Simplified constructions of almost peripheral graphs and improved embeddings into them

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Abstract

The center and the periphery of a graph are the sets of vertices with minimum and maximum eccentricity, respectively. A graph is called almost peripheral (AP) if all its vertices but one lie in the periphery. The r-AP index $AP_r(G)$ of a graph $G$ is the smallest number of vertices needed to add to $G$ to obtain an r-AP graph in which $G$ lies as an induced subgraph. In this paper new, simplified constructions of AP graphs are presented. It is proved that if $r \geq 2$ and $n \geq 2$, then $AP_r(K_n) \leq 4r - 3$. Moreover, if $G$ is not complete and has at least three vertices, then $AP_r(G) \leq 4r - 4$. In this way the previously best known bound $AP_r(G) \leq 4r - 2$ is improved.

Key words: eccentricity; diameter; almost peripheral graph; induced embedding

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1 Introduction

Graphs considered in this paper are finite and contain no loops or multiple edges. The distance $d_G(u, v)$ between vertices $u$ and $v$ of a graph $G$ is the shortest path distance. The eccentricity $ecc_G(u)$ of vertex $u$ is $\max\{d_G(u, v) : v \in V(G)\}$. The radius $rad(G)$
of $G$ and the diameter $\text{diam}(G)$ of $G$ are the minimum and the maximum eccentricity of the vertices of $G$, respectively. The center $C(G)$ and the periphery $P(G)$ are the sets of the vertices of $G$ of minimum and maximum eccentricity, respectively.

Central and peripheral vertices of graphs are of great importance in location theory and in investigations of (large) networks. Consequently, different classes of graphs and networks in which the center and the periphery have a special structure were introduced. These classes include self-centered graphs (alias eccentric graphs) [1, 3, 4, 14], their generalization to graphs whose center is a $k$-distance dominating set [5], and almost self-centered graphs [2, 8, 10]. The latter graphs (as well as almost peripheral graphs) turned out to be extremal graphs for a newly introduced measure of non-self-centrality introduced and studied in [15]. Eccentricity in graphs has also been studied from many additional aspects, cf. [6, 11, 13]. Finally, different derived graphs have been proposed based on the eccentricity such as radial graphs [7] or the recently introduced graphs with a bit unfortunate name “eccentric graphs” (which are not eccentric graphs in the above sense) [12].

In this paper we are interested in almost peripheral graphs that were introduced in [9] and in part motivated by location problems in which it is required that most of the resources do not lie in the center. A graph $G$ is called almost-peripheral, AP for short, if all but one of its vertices lie in the periphery, that is, if $|P(G)| = |V(G)| - 1$ holds. If $G$ is an AP graph with $\text{rad}(G) = r$ then we will say that $G$ is an $r$-AP graph.

We proceed as follows. In the next section we first give a new construction that from a given $r$-AP graph and an arbitrary graph produces a new $r$-AP graphs. Then we present, for any integer $r \geq 1$, an $r$-AP graph of order $4r - 1$. The present construction is significantly simpler than a related construction given in [10]. Then, in Section 3, we prove that the complete graph $K_n$ can be embedded as a subgraph into an $r$-AP graph $H$ of order $n + 4r - 3$, and that an arbitrary graph of order $n \geq 3$ that is not complete can be embedded as an induced subgraph into an $r$-AP graph of order $n + 4r - 4$. This improves the best earlier such embeddings from [10], where the host graph is of order $n + 4r - 2$. We conclude the paper with a couple of open problems.

Before we start, let us recall some additional concepts and notations needed. If $x$ is a vertex of $G$, then its closed neighborhood is denoted with $N[x]$. A subgraph $H$ of a graph $G$ is isometric if $d_H(u, v) = d_G(u, v)$ holds for any $u, v \in V(H)$. The vertex deleted $d-$ cube $Q_{d}^-, d \geq 1$, is obtained from the $d$-cube $Q_{d}$ by removing one of its vertices.
2 New constructions of AP graphs

If $G$ and $H$ are disjoint graphs and $S \subseteq V(G)$, then let $G \oplus_S H$ denote the graph obtained from the disjoint union of $G$ and $H$ by adding a join between $S$ and $V(H)$, that is, adding an edge $xy$ for each $x \in S$ and $y \in V(H)$. In [9, Theorem 2.3] it was proved that if $u$ is the center vertex of an $r$-AP graph $G$, $r \geq 1$, then $G \oplus \{u\} H$ is an $r$-AP graph for any graph $H$. We now prove a variation of this result that yields a much larger class of AP graphs.

**Theorem 2.1** If $u$ is a peripheral vertex of an $r$-AP graph $G$, $r \geq 1$, and $H$ is a graph, then $G \oplus N[u] H$ is an $r$-AP graph.

**Proof.** Let $u$ be an arbitrary vertex of $G$ that is not the center vertex, and set $K = G \oplus N[u] H$. Let $x$ and $x'$ be arbitrary vertices of $K$ and consider the following cases.

Suppose first that $x, x' \in V(G)$. Let $P$ be a shortest $x, x'$-path. If $P$ lies completely in $G$, then clearly $d_K(x, x') = d_G(x, x')$. Otherwise $P$ contains a vertex $y$ of $H$. Let $z$ be the last vertex of $P$ that is still in $G$ and let $z'$ be the first vertex of $P$ after $y$ that lies in $G$. Then $z, z' \in N[u]$. Since $P$ is a shortest path, we necessarily have $d_G(z, z') = 2$ and consequently $P$ contains the subpath $z - y - z'$. Replacing this subpath of $P$ with $z - u - z'$, a shortest $x, x'$-path in $G$ is obtained that has the same length as $P$. In conclusion,

$$d_K(x, x') = d_G(x, x'), \quad x, x' \in V(G). \quad (1)$$

Assume next that $x \in V(G), x \neq u$, and $x' \in V(H)$. Let $Q$ be a shortest $x, u$-path in $G$. If $u'$ is the neighbor of $u$ on $Q$ ((it is possible that $u' = x$), then clearly $u' \in N[u]$. Since $u'x' \in E(K)$ we conclude that

$$d_K(x, x') = d_G(x, u), \quad x \in V(G), x \neq u, x' \in V(H). \quad (2)$$

We also clearly have

$$d_K(x, x') \leq 2, \quad x, x' \in V(H). \quad (3)$$

Since $u$ is adjacent to every vertex of $H$, we infer from (1) that $\text{ecc}_K(u) = \text{ecc}_G(u) = r + 1$. In addition, from (1) and (2) we get that $\text{ecc}_K(x) = \text{ecc}_G(x)$ holds for every vertex $x \in V(G), x \neq u$. In particular, if $z$ is the center vertex of $G$, then $\text{ecc}_K(z) = \text{ecc}_G(z) = r$. Finally, from (2) and (3) we obtain that $\text{ecc}_K(x) = \text{ecc}_G(u) = r + 1$ holds for every vertex $x \in V(H)$. Hence $K$ is an $r$-AP graph with $C(K) = \{z\}$. \hfill \Box
In [9] a question was posed whether there exist \( r \)-AP graphs of order \( n < 4r + 1 \) for \( r \geq 4 \). A positive answer to this problem was given in [10] by demonstrating that for any \( r \geq 1 \) there exists an \( r \)-AP graph of order \( 4r - 1 \). We reprove here this answer with a significantly simpler construction.

**Proposition 2.2** For any integer \( r \geq 1 \) there exists an \( r \)-AP graph of order \( 4r - 1 \).

**Proof.** For \( r = 1 \) the path on three vertices is such a graph. Hence assume in the rest of the proof that \( r \geq 2 \). Let \( G_r \) be the graph constructed as follows. Start with a cycle \( C \) of length \( 4r - 2 \) and label its vertices consecutively with \( v_1, v_2, \ldots, v_{4r-2} \). Add a new vertex \( x \) and finalize the construction by adding the edges \( xv_1, xv_{2r-1}, \) and \( v_{r}v_{3r-1} \). The construction is illustrated in Fig. 1 on the graph \( r = 6 \).

![Figure 1: The graph \( G_6 \)](image)

It is straightforward to check that \( G_r \) is an \( r \)-AP graph with \( C(G_r) = \{v_r\} \). To verify this, define the cycles:

\[
C' = v_1 - v_2 - \cdots - v_r - v_{3r-1} - v_{3r} - \cdots v_{4r-2} - v_1, \\
C'' = v_r - v_{r+1} - \cdots - v_{2r-1} - v_{2r} - v_{2r+1} - \cdots v_{3r-1} - v_r, \\
C''' = v_1 - v_2 - \cdots - v_{2r-2} - v_{2r-1} - x - v_1,
\]

(cf. Fig. 1 again) and note that they are all isometric cycles of length \( 2r \). Since \( v_r \) lies in \( C' \cap C'' \cap C''' \) and these three cycles cover \( G_r \) we already get that \( \text{ecc}(v_r) = r \). To compute the other eccentricity it is useful to observe that also the cycle

\[
x - v_{2r-1} - v_{2r} - v_{2r+1} - \cdots - v_{4r-3} - v_{4r-2} - v_1 - x
\]

is isometric and that it is of length \( 2r + 2 \). So we are left with considering the distances between the vertices from \( C' \) and the vertices from \( C'' \). It is straightforward to verify
that \( d_{Gr}(v_i, v_{3r-1-i}) = r + 1 \) holds for \( i = 1, \ldots, r - 1 \) and that \( d_{Gr}(v_i, w) \leq r \) for any other vertex. Hence \( \text{ecc}(v_i) = \text{ecc}(v_{3r-1-i}) = r + 1 \) holds for \( i = 1, \ldots, r - 1 \). By symmetry, the same conclusion holds also for the remaining vertices to be considered.

Note that the graph \( G_2 \) constructed in Proposition 2.2 is the vertex-deleted 3-cube \( Q_3^- \).

3 Embeddings into \( r \)-AP graphs

If \( G \) is a graph and \( r \) a positive integer, then the \( r \)-AP index \( \AP_r(G) \) of \( G \) is

\[
\AP_r(G) = \min\{|V(H)| - |V(G)| : \ H \text{ is } r\text{-AP graph, } G \text{ induced in } H\}.
\]

Clearly, \( \AP_r(G) = 0 \) if and only if \( G \) is an \( r \)-AP graph. Moreover, if a graph \( G \) does not contain a unique universal vertex (equivalently \( \AP_1(G) > 0 \)), then adding a new vertex and joining it to all vertices of \( G \) yields an 1-AP graph. Consequently \( \AP_1(G) \leq 1 \) holds for every graph \( G \). For \( r \geq 2 \) it was proved in [9] that if \( G \) is an arbitrary graph on at least two vertices, then

\[
\AP_2(G) \leq 5,
\]

where equality holds if and only if \( G \) is a complete graph. It was further shown that for every \( r \geq 2 \) and every graph \( G \) we have \( \AP_r(G) \leq 4r - 1 \). This result was improved in [10] by proving that if \( G \) is an arbitrary graph, then

\[
\AP_r(G) \leq 4r - 3.
\]

Based on (5) it was asked in [10, Problem 4.1] whether for \( r \geq 3 \) there exists a graph \( G \) with \( \AP_r(G) = 4r - 2 \). In the following theorem we answer this problem in negative.

**Theorem 3.1** If \( r \geq 2 \) and \( n \geq 2 \) then \( \AP_r(K_n) \leq 4r - 3 \). Moreover, if \( G \) is not complete and has at least three vertices, then \( \AP_r(G) \leq 4r - 4 \).

**Proof.** In our construction we will essentially use the graphs \( G_r, r \geq 2 \), constructed in the proof of Proposition 2.2 (cf. Fig. 1). Let \( G \) be a graph and distinguish the following cases.

**Case 1:** \( G = K_n, n \geq 2 \).

In this case let \( H_{r,n} \) be the graph obtained from \( G_r \) and (a disjoint copy of) \( K_n \) by
identifying two vertices of $K_n$ with $v_{3r-1}$ and $v_{3r}$, respectively. Note that for $H_{r,2} = G_r$.

The graph $H_{6,5}$ is drawn in Fig. 2.

We claim that $H_{r,n}$ is an $r$-AP graph. Clearly, the vertices of the complete subgraph $K_n$ of $H_{r,n}$ do not decrease the eccentricities of the vertices from the subgraph $G_r$ of $H_{r,n}$. Moreover, since each vertex of this subgraph is at distance at most $r$ from at least one of the vertices $v_{3r-1}$ and $v_{3r}$ (for instance, $d_{H_{r,n}}(x, v_{3r}) = r$), we have $\text{ecc}_{H_{r,n}}(u) = \text{ecc}_{G_r}(u)$ holds for any vertex $u$ from the subgraph $G_r$ of $H_{r,n}$. Finally, if $v \in V(K_n) \setminus V(G_r)$, then $d_{H_{r,n}}(v, x) = r + 1$, so that $\text{ecc}_{H_{r,n}}(v) = r + 1$. Hence $H_{r,n}$ is an $r$-AP graph, where $C(H_{r,n}) = \{v_r\}$. Since $|V(G_r)| = 4r - 1$ it follows that $\text{AP}_r(K_n) \leq 4r - 3$ holds for any $r \geq 2$ and $n \geq 2$.

**Case 2:** $G$ contains an induced $P_3$.

Let $u$, $v$, and $w$ be the vertices of $G$ that induce $P_3$, where $uw \notin E(G)$. Let $X(r, G)$ be the graph obtained from $G_r$ and $G$ by identifying the path $u - v - w$ of $G$ with the path $v_{3r} - v_{3r-1} - v_{3r-2}$ and joining $v_r$ by an edge to every vertex of $V(G) \setminus \{u, v, w\}$. The graph $X(r, G)$ is schematically shown in Fig. 3.

The eccentricities of the vertices of $G_r$ do not change in $G_r \circ G(uvw)$. Moreover, if $z \in V(G) \setminus \{u, v, w\}$, then $\text{ecc}_{X(r, G)}(z) = \text{ecc}_{G_r}(v_r) + 1 = r + 1$. It follows that Hence $X(r, G)$ is an $r$-AP graph and since $|V(G_r)| = 4r - 1$ we conclude that $\text{AP}_r(G) \leq (4r - 1) - 3 = 4r - 4$.

The remaining case to consider is:

**Case 3:** $G$ is a disjoint union of complete graphs.

Let $G$ be the disjoint union of $K_{n_1}, \ldots, K_{n_k}$, where $n_1 \leq \cdots \leq n_k$. Then we distinguish the following subcases.
Figure 3: The graph $X(r, G)$

Suppose first that $n_k = 1$, that is, $G$ is edge-less. Since $|V(G)| \geq 3$, identify three vertices of $G$ with three independent vertices of $G_r$ pairwise different from $v_r$, and connect all the other isolated vertices of $G$ with $v_r$. In this way $G$ is embedded into an $r$-AP graph and hence $AP_r(G) \leq 4r - 4$.

If $n_k = 2$ and $n_{k-1} = 1$ (or if $n_k = 2$ and $n_{k-1} = 2$), then identify the $K_2$ and one independent vertex (or two $K_2$'s, respectively) of $G$ with a corresponding induced subgraph of $G_r$ and connect all the other vertices of $G$ with $v_r$ to reach the same conclusion.

Finally let $n_k \geq 3$. In this subcase construct first the graph $H_{r,n_1}$ as in Case 1. Further, if $n_{k-1} \geq 2$, then identify one vertex of the component $K_{n_{k-1}}$ with $v$, and connect all the vertices of the other $k - 2$ components with $v$. Otherwise we have $n_{k-1} = 1$, in which case identify one isolated vertex with a vertex of $G_r$ independent from the vertices $v_r$, $v_{3r-1}$, and $v_{3r}$, and connect all the other isolated vertices with $v_r$.

The proof is completed by observing that in all the cases we have constructed $r$-AP graphs and consequently $AP_r(G) \leq 4r - 4$. □

4 Conducing remarks

Note that the inequality (4) and its equality case imply that Theorem 3.1 is best possible for $r = 2$. Hence we pose:

Problem 4.1 Is Theorem 3.1 best possible for $r \geq 3$? More precisely:

(i) Is it true that $AP_r(K_n) = 4r - 3$ for $r \geq 3$ and $n \geq 2$?

(ii) Let $r \geq 3$. Does there exist a not complete graph $X_r$ such that $AP_r(X_r) = 4r - 4$?
The constructions from Theorem 3.1 in many cases yield graphs AP graphs with cut vertices. It would be interesting to see if there exist related embeddings into 2-connected AP graphs.

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