The general position number of the Cartesian product of two trees

Jing Tian\textsuperscript{a}, Kexiang Xu\textsuperscript{a}, Sandi Klavžar\textsuperscript{b,c,d}

\textsuperscript{a} College of Science, Nanjing University of Aeronautics & Astronautics, Nanjing, Jiangsu 210016, PR China
\textsuperscript{b} Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
\textsuperscript{c} Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
\textsuperscript{d} Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
jingtian526@126.com (J. Tian)
kexxu1221@126.com (K. Xu)
sandi.klavzar@fmf.uni-lj.si (S. Klavžar)

Abstract

The general position number of a connected graph is the cardinality of a largest set of vertices such that no three pairwise-distinct vertices from the set lie on a common shortest path. In this paper it is proved that the general position number is additive on the Cartesian product of two trees.

Keywords: general position set; general position number; Cartesian product; trees

AMS Math. Subj. Class. (2020): 05C05, 05C12, 05C35

1 Introduction

Let $d_G(x, y)$ denote, as usual, the number of edges on a shortest $x, y$-path in $G$. A set $S$ of vertices of a connected graph $G$ is a general position set if $d_G(x, y) \neq d_G(x, z) + d_G(z, y)$ holds for every $\{x, y, z\} \in \binom{S}{3}$. The general position number $gp(G)$ of $G$ is the cardinality of a largest general position set in $G$. Such a set is briefly called a gp-set of $G$. 
Before the general position number was introduced in [9], an equivalent concept was proposed in [14]. Much earlier, however, the general position problem has been studied by Körner [8] in the special case of hypercubes. Following [9], the graph theory general position problem has been investigated in [1, 3, 5, 6, 10, 11, 13].

The Cartesian product \( G \square H \) of vertex-disjoint graphs \( G \) and \( H \) is the graph with vertex set \( V(G) \times V(H) \), vertices \( (g, h) \) and \( (g', h') \) being adjacent if either \( g = g' \) and \( hh' \in E(H) \), or \( h = h' \) and \( gg' \in E(G) \). In this paper we are interested in \( \text{gp}(G \square H) \), a problem earlier studied in [3, 6, 10, 13]. More precisely, we are interested in Cartesian products of two (finite) trees. (For some of the other investigations of the Cartesian product of trees see [2, 12, 15].) An important reason for this interest is the fact that the general position number of products of paths is far from being trivial. First, denoting with \( P_{\infty} \) the two-way infinite path, one of the main results from [10] asserts that \( \text{gp}(P_{\infty} \square P_{\infty}) = 4 \). Denoting further with \( G^n \) the \( n \)-fold Cartesian product of \( G \), it was demonstrated in the same paper that \( 10 \leq \text{gp}(P_{\infty}^n) \leq 16 \). The lower bound 10 was improved to 14 in [6]. Very recently, these results were superseded in [7] by proving that if \( n \) is an arbitrary positive integer, then \( \text{gp}(P_{\infty}^n) = 2^{2n-1} \). Denoting with \( n(G) \) the order of a graph \( G \), in this paper we prove:

**Theorem 1.** If \( T \) and \( T^* \) are trees with \( \min\{n(T), n(T^*)\} \geq 3 \), then

\[
\text{gp}(T \square T^*) = \text{gp}(T) + \text{gp}(T^*).
\]

Theorem 1 widely extends the above mentioned result \( \text{gp}(P_{\infty} \square P_{\infty}) = 4 \). Further, the equality \( \text{gp}(P_{\infty}^n) = 2^{2n-1} \) shows that Theorem 1 has no obvious (inductive) extension to Cartesian products of more than two trees. Hence, to determine the general position number of such products remains a challenging problem.

In the next section we give further definitions, recall known results needed, and prove several auxiliary new results. Then, in Section 3 we prove Theorem 1.

## 2 Preliminaries

Let \( T \) be a tree. The set of leaves of \( T \) will be denoted by \( L(T) \), and let \( \ell(T) = |L(T)| \).

If \( u \) and \( v \) are vertices of \( T \) with \( \deg(u) \geq 2 \) and \( \deg(v) = 1 \), then the unique \( u, v \)-path is a branching path of \( T \). If \( u \) is not a leaf of \( T \), then there are exactly \( \ell(T) \) branching paths starting from \( u \); we say that the \( u \) is the root of these branching paths and that the degree 1 vertex of a branching path \( P \) is the leaf of \( P \).

**Lemma 1.** ([9]) If \( T \) is a tree, then \( \text{gp}(T) = \ell(T) \).

We next describe which vertices of a tree lie in some gp-set of the tree.
Lemma 2. A non-leaf vertex \( u \) in a tree \( T \) belongs to a gp-set of \( T \) if and only if \( T - u \) has exactly two components and at least one of them is a path.

Proof. First, let \( R \) be a gp-set of \( T \) containing the non-leaf vertex \( u \). Suppose that \( T - u \) has at least three components, say \( T_1, T_2 \) and \( T_3 \). Since \( R \) is a gp-set containing \( u \), \( R \) intersects with at most one of \( T_1, T_2 \) and \( T_3 \). Assume without loss of generality that \( R \cap V(T_2) = \emptyset \) and \( R \cap V(T_3) = \emptyset \). Choose vertices \( v \) and \( w \) in \( T \) such that \( v \in V(T_2) \) and \( w \in V(T_3) \). Then \((R - \{u\}) \cup \{v, w\}\) is a larger gp-set than \( R \) in \( T \), a contradiction. Hence \( T - u \) has exactly two components, say \( T_1 \) and \( T_2 \). Now suppose that neither \( T_1 \) nor \( T_2 \) is a path. Then as above, we have \( R \cap V(T_1) = \emptyset \) or \( R \cap V(T_2) = \emptyset \). By symmetry, we assume that \( R \cap V(T_2) = \emptyset \). Since \( T_2 \) is not a path, there are at least two leaves \( x_1 \) and \( x_2 \) in \( T_2 \). Then the set \((R - \{u\}) \cup \{x_1, x_2\}\) is a larger gp-set than \( R \), again, in \( T \). Therefore, at least one of \( T_1 \) and \( T_2 \) is a path.

Conversely, we observe that \( u \) is a non-leaf vertex on a pendant path in \( T \). Then \( u \) belongs to a gp-set in \( T \). \( \square \)

In \( G \square H \), if \( h \in V(H) \), then the subgraph of \( G \square H \) induced by the vertices \((g, h)\), \( g \in V(G) \), is a \( G \)-layer, denoted with \( G^h \). Analogously \( H \)-layers \( ^hH \) are defined. \( G \)-layers and \( H \)-layers are isomorphic to \( G \) and to \( H \), respectively. The distance function in Cartesian products is additive, that is, if \((g_1, h_1), (g_2, h_2) \in V(G \square H)\), then

\[
d_{G \square H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2).
\]

If \( u, v \in V(G) \), then the interval \( I_G(u, v) \) between \( u \) and \( v \) in \( G \) is the set of all vertices lying on shortest \( u, v \)-paths, that is,

\[
I_G(u, v) = \{w : d_G(u, v) = d_G(u, w) + d_G(w, u)\}.
\]

In what follows, the notations \( d_G(u, v) \) and \( I_G(u, v) \) may be simplified to \( d(u, v) \) and \( I(u, v) \) if \( G \) will be clear from the context. Equality \( \square \) implies that intervals in Cartesian products have the following nice structure, cf. \[3\] Proposition 12.4.

Lemma 3. If \( G \) and \( H \) are connected graphs and \((g_1, h_1), (g_2, h_2) \in V(G \square H)\), then

\[
I_{G \square H}((g_1, h_1), (g_2, h_2)) = I_G(g_1, g_2) \times I_H(h_1, h_2).
\]

Equality \( \square \) also easily implies the following fact (also proved in \[3\]).

Lemma 4. Let \( G \) and \( H \) be connected graphs and \( R \) a general position set of \( G \square H \). If \( u = (g, h) \in R \), then \( V(3H) \cap R = \{u\} \) or \( V(G^h) \cap R = \{u\} \).

For finite paths the already mentioned result \( \text{gp}(P_{\infty} \square P_{\infty}) = 4 \) reduces to:
Lemma 5. ([10]) If \( n_1, n_2 \geq 2 \), then
\[
\text{gp}(P_{n_1} \square P_{n_2}) = \begin{cases} 
4; & \min\{n_1, n_2\} \geq 3, \\
3; & \text{otherwise.}
\end{cases}
\]

To conclude the preliminaries we construct special maximal (with respect to inclusion) general position sets in products of trees.

Lemma 6. Let \( T \) and \( T^* \) be two trees with \( \min\{n(T), n(T^*)\} \geq 3 \), \( v_i \in V(T) \setminus L(T) \), and \( v_j^* \in V(T^*) \setminus L(T^*) \). Then \( (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) is a maximal general position set of \( T \square T^* \).

Proof. Set \( R = (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) and let \( V_0 = \{u, v, w\} \subseteq R \). We first consider the case when \( V_0 \subseteq L(T) \times \{v_j^*\} \) or \( V_0 \subseteq \{v_i\} \times L(T^*) \). By symmetry, assume that \( V_0 \subseteq L(T) \times \{v_j^*\} \). Then each vertex of \( V_0 \) is corresponding to a leaf of \( L(T) \) in the layer \( T^*_j \cong T \). Therefore \( u, v, w \) do not lie on a common geodesic in \( T \square T^* \).

In the following, without loss of generality, we can assume that \( u, w \in L(T) \times \{v_j^*\} \) with \( u = (v_k, v_j^*) \), \( w = (v_s, v_s^*) \) and \( v = (v_i, v_i^*) \in \{v_i\} \times L(T^*) \). By Equality (1), we have \( d(u, v) = d_T(v_k, v_i) + d_T(v_j^*, v_i^*) \) and \( d(u, w) = d_T(v_k, v_s) + d_T(v_j^*, v_s^*) \). Note that \( v_k, v_s \) are two distinct vertices in \( L(T) \) of \( T \) and \( v_i \in V(T) \setminus L(T) \). Then \( d_T(v_k, v_i) < d_T(v_k, v_s) + d_T(v_s, v_i) \) whenever \( v_i \) lies on the \( v_k, v_s \)-geodesic or outside \( v_k, v_s \)-geodesic of \( T \). This implies that \( d(u, v) < d(u, w) + d(w, v) \) in \( T \square T^* \). Therefore \( w \) does not lie on the \( u, v \)-geodesic in \( T \square T^* \). Analogously, neither \( u \) lies on the \( v, w \)-geodesic nor \( v \) lies on the \( u, w \)-geodesic of \( T \square T^* \). Thus \( u, v, w \) do not lie on a common geodesic in \( T \square T^* \), which implies that \( R \) is a general position set in \( T \square T^* \).

Next we prove the maximality of \( (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) as a general position set in \( T \square T^* \). Otherwise, there is a general position set \( R' \) in \( T \square T^* \) of order greater than \( \ell(T) + \ell(T^*) \) such that \( R \subset R' \). Then there exists a vertex \( z \in R' \setminus R \), say \( z = (v_p, v_q^*) \). If \( p = i \), then there exist two vertices \( (v_i, v_i^*), (v_i, v_i^*) \in R \) such that \( z \in I_{T \square T^*}((v_i, v_i^*), (v_i, v_i^*)) \) (since \( v_i \cong T \)). This is a contradiction showing that \( p \neq i \). Similarly, we have \( q \neq j \). Now we consider the positions of \( v_p \) in \( T \) and \( v_q^* \) in \( T^* \). Suppose first that \( v_p \in L(T) \), \( v_q^* \in L(T^*) \). Then there are two vertices \( (v_p, v_q^*), (v_i, v_i^*) \in R \) such that \( z \in I_{T \square T^*}((v_p, v_q^*), (v_i, v_i^*)) \) contracting that \( R \cup \{z\} \) is a general position set of \( T \square T^* \). If \( v_p \in L(T) \) and \( v_q^* \notin L(T^*) \), then we select a vertex \( v_q^* \in L(T^*) \) such that \( v_q^* \) is closer to the leaf of the corresponding branching path than \( v_q^* \) in \( T^* \). Then \( z \in I_{T \square T^*}((v_p, v_q^*), (v_i, v_i^*)) \), a contradiction. Similarly, \( v_p \notin L(T) \) and \( v_q^* \notin L(T^*) \). Now we select two vertices \( v_{p'} \in L(T) \) and \( v_{q'}^* \in L(T^*) \) such that \( v_{p'} \) is closer to the leaf of the branching path than \( v_p \) in \( T \) and \( v_{q'}^* \) is closer to the leaf of the branching path than \( v_q^* \) in \( T^* \). But then \( (v_p, v_{q'}^*) \in I_{T \square T^*}((v_{p'}, v_{q'}^*), (v_i, v_i^*)) \), a final contradiction. \( \square \)
3 Proof of Theorem 1

If $T$ and $T^*$ are both paths, then Theorem 1 holds by Lemma 5. In the following we may thus without loss of generality assume that $T^*$ is not a path. Lemma 6 implies that $\text{gp}(T \sqcup T^*) \geq \text{gp}(T) + \text{gp}(T^*)$, hence it remains to prove that $\text{gp}(T \sqcup T^*) \leq \text{gp}(T) + \text{gp}(T^*)$. Set $n = n(T)$, $n^* = n(T^*)$, $V(T) = \{v_1, \ldots, v_n\}$, and $V(T^*) = \{v^*_1, \ldots, v^*_n\}$.

Assume on the contrary that there exists a general position set $R$ of $T$ such that $|R| > \text{gp}(T) + \text{gp}(T^*)$. Since the restriction of $R$ to a $T$-layer of $T \sqcup T^*$ is a general position set of the layer (which is in turn isomorphic to $T$), the restriction contains at most $\text{gp}(T) = \ell(T)$ elements. Similarly, the restriction of $R$ to a $T^*$-layer contains at most $\text{gp}(T^*) = \ell(T^*)$ elements. We now distinguish the following cases.

Case 1. There exists a $T$-layer $T^v_j$ with $|V(T^v_j) \cap R| = \text{gp}(T)$, or a $T^*$-layer $T^v_i$ with $|V(T^v_i) \cap R| = \text{gp}(T^*)$.

By the commutativity of the Cartesian product, we may without loss of generality assume that there is a layer $T^v_i$ with $|R \cap V(T^v_i)| = \text{gp}(T^*)$. Let $R = R_1 \cup R_2$, where $R_1 = R \cap V(T^v_i)$ and $R_2 = R \setminus R_1$, that is, $R_2 = \bigcup_{t \in [n] \setminus \{i\}} (V(T^v_i) \cap R)$. Let further $S^*$ be the projection of $R \cap V(T^v_i)$ on $T^*$, that is, $S^* = \{v^*_j : (v_i, v^*_j) \in R_1\}$. Since $|R_1| = \text{gp}(T^*)$, our assumption implies $|R_2| \geq \text{gp}(T) + 1$. Then, as $\text{gp}(T) = \ell(T)$, there exist two different vertices $w = (v_p, v^*_p)$ and $w' = (v'_p, v^*_p)$ from $R_2$ such that $v_p$ and $v'_p$ lie on a same branching path $P$ of $T$. (Note that it is possible that $v_p = v'_p$.) We may assume that $d_T(v'_p, x) \leq d_T(v_p, x)$, where $x$ is the leaf of $P$. We proceed by distinguishing two subcases based on the position of $v^*_q$ and $v^*_q$ in $T^*$.

Case 1.1. There exists a branching path $P^*$ of $T^*$ that contains both $v^*_q$ and $v^*_q$.

Recall that $T^*$ is not a path. Lemma 2 implies that a vertex of a tree belongs to a gp-set if and only if it lies on a pendant path and has degree 1 or 2. Therefore, we can select $P^*$ with the root of degree at least 3. Assume that $d_{T^*}(v^*_q, y) \leq d_{T^*}(v^*_q, y)$, where $y$ is the leaf of $P^*$. (The reverse case can be treated analogously.) Since $S^*$ is a gp-set of $T^*$ which is not isomorphic to a path, there is a vertex $v^*_k \in S^*$ lying on $P^*$. So we may consider that $P^*$ is a branching path that contains $v^*_q$, $v^*_q$ and a vertex $v^*_k \in S^*$. (It is possible that some of these vertices are the same.) Let $z = (v_i, v^*_k)$. Then $z \in R_1$. We proceed by distinguishing the following subcases based on the position of $v_p$, $v'_p$ and $v_i$ in $T$.

Subcase 1.1.1. $v'_p \in I(v_i, v_p)$.

In this subcase, if $v^*_q$ is closer than $v^*_q$ to the leaf $y$ of $P^*$, then, by Lemma 3, $w' \in I_{T \sqcup T^*}(w, z)$, a contradiction.

If $v^*_k \in I(v^*_q, v^*_q)$, then since $\ell(T^*) \geq 3$, there exists $z' = (v_i, v^*_k) \in \{v_i\} \times S^*$ such
that \( v_k^*, v_q^* \in I(v_q^*, v_{k'}) \) in \( T^* \). Then we have

\[
\begin{align*}
d(w', z') &= d_T(v_{p'}, v_i) + d_T(v_q^*, v_{k'}) \\
&= d_T(v_{p'}, v_i) + d_T(v_q^*, v_k^*) + d_T(v_k^*, v_{k'}) \\
&= d(w', z) + d(z, z'),
\end{align*}
\]

which implies that \( z \in I_{T \cap T^*}(w', z') \), a contradiction.

**Subcase 1.1.2.** \( v_i \in I(v_p, v_{p'}) \).

In this subcase, if \( v_k^* \in I(v_q^*, v_{p'}) \) in \( P^* \), then \( z \in I_{T \cap T^*}(w, w') \) by Lemma 3, a contradiction.

Assume that \( v_k^* \) is closer than \( v_q^*, v_{p'}^* \) to the leaf of \( P^* \). Since \( |S^*| = \ell(T^*) \geq 3 \), there is a vertex \( z' = (v_i, v_k^*) \in \{v_i\} \times S^* \) such that \( v_q^*, v_{p'}^* \in I(v_k^*, v_{k'}) \) in \( T^* \). Let \( v_k^* \) be on a branching path \( P^* \) in \( T^* \) where \( P^* \neq P^* \). Note that \( \ell(T) + 1 \geq 3 \). There exists at least one vertex \( a = (v_x, v_y') \in R_2 \setminus \{w, w'\} \). Next we consider the positions of \( v_x, v_y^* \) in \( T, T^* \), respectively.

Suppose first that \( v_y^* \in V(P^* \cup P'^*) \). If \( v_x, v_p, v_{y'} \) and \( v_i \) lie on a path in \( T \), then there are five vertices \( w, w', z, z' \) and \( a \) in \( R_2 \), three of which lie on a common geodesic in \( T \cap T^* \), a contradiction. Note that if \( T \) is a path, then we are done as above. Therefore, assume that \( T \) is not isomorphic to a path in the following and the root of \( P \) has degree at least 3. Otherwise, \( v_x \notin P \) and \( v_x, v_p \) lie on a common branching path in \( T \). Let \( V_s \) be the set of vertices of \( T \) but not contained in \( T_{ip'} \) where \( T_{ip'} \) is the subtree of \( T - v_p \) containing \( v_i \) and \( v_{p'} \). If there is a vertex \( a' = (v_s, v_i^*) \in R_2 \) with \( v_s \in V_s \), then \( R_2 \) contains \( w, w', z, z' \) and \( a' \), three of which are on a common geodesic, a contradiction. Therefore, the first coordinate of any vertex in \( R_2 \) cannot be in \( V_s \). Assume that \( P' \neq P \) is any branching path containing \( v_p \) and a leaf both in \( T_{ip'} \) and \( T \). Then, besides \( w \), \( P' \cap T^* \) contains at most one vertex in \( R_2 \) of \( T \cap T^* \). Otherwise, \( P' \cap T^* \) contain two vertices \( h, h' \) in \( R_2 \). Then there exist two vertices \( h_0, h'_0 \in \{v_i\} \times S^* \) such that three vertices from \( \{h, h', h_0, h'_0, w\} \) lie on some geodesic in \( T \cap T^* \), a contradiction. (Here \( h_0 \) may be equal to \( h'_0 \).) Note that \( V_s \) contains at least two leaves of \( T \) since the root of \( P \) (just in \( V_s \)) has degree at least 3. Then \( T_{ip'} \) has at most \( \ell(T) - 2 \) leaves in \( T \). Since \( P \cap T^* \) contains two vertices \( w \) and \( w' \) in \( R_2 \), we have \( |R_2| \leq \ell(T) - 2 + 1 < \ell(T) = \text{gp}(T) \), a contradiction with the assumption.

Assume now that \( v_y^* \notin V(P^* \cup P'^*) \). Then there exists a vertex \( z'' = (v_i, v_{k''}) \in \{v_i\} \times S^* \) such that \( v_q^*, v_{k''} \) lie on a common branching path in \( T^* \). If \( v_y^* \) is closer to the leaf of the branching path than \( v_{k''} \) in \( T^* \), then \( v_i \in I(v_x, v_i) \) and \( v_{k''} \in I(v_{y''}, v_k^*) \). Therefore, by Lemma 3 we get \( z'' \in I_{T \cap T^*}(a, z) \), a contradiction. In the case that \( v_{k''} \) is closer to the leaf of the branching path than \( v_y^* \) in \( T^* \), we consider the positions of \( v_x, v_p, v_{y'} \) and \( v_i \) in \( T \). Let \( V_1 = \{z, z', w, w', a, z''\} \). Then \( V_1 \subseteq R_2 \). If \( v_x, v_p, v_{y'} \) and \( v_i \) lie on a path in \( T \), then there exist three vertices in \( V_1 \) lying on a common geodesic in
In this subcase, we may assume that $v$ is closer than $v'$ to the leaf of $k^*$. Similarly as above, a contradiction occurs.

**Subcase 1.1.3.** $v_p \in I(v_i, v_{p'})$.

In this subcase, since $\ell(T^*) \geq 3$, there exists a vertex $z' = (v_i, v_{k'}) \in \{v_i\} \times S^*$ such that $v_{k'} \notin P^*$ and $v_q^* \in I(v_{k'}, v_{q''})$ in $T^*$. Since

$$d(z', w') = d_T(v_i, v_{p'}) + d_{T^*}(v_{k'}, v_q^*)$$

$$= d_T(v_i, v_p) + d_{T^*}(v_{k'}, v_q^*) + d_{T^*}(v_p, v_{p'}) + d_{T^*}(v_q^*, v_{q''})$$

$$= d(z', w) + d(w, w'),$$

we have $w \in I_{T \square T^*}(z', w')$, a contradiction.

**Subcase 1.1.4.** $v_i \notin V(P)$ such that $v_i, v_p$ lie on a same branching path in $T$.

In this subcase, since $\ell(T^*) \geq 3$, there is a vertex $z' = (v_i, v_{k'}) \in \{v_i\} \times S^*$ such that $v_{k'} \in I(v_{k'}, v_{q'})$ in $T^*$. If $v_k^* \in I(v_{k'}, v_{q'}^*)$, then obviously $v_k^* \in I(v_q^*, v_{k'})$ and therefore,

$$d(w', z') = d_T(v_{p'}, v_i) + d_{T^*}(v_{q'}, v_{k'})$$

$$= d_T(v_{p'}, v_p) + d_{T^*}(v_{q'}, v_{k'}) + d_{T^*}(v_p, v_{p'}) + d_{T^*}(v_{k'}, v_{q''})$$

$$= d(w', z) + d(z', z').$$

We conclude that $z \in I_{T \square T^*}(w', z')$, a contradiction.

If $v_k^*$ is closer to the leaf of $P^*$ than $v_{q'}^*$, then we get a contradiction similarly as in Subcase 1.1.2.

**Case 1.2.** $v_q^*$ and $v_{q'}^*$ do not lie on a same branching path in $T^*$.

In this subcase, we may assume that $v_q^*$ and $v_{q'}^*$ lie on distinct branching paths $P^*$ and $P'^*$ in $T^*$, respectively. Since $\ell(T^*) \geq 3$ and $T^*$ is not isomorphic to a path, there exist two vertices $z = (v_i, v_{k'})$ and $z' = (v_i, v_{k''})$ from $\{v_i\} \times S^*$, such that $v_k^* \in P^*$ and $v_{k'}^* \in P'^*$. We consider the following subcases based on the positions of $v_{p'}$, $v_{p'}$ and $v_i$ in $T$.

**Subcase 1.2.1.** $v_{p'} \in I(v_i, v_{p'})$.

In this subcase, if $v_{k'}^*$ is closer than $v_q^*$ to the leaf of $P'^*$, then $v_{p'} \in I(v_p, v_i)$ and $v_{q'}^* \in I(v_{q'}, v_{k'})$. Lemma 3 gives $w' \in I_{T \square T^*}(w, z')$, a contradiction. On the other hand, if $v_{q'}^*$ is closer than $v_{k'}^*$ to the leaf of $P'^*$, then $v_i \in I(v_i, v_{p'})$ and $v_{k'}^* \in I(v_{k'}, v_{q'})$, hence Lemma 3 gives $z' \in I_{T \square T^*}(w', z)$, a contradiction again.

**Subcase 1.2.2.** $v_i \in I(v_p, v_{p'})$.

In this subcase, we first assume that $v_{q'}^*$ is closer than $v_{k'}^*$ to the leaf of $P'^*$. Then $v_i \in I(v_i, v_{p'})$ and $v_{p'} \in I(v_{k'}, v_{q'})$. Therefore, by Lemma 3, we get $z' \in I_{T \square T^*}(z, w')$ as a contradiction. Otherwise we suppose that $v_{k'}^*$ is closer than $v_{q'}^*$ to the leaf of $P'^*$. If $v_{q'}^*$
is closer than \( v_k^* \) to the leaf of \( P^* \), then \( v_i \in I(v_p, v_i) \) and \( v_k^* \in I(v_q^*, v_k^*) \). Therefore, by Lemma 3 we get \( z \in I_T(\bar{z}(w, z')) \), a contradiction. In the case that \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), we find a contradiction similarly as the proof of Subcase 1.1.2.

Subcase 1.2.3. \( v_p \in I(v_i, v_p') \).
In this subcase, if \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), then \( v_p \in I(v_i, v_p') \) and \( v_q^* \in I(v_k^*, v_q^*) \). So Lemma 3 gives \( w \in I_T(\bar{z}(z, w')) \), a contradiction. And if \( v_q^* \) is closer than \( v_k^* \) to the leaf of \( P^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_k^*, v_q^*) \), hence we get \( z \in I_T(\bar{z}(z', w)) \).

Subcase 1.2.4. \( v_i \notin V(P) \) such that \( v_i, v_p \) lie on a same branching path in \( T \).
First suppose that \( v_q^* \) is closer to the leaf than \( v_k^* \) in \( P^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_q^*, v_k^*) \). Thus, by Lemma 3 we get \( z \in I_T(\bar{z}(z', w)) \).

Assume that \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \). If \( v_q^* \) is closer to the leaf than \( v_k^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_q^*, v_k^*) \), which gives \( z' \in I_T(\bar{z}(z, w')) \). If \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), we can proceed similarly as in Subcase 1.1.4.

Now we turn to the second case.

Case 2. \(|R \cap V^{(v)}(T^*)| < \ell(T^*)\) for any \( k \in [n] \), and \(|R \cap V(T^{n'})| < \ell(T)\) for any \( t \in [n^*] \).
In this case, let \( v^* \) be a layer with \(|R \cap V^{(v)}(T^*)| = \max\{|R \cap V^{(v)}(T^*)| : k \in [n]\}\). Let \( R = R_1 \cup R_2 \) where \( R_1 = R \cap V^{(v)}(T^*) \) and \( R_2 = R \setminus R_1 \), that is, \( R_2 = \bigcup_{k \in [n], t \in [1]} (V^{(v)}(T^*) \setminus R) \). Set further \( S^* = \{v_{\bar{u}}^*: (v_i, v_{\bar{u}}^*) \in R_1\} \). Then \( 1 \leq |S^*| \leq \ell(T^*) - 1 \).

Assume first \( |S^*| = 1 \). Therefore \(|R \cap V^{(v)}(T^*)| \leq 1 \) for any \( k \in [n] \). Next we only need to consider \(|R \cap V(T^{n'})| \leq 1 \) for any \( j \in [n^*] \). (If \(|R \cap V(T^{n'})| \geq 2 \) for some \( j \in [n^*] \), by commutativity of \( T \square T^* \), the proof is similar to the subcase in which \( 2 \leq |S^*| \leq \ell(T^* - 1) \). Therefore, suppose that \(|R \cap V(T^{n'})| \leq 1 \) for any \( j \in [n^*] \). Then \(|R| \leq \min\{n, n^*\} \). We now claim that \(|R| \leq \ell(T) + \ell(T^*) \). If not, then since \(|R| \geq \ell(T) + \ell(T^*) + 1 \geq 6 \), there exist three vertices \( u = (v_p, v_{\bar{u}}^*), v = (v_p', v_{\bar{u}}'^*), w = (v_s, v_{\bar{u}}^*) \) from \( R \) such that \( v_p, v_p' \) lie on a same branching path in \( T \), and \( v_{\bar{u}}^*, v_{\bar{u}}'^* \) lie on a common branching path in \( T^* \). Note that there may be \( p' = s, q = \ell \). But we can always select a vertex \( h \in R \setminus \{u, v, w\} \) such that \( u, v, h \) or \( u, w, h \) lie on a same geodesic in \( T \square T^* \), which is a contradiction. So our result holds when \(|S^*| = 1 \).

Suppose second that \( 2 \leq |S^*| \leq \ell(T^*) - 1 \). As \(|R_1| = |S^*| \), we need to prove that \(|R_2| \leq \ell(T) + \ell(T^*) - |S^*| \). Assume on the contrary that \(|R_2| \geq \ell(T) + \ell(T^*) - |S^*| + 1 \). Since \(|S^*| \geq 2 \), there are two distinct vertices \( w = (v_i, v_{\bar{u}}^*) \) and \( w' = (v_i, v_{\bar{u}}'^*) \) from \( \{v_i\} \times S^* \). We distinguish the following cases based on the positions of \( v_{\bar{u}}^*, v_{\bar{u}}'^* \) in \( T^* \).

Case 2.1. \( v_{\bar{u}}^* \) and \( v_{\bar{u}}'^* \) lie on a same branching path \( P^* \) of \( T^* \).
In this subcase, we may without loss of generality assume that \( v_{\bar{u}}' \) is closer than \( v_{\bar{u}}^* \),
to the leaf of $P^*$. Let $T_{v_j^*}^i$ be the maximal subtree of $T^* - v_j^*$ containing $v_j^*$ and let $V_{v_j^*} = V(T^*) \setminus V(T_{v_j^*}^i)$. Let further $S_1^* = \{v_q^* : v_q^* \in I(v_j^*, v_j^*), v_i^* \in S^* \cap V(T_{v_j^*}^i)\}$. Now we prove the following claim.

**Claim 1.** If $z = (v_p, v_p^*) \in R_2$, then $v_i^* \in S_1^*$.

**Proof of Claim 1.** If not, suppose first that $v_i^* \in V(P^*)$ is closer than $v_j^*$ to the leaf of $P^*$. Then $v_i^* \in I(v_i, v_p)$ and $v_j^* \in I(v_j^*, v_j^*)$. Hence, $w' \in I_{T \square T^*}(w, z)$. And if $v_i^* \in V_{v_j^*}^i$, then $v_j^* \in I(v_i^*, v_j^*)$. Combining this fact with $v_i \in I(v_i, v_p)$, we have $w' \in I_{T \square T^*}(w', z)$.

This proves Claim 1.

By Claim 1, we have $| \bigcup_{v_i^* \in S_1^*} (V(T_{v_i^*}^i) \cap R) | \geq \ell(T) + \ell(T^*) - |S^*| + 1 \geq \ell(T) + 1$.

Then there exist two vertices $z = (v_p, v_p^*)$ and $z' = (v_p', v_p'^*)$ from $\bigcup_{v_i^* \in S_1^*} (V(T_{v_i^*}^i) \cap R)$ such that $v_p^*, v_p'^* \in S_1^*$ and $v_p, v_p'$ lie on a same branching path $P$ in $T$. Without loss of generality, let $v_p'$ be closer than $v_p$ to the leaf of $P$, and let $v_i^*, v_i'^* \in I(v_i^*, v_j^*)$ (by the definition of $S_1^*$). We consider the following subcases according to the positions of $v_i, v_p, v_p'$ in $T$.

**Subcase 2.1.1.** $v_p' \in I(v_i, v_p)$.

If $v_p'$ is closer than $v_i^*$ to $v_j^*$ in $P^*$, then we have $v_p' \in I(v_i, v_p)$ and $v_i'^* \in I(v_i^*, v_j^*)$. Therefore, $z' \in I_{T \square T^*}(z, w')$. And if $v_p'$ is closer than $v_i^*$ to $v_j^*$ in $P^*$, then we have $v_p' \in I(v_i, v_p)$ and $v_i'^* \in I(v_i^*, v_j^*)$ and so $z' \in I_{T \square T^*}(z, w)$.

**Subcase 2.1.2.** $v_i \in I(v_p, v_p')$.

Note that $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$. Then there exists at least a vertex $a = (v_x, v_y^*) \in \bigcup_{v_i^* \in S_1^*} (V(T_{v_i^*}^i) \cap R)$ different from $z$ and $z'$. Based on the position of $v_y^* (v_y^* \in P^* \text{ or } v_y^* \notin P^* \text{ in } T^*)$ and the positions of $v_x, v_i, v_p$ and $v_p'$ in $T$, we get contradictions using a similar proof as in Subcase 1.1.2.

**Subcase 2.1.3.** $v_p \in I(v_i, v_p')$.

If $v_p'$ is closer than $v_i^*$ to $v_j^*$ in $P^*$, then $v_p \in I(v_i, v_p')$ and $v_i'^* \in I(v_i^*, v_j^*)$, therefore $z \in I_{T \square T^*}(w, z')$. And if $v_i'^* \in I(v_i^*, v_j^*)$ in $T^*$, then $v_p \in I(v_i, v_p')$ and $v_i'^* \in I(v_i^*, v_j^*)$, hence $z \in I_{T \square T^*}(w, z')$.

**Subcase 2.1.4.** $v_i \notin V(P)$ such that $v_i, v_p$ lie on a same branching path in $T$.

Since $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$, there exists a vertex $(v_x, v_y^*) \in \bigcup_{v_i^* \in S_1^*} (V(T_{v_i^*}^i) \cap R)$. Proceeding similarly as in Subcase 1.1.4, we get required contradictions. But then $| \bigcup_{v_i^* \in S_1^*} (V(T_{v_i^*}^i) \cap R) | \leq \ell(T) + \ell(T^*) - |S^*|$, a contradiction with the assumption.

**Case 2.2.** $v_j^*, v_j^*$ lie on different branching paths $P^*, P'^*$ in $T^*$, respectively.

In this subcase, let $S_2^*$ be a set of vertices of $v_i T^*$ closer to the leaf of a branching path than $v_j^*$ for any $v_j^* \in S^*$. Note that $S^* \cap S_2^* = \emptyset$. We prove the following claim.

**Claim 2.** If $(v_p, v_p^*) \in R_2$, then $v_i^* \in V(T^*) \setminus (S^* \cup S_2^*)$.
Proof of Claim 2. Lemma 4 implies $v^*_t \notin S^*$. Assume that $v^*_t \in S^*_2$ lies on a same branching path for some $v^*_g \in T^*$. Note that $|S^*| \geq 2$. Then there exists another vertex $v^*_g'$ such that $v^*_g' \in I(v^*_t, v^*_g)$. Combining this fact with $v_i \in I(v_i, v_p)$, we arrive at a contradiction $w \in I_{T \square T^*}(z, w')$. This proves Claim 2.

Let now $S^*_1 = \{v^*_q : v^*_q \in I(v^*_g, v^*_g'), v^*_g, v^*_g' \in S^*\}$. By a parallel reasoning as in Subcase 2.1 and with Claim 2 in hands we infer that $|\bigcup_{v^*_t \in S^*_1} (V(Tv^*_t) \cap R)| \leq \ell(T)$.

Let $S = \{(v_k) : (v_k, v^*_t) \in \bigcup_{v^*_t \in S^*_1} (V(Tv^*_t) \cap R)\}$ and set $S^{**} = V(T^*) \setminus (S^* \cup S^*_1)$. From the assumption we have $|\bigcup_{v^*_t \in S^{**}} (V(Tv^*_t) \cap R)| \geq \ell(T) + \ell(T^*) - |S| - |S^*| + 1$. So there exists a vertex $z = (v_p, v^*_t) \in \bigcup_{v^*_t \in S^{**}} (V(Tv^*_t) \cap R)$, and we can always select two distinct vertices $u = (v_h, v^*_g)$ and $v = (v_h', v^*_g')$ from $R$ such that $v_p$ and $v_h$ lie on a same branching path in $T$, while $v^*_t$ and $v^*_g$ lie on a common branching path in $T^*$. But we can choose another vertex $w \in R$ such that either $u, w, z$ or $u, v, z$ lie on a same geodesic in $T \square T^*$ as a contradiction. Therefore,

$$|\bigcup_{v^*_t \in S^{**}} (V(Tv^*_t) \cap R)| \leq \ell(T) + \ell(T^*) - |S| - |S^*|.$$ 

and we are done.

Acknowledgements

Kexiang Xu is supported by NNSF of China (grant No. 11671202, and the China-Slovene bilateral grant 12-9). Sandi Klavžar acknowledges the financial support from the Slovenian Research Agency (research core funding P1-0297, projects J1-9109, J1-1693, N1-0095, and the bilateral grant BI-CN-18-20-008).

References


