Coloring Sierpiński graphs and Sierpiński gasket graphs

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Abstract
Sierpiński graphs $S(n, 3)$ are the graphs of the Tower of Hanoi puzzle with $n$ disks, while Sierpiński gasket graphs $S_n$ are the graphs naturally defined by the finite number of iterations that lead to the Sierpiński gasket. An explicit labeling of the vertices of $S_n$ is introduced. It is proved that $S_n$ is uniquely 3-colorable, that $S(n, 3)$ is uniquely 3-edge-colorable, and that $\chi'(S_n) = 4$, thus answering a question from [15]. It is also shown that $S_n$ contains a 1-perfect code only for $n = 1$ or $n = 3$ and that every $S(n, 3)$ contains a unique Hamiltonian cycle.

Key words: Sierpiński graphs; Sierpiński gasket graphs; chromatic number; chromatic index; 1-perfect code, Hamiltonian cycle

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1 Introduction

Topological studies of the Lipscomb’s space [11, 12] led in [8] to the definition of Sierpiński graphs $S(n, k)$. Another motivation for the introduction of these graphs is the fact that the graph $S(n, 3)$ is isomorphic to the graph of the Tower of Hanoi puzzle with $n$ disks [8], see also [5]. Sierpiński graphs were also independently studied in [14], where it is shown that they arise in a natural way from regular graphs.

The graphs $S(n, k)$ have many appealing properties and were studied from different points of view. They possess (essentially) unique 1-perfect codes [9], a result proved before for $S(n, 3)$ in [2]. Alternative arguments for the uniqueness of 1-perfect codes in $S(n, k)$ were recently presented in [3] in order to determine

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their optimal $L(2,1)$-labelings. Moreover, (regularizations of) Sierpiński graphs are the first nontrivial families of graphs of “fractal” type for which the crossing number is known [10], while in [13] several metric invariants of these graphs are determined.

Hinz and Schief [7] used the connection between the graphs $S(n,3)$ with the Sierpiński gasket to compute the average distance of the latter, see also [1]. The graphs that are obtained in a natural way by $n$ iterations of the process that leads to the Sierpiński gasket are called Sierpiński gasket graphs and denoted $S_n$. Teguia and Godbole [15] studied several properties of these graphs, in particular the chromatic number, the domination number, and the pebbling number.

In this paper we continue studies of the Sierpiński graphs $S(n,3)$ and the Sierpiński gasket graphs $S_n$. We first introduce an explicit labeling of the vertices of $S_n$ that is obtained by “contracting” the Sierpiński labeling of $S(n,3)$. In the central part of the paper—Section 3—vertex- and edge-colorings of $S(n,3)$ and $S_n$ are treated. It is in particular shown that $S_n$ is uniquely 3-colorable (hence strengthening a result from [15]), that $S(n,3)$ is uniquely 3-edge-colorable, and that the chromatic index of $S_n$ is 4 (hence answering a question from [15]). We conclude the paper by observing that $S_n$ contains a 1-perfect code only for $n = 1$ and $n = 3$ and that every $S(n,3)$ contains a unique Hamiltonian cycle.

As usual, $\chi(G)$, $\chi’(G)$, and $\gamma(G)$ denote the chromatic number of $G$, the chromatic index of $G$, and the domination number of $G$, respectively. A 1-perfect code (also known as an efficient dominating set) in a graph $G$ is a vertex subset of $G$ such that the closed neighborhoods of its elements form a partition of $V(G)$. For any other graph theoretic concept not defined here we refer to [16].

2 Sierpiński graphs and Sierpiński gasket graphs

The Sierpiński graphs $S(n,3)$, $n \geq 1$, are defined in the following way:

$$V(S(n,3)) = \{1,2,3\}^n,$$

two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ being adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \ldots, h - 1$;
(ii) $u_h \neq v_h$; and
(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

We will shortly write $\langle u_1 \ldots u_n \rangle$ for $(u_1, \ldots, u_n)$. (On figures this convention is further shortened to $u_1 \ldots u_n$.) The graph $S(4,3)$ is shown in Fig. 1.

The vertices $\langle 1 \ldots 1 \rangle$, $\langle 2 \ldots 2 \rangle$, and $\langle 3 \ldots 3 \rangle$ are called the extreme vertices of $S(n,3)$. For $i = 1, 2, 3$ let $S(n + 1,3)_i$ be the subgraph of $S(n + 1,3)$ induced by the vertices of the form $\langle i \ldots \rangle$. Clearly, $S(n + 1,3)_i$ is isomorphic to $S(n,3)$.  

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The Sierpiński gasket graphs $S_n$, $n \geq 1$, are defined geometrically as the graphs whose vertices are the intersection points of the line segments of the finite Sierpiński gasket $\sigma_n$ and line segments of the gasket as edges, see [15]. The Sierpiński gasket graph $S_4$ is shown in Fig. 2.

We now give an alternative description of the graphs $S_n$ that yields an explicit labeling of the vertices of $S_n$. For this sake observe that $S_n$ can be obtained from $S(n,3)$ by contracting all of its edges that lie in no triangle. Let $\langle u_1 \ldots u_r, i \ldots j \rangle$ and $\langle u_1 \ldots u_r, j \ldots i \rangle$ be endvertices of such an edge, then we will denote the corresponding vertex of $S_n$ with $\langle u_1 \ldots u_r \rangle \{i,j\}$. Note that $r \leq n - 2$ for such an edge. Then $S_n$ is the graph with three special vertices $\langle 1 \ldots 1 \rangle$, $\langle 2 \ldots 2 \rangle$, and $\langle 3 \ldots 3 \rangle$, called extreme vertices of $S_n$, together with the vertices of the form $\langle u_1 \ldots u_r \rangle \{i,j\}$,
where $0 \leq r \leq n - 2$, and all the $u_k$’s, $i$ and $j$ are from $\{1, 2, 3\}$. Let us call this labeling the quotient labeling of $S_n$. (See Fig. 2 where the quotient labeling of $S_4$ is shown.) For a vertex $u = \langle u_1 \ldots u_r \rangle \{i, j\}$ of $S_n$ we will also say that $\langle u_1 \ldots u_r \rangle$ is the prefix of $u$.

Figure 2: $S_4$ with its quotient labeling

For $i = 1, 2, 3$ let $S_{n,i}$ be the subgraph of $S_{n+1}$ induced by $\langle i \ldots i \rangle$, $\{i, j\}$, $\{i, k\}$, where $\{i, j, k\} = \{1, 2, 3\}$, and all the vertices whose prefix starts with $i$. Note that $S_{n,i}$ is isomorphic to $S_n$.

To explicitly describe adjacencies with respect to the quotient labeling of $S_n$, $n \geq 2$, note first that an extreme vertex $\langle i \ldots i \rangle$ of $S_n$ is adjacent to vertices $\langle i \ldots i \rangle \{i, j\}$ and $\langle i \ldots i \rangle \{i, k\}$, where $\{i, j, k\} = \{1, 2, 3\}$ and the prefixes are of length $n - 2$. In particular, in $S_2$, an extreme vertex $\langle ii \rangle$ is adjacent to $\{i, j\}$ and $\{i, k\}$. To describe neighbors of other vertices we need the following notations. For a vertex $u = \langle u_1 \ldots u_n \rangle$ of $S(n, 3)$ and $s \leq n - 2$ let $u(s) = \langle u_1 \ldots u_s \rangle$. Let $u = \langle u_1 \ldots u_r \rangle \{i, j\}$ be a vertex of $S_n$. Then let

$$\overline{u} = \langle u_1 \ldots u_r i j \ldots j \rangle$$

and

$$\overline{u} = \langle u_1 \ldots u_r j i \ldots i \rangle$$
be the endvertices of the edge of $S(n,3)$ contracted to $u$.

**Proposition 2.1** Let $n \geq 2$, let $u = \langle u_1 \ldots u_r \rangle \{i,j\}$ be a vertex of $S_n$ and let \\
\{i,j,k\} = \{1,2,3\}.

(i) If $0 \leq r \leq n - 3$ then $u$ is adjacent to \\
$\overline{u}_{(n-2)} \{i,j\}, \overline{u}_{(n-2)} \{j,k\}, \overline{u}_{(n-2)} \{i,j\},$ and $\overline{u}_{(n-2)} \{i,k\}$.

(ii) If $r = n - 2$ then $u$ is adjacent to $\overline{u}_{(n-2)} \{i,k\}, \overline{u}_{(n-2)} \{j,k\}$, to \\
$$
\begin{cases} \\
\overline{u}_{(t-1)} \{i,u_t\}, & t \text{ is the largest index with } u_t \neq i, 1 \leq t \leq n - 2; \\
\{i \ldots i\}, & \text{no such } t \text{ exists};
\end{cases}
$$

and to \\
$$
\begin{cases} \\
\overline{u}_{(s-1)} \{j,u_s\}, & s \text{ is the largest index with } u_s \neq j, 1 \leq s \leq n - 2; \\
\{j \ldots j\}, & \text{no such } s \text{ exists};
\end{cases}
$$

**Proof.** (i) Since $r \leq n - 3$, $\overline{u} = \langle u_1 \ldots u_i \ldots i \ldots j \ldots j \rangle$ and $\overline{u} = \langle u_1 \ldots u_r \ldots i \ldots i \ldots i \rangle$, where $\overline{u}$ ends with at least two $j$'s and $\overline{u}$ ends with at least two $i$'s. Hence $\overline{u}$ is in $S(n,3)$ adjacent to $\langle u_1 \ldots u_i \ldots j \ldots j \rangle$ and to $\langle u_1 \ldots u_i \ldots j \ldots k \rangle$. These two vertices contract to $\overline{u}_{(n-2)} \{i,j\}$ and $\overline{u}_{(n-2)} \{j,k\}$, respectively. Similarly, $\overline{u}$ is adjacent to $\langle u_1 \ldots u_{r-1} \ldots i \ldots j \ldots j \rangle$ and to $\langle u_1 \ldots u_{r-1} \ldots j \ldots i \ldots k \rangle$ that contract $\overline{u}_{(n-2)} \{i,j\}$ and $\overline{u}_{(n-2)} \{i,k\}$, the two remaining vertices of $S_n$ adjacent to $u$. (Note that the argument also holds for $r = 0$.)

(ii) Let $r = n - 2$. Then $\overline{u} = \langle u_1 \ldots u_{n-2} \ldots i \ldots j \ldots j \rangle$ and $\overline{u} = \langle u_1 \ldots u_{n-2} \ldots i \ldots i \ldots i \rangle$. The vertex $\overline{u}$ is in $S(n,3)$ adjacent to $\langle u_1 \ldots u_{n-2} \ldots i \ldots j \ldots j \rangle$ that contracts to $\overline{u}_{(n-2)} \{i,k\}$ and is also adjacent to $x = \langle u_1 \ldots u_{n-2} \ldots i \ldots i \ldots i \ldots i \rangle$. If $u_1 = \cdots = u_{n-2} = i$, then $\overline{u}$ is adjacent to the extreme vertex $x = \langle i \ldots i \ldots i \rangle$ of $S(n,3)$, therefore $u$ is adjacent to the extreme vertex $\langle i \ldots i \rangle$ of $S_n$. Suppose not all $u_t$'s are equal to $i$, and let $t$ be the largest index such that $u_t \neq i$. Then $x = \langle u_1 \ldots u_t \ldots i \ldots i \ldots i \rangle$, where $1 \leq t \leq n - 2$. In this case $u$ is also adjacent to $\overline{u}_{(t-1)} \{i,u_t\}$.

The other two neighbors of $u$ that arise from the neighbors of $\overline{u}$ are obtained analogously as the neighbors induced by $\overline{u}$. The details are left to the reader. $\square$

We point out that in Proposition 2.1 the case $n = 2$ is treated in case (ii).

To conclude the section note that from the quotient labeling we can immediately infer that $S_n$ contains $3 + \sum_{t=0}^{n-2} 3 \cdot 3^t = \frac{3}{2}(3^{n-1} + 1)$ vertices.
3 Vertex- and edge-colorings

In this section we determine the chromatic number and the chromatic index of the Sierpiński graphs and the Sierpiński gasket graphs.

It is easy to see that for any \( n \geq 1 \), \( \chi(S(n, 3)) = 3 \). As observed by Parisse [13], a natural 3-coloring of \( S(n, 3) \) can be obtained by setting
\[
c(\langle u_1 \ldots u_n \rangle) = u_n
\]
for any vertex \( \langle u_1 \ldots u_n \rangle \) of \( S(n, 3) \). Tegui and Godbole [15, Proposition 2] showed an analogous result for the graphs \( S_n \), namely \( \chi(S_n) = 3 \), \( n \geq 1 \). We can strengthen the latter result as follows.

**Theorem 3.1** \( S_n \) is uniquely 3-colorable for any \( n \geq 1 \).

**Proof.** We prove the theorem by induction on \( n \) and pose the following stronger induction assumption: \( S_n \) is uniquely 3-colorable and in every 3-coloring the extreme vertices receive different colors.

The claim is clearly true for \( S_1 = K_3 \). Suppose it holds for \( S_n \), \( n \geq 2 \), and consider an arbitrary 3-coloring \( c \) of \( S_{n+1} \). By the induction assumption, \( S_{n+1,1} \) is uniquely 3-colorable and we may without loss of generality assume that \( c(\langle 1 \ldots 1 \rangle) = 1 \), \( c(\langle 1, 2 \rangle) = 2 \), and \( c(\langle 1, 3 \rangle) = 3 \). Then, considering \( S_{n+1,2} \), the induction assumption implies \( c(\langle 2, 3 \rangle) \neq 2 \). Similarly because of \( S_{n+1,3} \), we infer \( c(\langle 2, 3 \rangle) \neq 3 \). So \( c(\langle 2, 3 \rangle) = 1 \) and therefore \( c(\langle 2 \ldots 2 \rangle) = 3 \) and \( c(\langle 3 \ldots 3 \rangle) = 2 \). Induction completes the argument. \( \square \)

In the rest of this section we consider edge-colorings. To show that Sierpiński graphs are uniquely 3-edge-colorable we prove the following.

**Theorem 3.2** Let \( n \geq 1 \). The 3-colorings of \( S_n \) are in a 1-1 correspondence with the 3-edge-colorings of \( S(n,3) \).

**Proof.** For \( i, j \in \{1, 2, 3\} \), \( i \neq j \), let \( \overline{ij} = \{1, 2, 3\} \setminus \{i, j\} \). In addition, for a vertex \( u \in S(n, 3) \) let \( \tilde{u} \) be the vertex of \( S_n \) to which \( u \) is mapped while contracted \( S(n, 3) \) to \( S_n \).

Let \( c \) be a 3-coloring of \( S_n \). Then for an arbitrary edge \( uv \) of \( S(n, 3) \) set
\[
c'(uv) = \begin{cases} 
    \{c(\tilde{u}), c(\tilde{v})\}, & \tilde{u} \neq \tilde{v}; \\
    c(\tilde{u}), & \tilde{u} = \tilde{v};
\end{cases}
\]
We claim that \( c' \) is an edge-coloring of \( S(n, 3) \). Let \( uv \) and \( vw \) be adjacent edges of \( S(n, 3) \). Suppose first that they belong to a common triangle. Then \( \tilde{u}, \tilde{v}, \) and \( \tilde{w} \) are pairwise different vertices of \( S_n \) which implies that \( c'(uv) \neq c'(vw) \). The
other case to consider is when, without loss of generality, $uw$ belongs to a triangle while $vw$ belongs to no triangle of $S(n,3)$. Then $c'(uv) = c(\tilde{u}) = c(\tilde{v})$, while $c(vw) = \{c(\tilde{v}), c(w)\} \neq c(\tilde{u}) = c'(uv)$.

Let now $c'$ be a 3-edge-coloring of $S(n,3)$. For a vertex $\langle u_1 \ldots u_r \rangle \{i,j \}$ of $S_n$ set
\[ c(\langle u_1 \ldots u_r \rangle \{i,j \}) = c'(\langle u_1 \ldots u_r, ij \ldots j \rangle \langle u_1 \ldots u_r, ji \ldots i \}). \]
Suppose $c(\tilde{u}) = c(\tilde{v})$ holds for adjacent vertices $\tilde{u}$ and $\tilde{v}$ of $S_n$. Since $\tilde{u}$ and $\tilde{v}$ are adjacent they lie in a common triangle $T = \tilde{u}\tilde{v}\tilde{w}$ that in turn corresponds to a triangle $T = uvw$ of $S(n,3)$. Then $c(\tilde{u}) = c(\tilde{v})$ implies that the two edges of $S(n,3)$ that are adjacent to $u$ and $v$, and do not lie in $T$, receive the same $c'$-color. But then none of the edges of $T$ can be colored with this color, a contradiction.

We have thus shown that $c$ properly colors the subgraph of $S_n$ induced by all but the three extreme vertices. Clearly, $c$ can be uniquely extended to a 3-coloring of $S_n$. \qed

Combining Theorems 3.1 and 3.2 we immediately get:

**Corollary 3.3** $S(n,3)$ is uniquely 3-edge-colorable for any $n \geq 1$.

We next determine the chromatic index of the Sierpiński gasket graphs, a question posed in [15].

Clearly, $\chi'(S_n) \geq 4$ for $n \geq 2$. Consider a 4-edge coloring of $S_2$. The edges of its middle triangle receive different colors and their color classes contain at most two edges each. Since any color class contains at most 3 edges and $S_2$ has 9 edges, we infer that no color classes contain two edges while the remaining color class has 3 edges. But then the later edges alternate on the outer 6-cycle and we conclude that a 4-edge-coloring of $S_2$ is unique modulo permutations of the colors and the shift of the color class with three elements on the outer cycle. In particular, a 4-edge-coloring of $S_2$ is uniquely defined with the colors of its outer 6-cycle, hence in our figures we will color only such edges.

Note that $\chi'(S_1) = 3$, while for the other Sierpiński gasket graphs we have the following result.

**Theorem 3.4** For any $n \geq 2$, $\chi'(S_n) = 4$.

**Proof.** It suffices to construct an edge-coloring with four colors for any $n \geq 2$.

Let $c$ be an edge-coloring of $S_n$, then let $C_1$, $C_2$, and $C_3$ be the sets of colors assigned by $c$ to the two edges incident with $\langle 1 \ldots 1 \rangle$, $\langle 2 \ldots 2 \rangle$, and $\langle 3 \ldots 3 \rangle$, respectively. To prove the theorem we pose the following stronger claim.

**Claim:** If $n$ is even, then there exists a 4-edge-coloring $c$ of $S_n$ such that $C_1 = \{1,2\}$, $C_2 = \{1,3\}$, and $C_3 = \{1,4\}$. If $n$ is odd, then there exists a 4-edge-coloring $c$ of $S_n$ such that $C_1 = \{1,2\}$, $C_2 = \{1,3\}$, and $C_3 = \{2,3\}$.

\[ 7 \]
The claim is true for \( n = 2, 3 \) as demonstrated in Fig. 3.

Let \( n \) be even. Then color \( S_{n+1} \) as follows. Let \( c' \) be a coloring of \( S_{n+1,1} \) such that \( C_1' = \{1, 2\}, C_2' = \{1, 3\}, \) and \( C_3' = \{1, 4\} \). Let \( c'' \) be a coloring of \( S_{n+1,2} \) such that \( C_1'' = \{4, 2\}, C_2'' = \{4, 1\}, \) and \( C_3'' = \{4, 3\} \). Finally, let \( c''' \) be a coloring of \( S_{n+1,3} \) such that \( C_1''' = \{2, 3\}, C_2''' = \{2, 1\}, \) and \( C_3''' = \{2, 4\} \). See the left-hand side of Fig. 4. All these colorings exist by the induction assumption and by appropriate permutations of colors. Combine \( c', c'' \), and \( c''' \) to obtain a 4-edge-coloring of \( S_{n+1} \). Finally exchange colors 3 and 4 to obtain a desired coloring.

For \( n \) odd we proceed similarly. Let \( c' \) be a coloring of \( S_{n+1,1} \) such that \( C_1' = \{1, 2\}, C_2' = \{1, 3\}, \) and \( C_3' = \{2, 3\} \), let \( c'' \) be a coloring of \( S_{n+1,2} \) such that \( C_1'' = \{2, 4\}, C_2'' = \{2, 3\}, \) and \( C_3'' = \{4, 3\} \), and \( c''' \) a coloring of \( S_{n+1,3} \) such that \( C_1''' = \{1, 4\}, C_2''' = \{1, 2\}, \) and \( C_3''' = \{4, 2\} \), see the right-hand side of Fig. 4. Again, there colorings exist by the induction assumption and by appropriate permutations of colors. Combine \( c', c'', \) and \( c''' \) to color \( S_{n+1} \) and exchange colors 1 and 2 to obtain a desired coloring of \( S_{n+1} \).
4 On codes and Hamiltonicity

In the final section we present two additional aspects of Sierpiński (gasket) graphs.

It is proved in [15] that for every $n \geq 4$, $\gamma(S_n) = 3\gamma(S_{n-1})$. This enables us to quickly prove the following result that is in a strike contrast to the fact already mentioned in the introduction that every Sierpiński graph $S(n,k)$ contains essentially a unique 1-perfect code. Recall that if $C$ is a 1-perfect code of a graph $G$, then $|C| = \gamma(G)$, see [4, Theorem 4.2].

**Proposition 4.1** $S_n$ contains a 1-perfect code if and only if $n = 1$ or $n = 3$.

**Proof.** It is straightforward to verify the result for $n \leq 3$.

Let $n \geq 4$ and suppose that $C$ is a 1-perfect code of $S_n$. Assume $(1\ldots 1) \in C$ and consider the vertex $(1\ldots 1)\{2,3\}$ with the prefix of length $n - 2$. Then it can be dominated either with itself, with $(1\ldots 1)\{1,2\}$, or with $(1\ldots 1)\{1,3\}$, where the last two vertices have prefixes of length $n - 3$. However, none of these vertices qualifies for $C$, so $(1\ldots 1) \notin C$. Analogously $(2\ldots 2) \notin C$ and $(3\ldots 3) \notin C$. Hence every vertex of $C$ is of degree 4 and as $S_n$ contains $3(3^{n-1}+1)/2$ vertices, we infer that $|C| = 3(3^{n-1}+1)/10$. Since $|C| = \gamma(S_n)$, the above result of Tegúia and Godbole implies that $\gamma(S_n) = |C| = 3^{n-2}$ for $n \geq 3$. But then $3(3^{n-1}+1)/10 = 3^{n-2}$, which reduces to $3^{n-2} = 3$ with $n = 3$ as the solution. \hfill □

We conclude the paper by proving that the Sierpiński graphs $S(n,3)$ contain unique Hamiltonian cycles. For this sake we first show:

**Lemma 4.2** Let $n \geq 1$ and let $u,v$ be extreme vertices of $S(n,3)$. Then there exists a unique Hamiltonian $u,v$-path.

**Proof.** The statement is clearly true for $n = 1$. Suppose it holds for $n \geq 2$ and without loss of generality consider the extreme vertices $(1\ldots 1)$ and $(2\ldots 2)$ of $S(n+1,3)$. By the induction assumption, there exists a unique Hamiltonian path $P$ between $(1\ldots 1)$ and $(13\ldots 3)$ in $S(n+1,3)_1$, a unique Hamiltonian path $Q$ between $(31\ldots 1)$ and $(32\ldots 2)$ in $S(n+1,3)_3$, and a unique Hamiltonian path $S$ between $(23\ldots 3)$ and $(2\ldots 2)$ in $S(n+1,3)_2$. Then

$$(1\ldots 1)P(13\ldots 3)(31\ldots 1)Q(32\ldots 2)(23\ldots 3)S(2\ldots 2)$$

is a Hamiltonian path in $S(n+1,3)$. To see that it is unique, observe that $(12\ldots 2)$ must appear before $(13\ldots 3)$ on any Hamiltonian $(1\ldots 1),(2\ldots 2)$-path. Indeed, suppose this is not the case. Then if we proceed from $(13\ldots 3)$ to $(31\ldots 1)$, the vertex $(12\ldots 2)$ would appear on the Hamiltonian path just after $(21\ldots 1)$ which is clearly not possible. And if we proceed from $(13\ldots 3)$ to a vertex of $S(n+1,3)_1$, then the vertex $(32\ldots 2)$ would appear on the Hamiltonian path.
just after \( \langle 23 \ldots 3 \rangle \), which is also not possible. Similarly, \( \langle 3 \ldots 3 \rangle \) must appear before \( \langle 32 \ldots 2 \rangle \) on any Hamiltonian \( \langle 1 \ldots 1 \rangle, \langle 2 \ldots 2 \rangle \)-path. Induction completes the argument. \( \Box \)

**Theorem 4.3** \( S(n,3), n \geq 1 \), contains a unique Hamiltonian cycle.

**Proof.** The case \( n = 1 \) is trivial. For \( n > 1 \) construct a Hamiltonian cycle of \( S(n,3) \) by combining a Hamiltonian \( \langle 12 \ldots 2 \rangle, \langle 13 \ldots 3 \rangle \)-path in \( S(n,3)_1 \), a Hamiltonian \( \langle 31 \ldots 1 \rangle, \langle 32 \ldots 2 \rangle \)-path in \( S(n,3)_3 \), and a Hamiltonian \( \langle 23 \ldots 3 \rangle, \langle 21 \ldots 1 \rangle \)-path in \( S(n,3)_2 \). By Lemma 4.2, this Hamiltonian cycle is unique. \( \Box \)

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**References**


