an Oka manifold if and only if Z is an Oka manifold. This holds in particular if $h : Y \to Z$ is a holomorphic fibre bundle with Oka fibre.*

It follows from definitions that if $h : Y \to Z$ is a holomorphic submersion enjoying POPAI then every fibre $h^{-1}(z)$, $z \in Z$ is an Oka manifold. The converse is not true. For example, if $g : Z \to \mathbb{C}$ is a continuous function on a domain $Z \subset \mathbb{C}$ then every fibre of the coordinate projection $h : Y = \{(z, w) \in Z \times \mathbb{C} : w \neq g(z)\} \to Z$ is the Oka manifold $\mathbb{C}^*$, but h is an Oka map if and only if g is a holomorphic function (see [28, Corollary 7.4.10]).

The following result pointed out by Kusakabe shows that a manifold is Oka if it admits sufficiently many projections having the Oka property.

Proposition 4.3 (Lemma 5.1 in [57]). *If Y is a complex manifold such that for any point $y \in Y$ there exist holomorphic submersions $h_j : Y \to Z_j$ $(j = 1, \ldots, k)$ enjoying* POPAI *such that $T_yY = \sum_{j=1}^k T_y h_j^{-1}(h_j(y))$, then Y is an Oka manifold.*

Proof. Let X be a Stein manifold and $f : X \to Y$ be a holomorphic map. Fix a point $x_0 \in X$ and let $y = f(x_0) \in Y$. If $h_j : Y \to Z_j$ are as above, the Oka property of h_j furnishes for every $j = 1, \ldots, k$ a fibre dominating holomorphic spray $F_j : X \times \mathbb{C}^{N_j} \to Y$ over f with $h_j \circ F = h_j \circ f$ (see [28, Corollary 8.8.7]). In particular, we have that $\partial_t|_{t=0} F_j(x_0, t)(\mathbb{C}^{N_j}) = T_y h_j^{-1}(h_j(y))$. It follows that the collection of sprays $F_1, \ldots, F_k$ dominates T_yY, and hence Y is Oka by Corollary 3.8. □

We now describe the main result of Kusakabe's paper [54]. As pointed out in the introduction to his paper, the two main types of maps which are known to satisfy POPAI are (stratified) holomorphic fibre bundles with Oka fibres, and (stratified) subelliptic holomorphic submersions. None of these two families is a subfamily of the other one: there are noncompact Oka manifolds which fail to be subelliptic (see Section 5), and there are subelliptic holomorphic submersions which are not locally trivial at any base point, e.g. a complete family of complex tori (see [64, Theorem 16]). Furthermore, Kusakabe gave an example of a holomorphic submersion which enjoys POPAI but does not belong to any of the above classes (see [54, Proposition 5.10]). It is therefore of interest to find a characterization of POPAI which unifies the theory, in the same way as CAP and Ell_1 characterize Oka manifolds (cf. Theorem 3.3). Kusakabe introduced the following notion.

Definition 4.4 (Definition 1.2 in [54])**.** A holomorphic submersion $h : Y \to Z$ of complex spaces is *convexly elliptic* if there is an open cover $\{U_i\}_{i\in I}$ of Z such that for any $n \in \mathbb{N}$, compact convex set $K \subset \mathbb{C}^n$ and holomorphic map $f \in \mathscr{O}(K, Y)$ with $f(K) \subset h^{-1}(U_i)$ for some $i \in I$ there exists a holomorphic map $F \in \mathscr{O}(K \times \mathbb{C}^N, Y)$ such that

(i) $F(\cdot\,, 0) = f$ and $h \circ F(z,t) = h \circ f(z)$ for all $z \in K$ and $t \in \mathbb{C}^N$, and
(ii) $F(z, \cdot\,) : \mathbb{C}^N \to h^{-1}(h(f(z)))$ is a submersion at $0 \in \mathbb{C}^N$ for all $z \in K$.

A map F as in the above definition is a *fibre dominating spray over f*. Note that convex ellipticity is a fibred version of condition C-Ell$_1$ (cf. Definition 3.1 (c)).

Theorem 4.5. (Kusakabe [54, Theorem 1.3]) *A holomorphic submersion of complex spaces enjoys* POPAI *if and only if it is convexly elliptic. In particular, a holomorphic submersion is an Oka map if and only if it is a convexly elliptic Serre fibration.*

This is a generalization of Theorem 3.3 characterizing Oka manifolds by C-Ell$_1$, and the proof (see [54, Sections 3-4]) is similar. First, the problem is reduced to the main special case which pertains to sections of a holomorphic submersion $h : Y \to Z$. In this case and assuming that the base Z is Stein, an axiomatic characterization of POPAI is provided by the *homotopy approximation property*, HAP, first introduced in [24, Proposition 2.1]. (See also [28, Definition 6.6.5 and Theorem 6.6.6].) This condition, which is local on the base, is an axiomatization of the homotopy Runge theorem [28, Theorem 6.6.2][3]. The gist of Kusakabe's proof of Theorem 4.5 is to show that HAP is implied by convex ellipticity, in a similar way as CAP is implied by condition C-Ell$_1$. We refer to [54] for the details.

5. Oka complements

A long-standing problem in Oka theory asked whether the complement of every compact convex set K in $\mathbb{C}^n$ for $n > 1$ and Oka manifold (see [28, Problem 7.6.1]). In May 2020, Kusakabe answered this problem affirmatively and in much bigger generality. For the notion of density property, see Varolin [76] or [28, Definition 4.10.1]. (On a Stein manifold, density property implies the Andersén-Lempert theorem, see [28, Theorem 4.10.5]. Every such manifold has dimension > 1 and is an Oka manifold.)

Theorem 5.1. (Kusakabe, [57, Theorem 1.2 and Corollary 1.3].) *For any compact holomorphically convex set K in a Stein manifold Y with the density property the complement $Y \setminus K$ is an Oka manifold. In particular, if K is a compact polynomially convex set in $\mathbb{C}^n$ for $n > 1$ then $\mathbb{C}^n \setminus K$ is an Oka manifold.*

This is the first result in the literature which provides a large class of Oka domains in $\mathbb{C}^n$ for any $n > 1$, as well as in all Stein manifolds with the density property. As noted in [57, Corollary 1.4], it follows from [29, Theorem 1.1] that $Y \setminus K$ (like any Oka manifold) is the image of a strongly dominating holomorphic map $\mathbb{C}^n \to Y \setminus K$ with $n = \dim Y$.

Kusakabe's proof of Theorem 5.1 uses the characterization of Oka manifolds by C-Ell$_1$; see Theorem 3.3. Take a compact convex set $L \subset \mathbb{C}^N$ and a holomorphic map $f : L \to Y$ such that $f(z) \in Y \setminus K$ for all $z \in L$. He constructed a holomorphically varying family

[3] As pointed out in [54, Remark 3.6], HAP is not stated correctly in [28, Definition 6.6.5]: the same condition must hold for any local holomorphic spray of sections with parameter in a ball $\mathbb{B} \subset \mathbb{C}^n$, for this is needed when gluing sprays. Equivalently, the stated condition must apply to the trivial extensions $Z \times \mathbb{B} \to X \times \mathbb{B}$. This holds for any subelliptic submersions $h : Z \to X$ in view of [28, Theorem 6.6.2].

$f(z) \in \Omega_z \subset Y \setminus K$ $(z \in L)$ of nonautonomous basins with uniform bounds (i.e., basins of random sequences of automorphisms of Y which are uniformly attracting at $f(z) \in Y \setminus K$); these are elliptic manifolds as shown by Fornæss and Wold [15], hence Oka. It is then possible to find a dominating holomorphic spray $F : L \times \mathbb{C}^n \to Y$ over $f = F(\cdot\,, 0)$ such that $F(z, \zeta) \in \Omega_z$ for all $z \in L$ and $\zeta \in \mathbb{C}^n$. Thus, $Y \setminus K$ satisfies condition C-Ell$_1$.

Soon thereafter, E. F. Wold and the author pointed out in [40] that one can choose F as above such that $F(z, \cdot\,) : \mathbb{C}^n \to Y \setminus K$ is a Fatou-Bieberbach map for every $z \in L$.

Theorem 5.2 (Theorems 1.1 and 3.1 in [40]). *Let K be a compact holomorphically convex set in a Stein manifold Y with the density property, L be a compact convex set in $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and $f : U \to \mathbb{C}^n$ be a holomorphic map on an open neighbourhood $U \subset \mathbb{C}^N$ of L such that $f(z) \in Y \setminus K$ for all $z \in L$. Then there are an open neighbourhood $V \subset U$ of L and a holomorphic map $F : V \times \mathbb{C}^n \to Y$ with $n = \dim Y$ such that for every $z \in V$ we have that $F(z, 0) = f(z)$ and the map $F(z, \cdot\,) : \mathbb{C}^n \to Y \setminus K$ is injective.*

If $Y = \mathbb{C}^n$ with $n > 1$ then the same conclusion holds if L is polynomially convex.

It was proven by Andrist, Shcherbina and Wold [6] that in a Stein manifold of dimension at least three every compact holomorphically convex set K with an infinite derived set K' (the set of limit points of K) has a nonelliptic complement. Together with Theorem 5.1 this implies the following corollary.

Corollary 5.3. *For every compact polynomially convex set $K \subset \mathbb{C}^n$ $(n \geq 3)$ with infinitely many limit points, the complement $\mathbb{C}^n \setminus K$ is Oka but not weakly subelliptic. The analogous result holds in any Stein manifold with the density property of dimension ≥ 3.*

The first known examples of Oka manifolds which fail to be subelliptic were given by Kusakabe in [56]. One his main results there is the following.

Theorem 5.4 (Theorem 1.2 in [56]). *If $S \subset \mathbb{C}^n$ $(n > 1)$ is a closed tame countable set with discrete derived set S', then the complement $\mathbb{C}^n \setminus S$ is Oka.*

Previously it was known that the complement of a closed tame subvariety of $\mathbb{C}^n$ of codimension at least 2 (in particular, of a closed tame discrete subset) is elliptic and hence Oka; see [28, Proposition 5.6.17].

In the same paper, Kusakabe constructed examples of compact countable sets in $\mathbb{C}^n$ with nondiscrete derived sets having nonelliptic Oka complements. An example is the following. Let $\mathbb{N}^{-1} = \{1/j : j \in \mathbb{N}\}$ and $\overline{\mathbb{N}}^{-1} = \mathbb{N}^{-1} \cup \{0\} \subset \mathbb{C}$. Then for every $n \geq 3$ the domain

$$X = \mathbb{C}^n \setminus \left((\overline{\mathbb{N}}^{-1})^2 \times \{0\}^{n-2} \right)$$

is an Oka manifold which is not weakly subelliptic (see [56, Corollary 1.4]).

The corresponding problem in complex dimension 2 remains open.

Problem 5.5. Is there a compact subset K of $\mathbb{C}^2$ whose complement $\mathbb{C}^2 \setminus K$ is Oka but is not elliptic or (weakly) subelliptic?

Recall that a closed unbounded set is said to be polynomially convex if it is exhausted by an increasing sequence of compact polynomially convex sets. Theorem 5.1 is a special case of the following result of Kusakabe [57, Theorem 1.6]. See also [57, Theorem 4.2] for a more general result in this direction.

Theorem 5.6. *If S is a closed polynomially convex subset of $\mathbb{C}^n$ $(n \geq 2)$ such that*

$$S \subset \big\{(z,w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : |w| \leq c(1+|z|)\big\} \tag{5.1}$$

for some $c > 0$, then $\mathbb{C}^n \setminus S$ is an Oka manifold.

Sketch of proof of Theorem 5.6. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{n-2}$ denote the coordinate projection $\pi(z,w) = z$. To prove that $\mathbb{C}^n \setminus S$ is Oka, it suffices to show that the restricted projection $\pi : \mathbb{C}^n \setminus S \to \mathbb{C}^{n-2}$ has the Oka property (POPAI; see Section 4). Indeed, if this projection is an Oka map (i.e., it is also a Serre fibration), it follows from Theorem 4.2 that $\mathbb{C}^n \setminus S$ is Oka (since the base $\mathbb{C}^{n-2}$ is Oka). In the general case, note that condition (5.1) holds for all linear projections $\mathbb{C}^n \to \mathbb{C}^{n-2}$ sufficiently close to π. This gives finitely many linear projections $\mathbb{C}^n \setminus S \to \mathbb{C}^{n-2}$ enjoying POPAI such that their kernel subspaces span $\mathbb{C}^n$, and hence the conclusion follows from Proposition 4.3.

In order to show that $\mathbb{C}^n \setminus S$ satisfies POPAI, it suffices to verify convex ellipticity; see Theorem 4.5. This means that for any compact convex set $L \subset \mathbb{C}^N$ and holomorphic map $f = (f', f'') : L \to \mathbb{C}^n \setminus S$ (with $f' : L \to \mathbb{C}^{n-2}$ and $f'' : L \to \mathbb{C}^2$) there is a fibre-dominating spray $F : L \times \mathbb{C}^m \to \mathbb{C}^n \setminus S$ over f such that $\pi \circ F = f'$ (see Definition 4.4). By taking the pullback of $\pi : \mathbb{C}^n \to \mathbb{C}^{n-2}$ by the base map $f' : L \to \mathbb{C}^{n-2}$, all relevant properties are preserved and the problem gets reduced to the one where f is a holomorphic map from a neighbourhood of $L \subset \mathbb{C}^N$ to $\mathbb{C}^n$ such that $f(z) \in \mathbb{C}^n \setminus S_z$ $(z \in L)$ where S_z is the fibre of S over z. (Here, S is the new set obtained from the initial one by the pullback.) A spray F with the desired properties can be obtained with $m = n$ as a holomorphic family of Fatou-Bieberbach maps into $\mathbb{C}^n \setminus S_z$ depending on $z \in L$ by using the version of Theorem 5.2 for variable fibres S_z; see [40, Remark 2.2]. □

In [57], Kusakabe gave several interesting applications of these results; let us mention a few without stating them explicitly.

Gromov showed in [48, 0.5.B] that the complement $\mathbb{C}^n \setminus A$ of any closed algebraic subvariety of codimension ≥ 2 is Oka; see also [28, Proposition 5.6.10 and Sect. 6.4]. Since any such subvariety A satisfies condition (5.1) in a suitable linear coordinate system on $\mathbb{C}^n$, it has a basis of closed neighbourhoods in $\mathbb{C}^n$ with Oka complements [57, Corollary 5.3]. Similar results hold for tame discrete sets in $\mathbb{C}^n$; see [57, Corollaries 5.5 and 5.7].

Furthermore, Kusakabe's results imply that the complement of any closed rectifiable curve in $\mathbb{C}^n$ for $n \geq 3$ is Oka. For rectifiable arcs in $\mathbb{C}^n$ this holds for all $n > 1$ since they are polynomially convex (see [57, Corollary 1.8]). The proof for closed curves (when $n \geq 3$) combines Theorem 5.6 (cutting the curve with a suitably chosen complex hyperplane and applying a change of coordinates on the complement to get a set satisfying the hypothesis of Theorem 5.6) with the localization theorem given by Theorem 3.6.

These results and examples give rise to the following question, reminiscent of the classical Levi problem concerning a geometric characterization of domains of holomorphy.

Problem 5.7. Let K be a compact set with reasonably nice boundary in $\mathbb{C}^n$ for some $n > 1$. Is there a characterization of $\mathbb{C}^n \setminus K$ being an Oka manifold in terms of the geometric properties of bK?

The only obvious thing at the moment is that if $\mathbb{C}^n \setminus K$ is Oka then K cannot have a strongly pseudoconcave boundary point, because this would yield a nonconstant bounded plurisubharmonic function on $\mathbb{C}^n \setminus K$.

6. Oka theory for algebraic manifolds

The algebraic Oka theory concerns (regular) algebraic maps from affine algebraic manifolds (the algebraic analogues of Stein manifolds) to algebraic manifolds. The following are algebraic analogues of basic Oka conditions (see [28, Sect. 5.15]).

Definition 6.1. Let Y be an algebraic manifold, and let X denote an arbitrary affine algebraic manifold.

(a) Y enjoys the *(basic) algebraic Oka property* (aBOP) if every continuous map $X \to Y$ is homotopic to an algebraic map.
(b) Y enjoys the *algebraic approximation property* (aAP) if every continuous map $X \to Y$ which is holomorphic on a neighbourhood of a compact holomorphically convex subset K of X can be approximated uniformly on K by algebraic maps $X \to Y$.
(c) Y enjoys the *algebraic interpolation property* (aIP) if every algebraic map from an algebraic subvariety of X to Y has an algebraic extension $X \to Y$ if it has a continuous extension. [4]

A few examples of algebraic manifolds which are Oka in the holomorphic sense but aBOP fails are mentioned in [28, Examples 6.15.7, 6.15.8]. In addition, one of the cornerstones of the classical Oka-Grauert theory, that the holomorphic classification of vector bundles over Stein manifolds agrees with their topological classification, fails in the algebraic category already for affine algebraic curves of genus $g > 0$. Hence, it is not very surprising that there are almost no algebraic manifolds satisfying these conditions, except perhaps in the class of affine algebraic manifolds. The following result is due to Lárusson and Truong [66, Theorem 2].

Theorem 6.2. *If Y is an algebraic manifold which contains a rational curve $\mathbb{CP}^1$ or is compact, then Y does not satisfy any of the properties* aBOP, aAP, aIP.

Lárusson and Truong also pointed out the following (cf. [66, Theorem 1]).

Theorem 6.3. *The following conditions are equivalent for an algebraic manifold Y.*

(a) *Algebraic subellipticity,* aSEll. *(See* [28, Definition 5.6.13 (e)].*)*
(b) *Algebraic* Ell_1. *(Replace holomorphic maps in Definition 3.1 (b) by algebraic maps.)*
(c) *Algebraic homotopy approximation property,* aHAP*: If $f : X \to Y$ is an algebraic map from an affine algebraic variety X, K is a compact holomorphically convex set in X and $f_t : U \to Y$ $(t \in [0,1)$ is a homotopy of holomorphic maps on an open neighbourhood of K with $f_0 = f|_U$, then $\{f_t\}_{t\in[0,1]}$ can be approximated uniformly on $K \times [0,1]$ be algebraic maps $F : X \times \mathbb{C} \to Y$ with $F(\cdot\,,0) = f : X \to Y$.*

The implication (a)⇒(c) was proved by the author in [21, Theorem 3.1] (the proof is also given in [28, Theorem 6.15.1]). The implication (c)⇒(b) is immediate, and (b)⇒(a) follows from Gromov's localization theorem for algebraically subelliptic manifolds (see [48, 3.5.B, 3.5.C] and [28, Proposition 6.4.2]).

We mention the following special case in order to introduce condition aCAP.

[4] Note that properties aAP and aIP are similar to the algebraic versions of the corresponding properties BOPA and BOPI, respectively, in the holomorphic category; however, in aAP and aIP one does not ask for the existence of homotopies connecting the initial map to the final map.

Corollary 6.4 (Corollary 6.15.2 in [28])**.** *Every algebraically subelliptic manifold Y satisfies the following* algebraic convex approximation property*:*

aCAP: *Every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ can be uniformly approximated by regular algebraic maps $\mathbb{C}^n \to Y$.*

Remark 1. Lárusson and Truong proposed in [66] to call a manifold satisfying the equivalent conditions in Theorem 6.3 an *algebraically Oka manifold*, aOka. A slight reservation to this choice of term could be that algebraically subelliptic manifolds do not abide by the philosophy that Oka properties refer to the existence of solutions of analytic (or, in this case, algebraic) problems in the absence of topological obstructions. Indeed, condition (c) in Theorem 6.3 only provides a relative Oka principle for algebraic maps under holomorphic deformations, while Theorem 6.2 shows that most such manifolds do not have the absolute Oka properties such as aBOP. $\square$

Note that every algebraically subelliptic manifold is an Oka manifold, and this class of manifolds appears in several interesting applications, some of which are already indicated in [28]. A further list of properties of this class, and relations with other properties such as (local) algebraic flexibility in the sense of Arzhantsev et al. [9], can be found in [66, Remark 2]. Lárusson and Truong also gave the following new examples in this class; previously it was known that such manifolds are Oka (see [28, Theorem 5.6.12]).

Theorem 6.5 (Theorem 3 in [66])**.** *Every smooth nondegenerate toric variety is locally flexible, and hence algebraically subelliptic.*

Finally, we mention that Kusakabe proved in [58, Theorem 1.2] the jet transversality theorem for regular algebraic maps from affine algebraic manifolds to a certain subclass of algebraically subelliptic manifolds. A local version of the transversality theorem for algebraic maps to *all* algebraically subelliptic manifolds was proved by the author in 2006 (see [21, Theorem 4.3] and [28, Theorem 8.8.6]); here, local means that one can achieve the transversality condition on any compact subset of the source manifold. This local version suffices for many natural applications, such as those described in [28, Sect. 9.14].

On the theme of Oka properties of blow-ups of algebraic manifolds, we mention the following recent result of Kusakabe.

Theorem 6.6 (Corollary 4.3 in [55])**.** *Let Y be an algebraic manifold and $A \subset Y$ be a closed algebraic submanifold. If Y enjoys* aCAP *(in particular, if Y is algebraically subelliptic), then the blow-up $\mathrm{Bl}_A Y$ also enjoys* aCAP *and hence is an Oka manifold.*

Kusakabe proved this result by reducing it to [65, Theorem 1] by Lárusson and Truong. Note that in Theorem 6.6 it is not claimed that $\mathrm{Bl}_A Y$ is algebraically subelliptic. Previous results on Oka property of blow-ups are due to Lárusson and Truong [65] (for algebraic manifolds covered by Zariski open sets equivalent to complements of codimension ≥ 2 algebraic subvarieties in affine spaces; see also [28, Theorem 6.4.8] where this is called Class $\mathcal{A}$) and Kaliman, Kutzschebauch, and Truong [50].

Another recent result concerning blow-ups of certain complex linear algebraic groups along tame discrete subsets is due to Winkelmann [77, Theorem 2.9].

Theorem 6.7. *If D is a closed tame discrete subset in a character-free complex linear algebraic group G then the complement $G \setminus D$ and the blow-up $\mathrm{Bl}_D G$ are Oka manifolds.*

Remark 2. The special case of Theorem 6.7 with $G = \mathbb{C}^n$ $(n > 1)$ is stated as [77, Lemma 9.1]. This was known before: see [28, Proposition 6.4.12] for the blow-up, while the Oka property of the complement is a special case of [28, Proposition 5.6.17]. Both these results already appeared in [26]. Furthermore, [77, Theorem 8.2] is the main result of [18] and it appears as [28, Corollary 5.6.14], while [77, Proposition 8.3] is also seen by noting that such manifold X is obviously weakly elliptic and hence Oka by [28, Corollary 5.6.14]. □

7. Oka pairs of sheaves and a homotopy theorem for Oka theory

Luca Studer made several contributions to Oka theory in his PhD dissertation. One of them, presented in the paper [74], provides a splitting lemma which enables one to glue local sections of coherent analytic sheaves. Splitting lemmas are of key importance in the proof of all Oka principles. Those in the work by Gromov [48] and in my joint works with Prezelj [39, 38], and their generalizations presented in [28], pertain to sections of the sheaf of holomorphic sections of a holomorphic submersion and to its subsheaf of sections vanishing (perhaps to a higher order) on a subvariety. Studer proved a splitting lemma for sections of an arbitrary coherent analytic sheaf. As applications he obtained shortcuts in the proofs of Forster and Ramspott's Oka principle for admissible pairs of sheaves (see [17]) and of the interpolation property of sections of elliptic submersions, an extension of Gromov's results obtained by Forstnerič and Prezelj [38]. The main technical part of his proof is a lifting theorem (see [74, Theorem 1]) which reduces the splitting problem to that for sections of a free sheaf which is already well understood.

The second main result of Studer is a homotopy theorem based on Oka theory, presented in [73]. He pointed out that all proofs of Oka principles can be divided into an analytic first part and a purely topological second part which can be formulated very generally, thus providing a reduction of the proofs to the analytic key difficulties. This general topological statement is [73, Theorem 1]: its assumptions state which key properties one has to show in the first part of the proof of an Oka principle, and its conclusion is an Oka principle. This extends Gromov's homomorphism theorem from [47] so that it applies in complex analytic settings and carries out ideas sketched in [48] and developed in [39] and [28, Chapter 6].

Studer gave an even more general result, [73, Theorem 2], with no particular reference to complex analysis. Let X be a paracompact Hausdorff space that has an exhaustion by finite dimensional compact subsets, and let $\Phi \hookrightarrow \Psi$ be a local weak homotopy equivalence of sheaves of topological spaces on X. He shows that under suitable conditions on Φ and Ψ the inclusion $\Phi(X) \hookrightarrow \Psi(X)$ of spaces of sections is a weak homotopy equivalence. The relevant conditions on the sheaves reflect what is happening when approximating and gluing sprays of sections in [48, 39]. His proof is essentially an abstraction of the proof of the Oka principle for (sub-) elliptic submersions given in [39, 28]. He then shows how the known examples (rather, applications of) the Oka principle are special case of this more general theorem. We refer to the original paper for precise statements.

Remark 3. In the second paragraph in [73, A.2] Studer comments on the problem of showing that, for a holomorphic submersion $h : Z \to X$ onto a reduced complex space, the inclusion $\Phi \hookrightarrow \Psi$ of the sheaf Φ of holomorphic sections into the sheaf Ψ of continuous sections is a local weak homotopy equivalence. Explicitly, the following holds.

Lemma 7.1. *Let $h : Z \to X$ be a holomorphic submersion onto a reduced complex space. For every point $x_0 \in X$, open neighbourhood $U \subset X$ of x_0, closed ball B in some $\mathbb{R}^n$, and continuous map $f : B \to \Psi(U)$ (the space of continuous sections of $Z \to X$ over U) there* *are a neighbourhood V of x_0 with $V \subset U$ and a homotopy $f_t : B \to \Psi(V)$ $(t \in [0,1])$ such that $f_0 = f|_V$, $f_t|_{bB}$ is independent of t (hence it equals $f|_{bB}$ for all $t \in [0,1]$), and f_1 has values in $\Phi(V)$ (the space of holomorphic sections of $Z \to X$ over V).*

Studer says the following in [73, A.2]: *Gromov seems to have taken this result for granted in* [48]. *In the more detailed work* [39]*, local weak homotopy equivalences are not introduced. Instead, the analogue of our Proposition 2.9 is stated in the special case of holomorphic submersions, namely Proposition 4.7. The validity of Proposition 4.7 in* [39] *was carefully checked in the Ph. D. dissertation of J. Prezelj.*

This might leave the impression that [39, Proposition 4.7] is not proved and one must read Prezelj's dissertation (in Slovenian). This calls for a clarification. The cited proposition gives a homotopy connecting a global continuous section to a complex of holomorphic sections of a holomorphic submersion; it is proved both in [39] and [28, Proposition 6.10.1]. On the other hand, the parametric case of the same result, which is the first step in the proof of the weak homotopy equivalence principle for the inclusion of spaces of sections, is not explicitly stated and proved in [39], but it is present in Prezelj's dissertation. There is a comment in [39] that the details of the parametric case were provided only in the nontrivial analytic parts of the proof, and those were given in the previous paper [37] and subsequently in the monograph [28]. I have taken Lemma 7.1 as an obvious application of partitions of unity in the parameter. In view of Studer's remark I include a proof here; it implies the parametric case of [39, Proposition 4.7] (cf. [73, proof of Proposition 2.9]).

Proof of Lemma 7.1. Precomposing f by smooth map $B \to B$ which retracts a spherical collar $A \subset B$ around the sphere bB onto bB, we may assume that $f(A) \subset \Phi(U)$.

When $Z = X \times \mathbb{C}^m \to X$ is a trivial submersion, we take a continuous function $\chi : B \to [0,1]$ which equals 1 near bB and equals zero on $B \setminus A$ and define

$$(7.1)\quad \tilde f(p) = \chi(p)f(p) + (1-\chi(p))f(p)(x_0) \ \ (p \in B), \qquad f_t = t\tilde f + (1-t)f \ \ (t \in [0,1]).$$

Here, $f(p)(x)$ is the value of the map $f(p)$ at the point $x \in U$. It is trivial to verify that $\tilde f(B) \subset \Phi(U)$, $\tilde f = f$ near bB, $\tilde f(p)(x_0) = f(p)(x_0)$ for all $p \in B$, the homotopy f_t is fixed near bB and is fixed at the point $x_0 \in X$ for all $p \in B$, and $f_1 = \tilde f$.

Consider now the general case. Up to shrinking U around x_0, there are finitely many pairs of compact sets $P_j \subset P'_j \subset B$ $(j = 1, \ldots, k)$ such each P_j is contained in the interior of P'_j, $\bigcup_{j=1}^k P_j = B$, and there is a submersion chart $Z_j \subset Z$ which is fibrewise biholomorphic to $U \times \mathbb{B}$, where $\mathbb{B} \subset \mathbb{C}^m$ is the ball and h is given in the coordinates $(x,z) \in U \times \mathbb{B}$ by $(x,z) \mapsto x$, such that the union of ranges of sections $f(p)$ for $p \in P'_j$ is contained in Z_j. For $p \in P_j$ we may treat sections $f(p)$ as maps $U \to \mathbb{B}$. Choose a function $\chi : B \to [0,1]$ as above and define $\tilde f$ as in (7.1) for all $p \in P'_1$. Choose a continuous function $\psi : B \to [0,1]$ supported in P'_1 such that $\psi = 1$ on a neighbourhood of P_1 and set

$$f_t(p) = t\psi(p)\tilde f(p) + (1 - t\psi(p))f(p), \quad p \in P'_1,\ t \in [0,1].$$

The homotopy is fixed near $bB \cup bP'_1$ (where $f_t = f$), and for all $p \in P_1$ and $t \in [0,1]$ we have that $f_t(p)(x_0) = f(p)(x_0)$ and $f_1(p) = \tilde f(p) \in \Phi(U)$. Hence, f_1 satisfies the conditions of the lemma on the parameter set $B_1 = \overline{B \setminus P_1}$. Performing the same procedure on the second set P_2 (and shrinking $U \ni x_0$ if necessary) gives a homotopy connecting f_1 to f_2 which is fixed for parameter values p on a neighbourhood of $P_1 \cup bB_1$ and such that $f_2(p) \in \Phi(U)$ for all $p \in P_1 \cup P_2$. After k steps of this kind we are done. □

Note that the argument in the above proof is a special case of the *method of successive patching* (see [28, p. 78]) which is used in the proofs of Lemma 6.5.3 and Theorem 6.6.2 in [28, pp. 282, 286], and possibly elsewhere in the book. What makes the proof of Lemma 7.1 particularly simple is that one may shrink the neighborhood $U \subset X$ of the given point $x_0 \in X$ as much as desired.

8. Carleman and Arakelian theorems for manifold-valued maps

The basic Oka property with approximation (BOPA) is one of the classical Oka properties of a complex manifold Y which characterizes the class of Oka manifolds (see Sect. 2). It refers to the possibility of approximating any holomorphic map $f \in \mathscr{O}(K,Y)$, where K is a compact $\mathscr{O}(X)$-convex set in a Stein manifold (or Stein space) X, uniformly on K by entire maps $F \in \mathscr{O}(X,Y)$ provided that f extends continuously from K to X.

Recently, B. Chenoweth [12] proved several Carleman-type approximation theorems in the same context. In general, Carleman approximation (after the work of T. Carleman [11] from 1927) refers to approximation of holomorphic functions and maps in fine Whitney topologies on certain unbounded sets in Stein manifolds.

Let X be a complex manifold. For any compact set C in X we set

$$h(C) := \overline{\widehat{C}_{\mathscr{O}(X)} \setminus C}.$$

A closed not necessarily compact set $E \subset X$ is said to be $\mathscr{O}(X)$-convex if it is exhausted by an increasing sequence of compact $\mathscr{O}(X)$-convex sets.

Definition 8.1. A closed set $E \subset X$ in a complex manifold X has the *bounded exhaustion hulls property* if for any compact $\mathscr{O}(X)$-convex set $K \subset X$ there exists a compact set $K' \subset X$ such that for any compact set $L \subset E$ we have that

$$h(K \cup L) \subset K'.$$

The following is a special case of Chenoweth's main result in [12] (2019). We refer to the original paper for additional results and corollaries.

Theorem 8.2 (Chenoweth [12]). *Let X be a Stein manifold and Y be an Oka manifold. If $K \subset X$ is a compact $\mathcal{O}(X)$-convex set and E is a closed totally real submanifold of X of class $\mathscr{C}^r$ $(r \in \mathbb{N})$ with the bounded exhaustion hulls property (see Definition 8.1) such that $K \cup E$ is $\mathcal{O}(X)$-convex, then for any $k \in \{0,1,\ldots,r\}$ the set $K \cup E$ admits $\mathscr{C}^k$-Carleman approximation of maps $f \in \mathscr{C}^k(X,Y)$ which are holomorphic on small neighborhoods of K. If in addition K is the closure of a strongly pseudoconvex domain then the same holds if f is $\overline{\partial}$-flat to order k on $K \cup E$.*

This is proved by inductively applying the Mergelyan theorem for admissible sets in Stein manifolds (see [28, Theorem 3.8.1] or [14, Theorem 34]) together with the basic Oka property (BOPA) for maps from Stein manifolds to Oka manifolds; see [28, Theorem 5.4.4]. These two methods are intertwined at every step of the induction procedure.

The special case of Theorem 8.2 for functions (i.e., for $Y = \mathbb{C}$) is due to Manne, Wold, and Øvrelid [68], and the necessity of the bounded exhaustion hulls condition was shown by Magnusson and Wold [67].

Given a closed unbounded set E in a Stein manifold X, one can ask when is it possible to uniformly approximate any continuous function on E which is holomorphic on the interior of E by functions holomorphic on X. This type of approximation is named after N. U. Arakelian [8] who proved that for a closed subset E of a planar domain $X \subset \mathbb{C}$, uniform approximation on E is possible if and only if E is holomorphically convex in X (which in this case means that E has no holes in X) and its complement $\widehat{X} \setminus E$ in the one-point compactification $\widehat{X} = X \cup \{\infty\}$ is locally connected at ∞. For a closed set E in a Riemann surface X, the latter property is equivalent to E having the bounded exhaustion hulls property. A set E with these two properties is called an *Arakelian set*. (See also [14, Theorem 10] and the related discussion.) The following seems to be the first known extension of Arakelian's theorem to manifold-valued maps.

Theorem 8.3 (Forstnerič [30])**.** *If E is an Arakelian set in a domain $X \subset \mathbb{C}$ and Y is a compact complex homogeneous manifold, then every continuous map $X \to Y$ which is holomorphic in $\mathring{E}$ can be approximated uniformly on E by holomorphic maps $X \to Y$.*

The analogous result holds if X is an open Riemann surface which admits bounded holomorphic solution operators for the $\overline{\partial}$-equation; see [30, Theorem 5.3]. On plane domains one can use the classical Cauchy-Green operator. Similar problems appear already in the Arakelian theorem for functions which does not hold for every open Riemann surface as shown by examples in [42] and [10, p. 120]. Note also that Carleman approximation in the fine topology is impossible in general if the interior of E is not relatively compact, or at least its connected components are not relatively compact. Nothing seems to be known concerning Arakelian approximation on closed sets whose interior is not relatively compact in higher dimensional Stein manifolds.

The scheme of proof of Theorem 8.3 in [30] follows the proof of the classical Arakelian's theorem by Rosay and Rudin [71]. The main new analytic ingredient is a technique for gluing sprays with uniform bounds on certain noncompact Cartan pairs. The proof does not apply to general Oka target manifolds, not even to noncompact homogeneous manifolds. This is natural since approximation problems of Arakelian type for maps to noncompact manifolds may crucially depend on the choice of metrics on both spaces.

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