On graphs with small game domination number

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Abstract

The domination game is played on a graph G by Dominator and Staller. The two players are taking turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated; the game ends when no move is possible. The game is called D-game when Dominator starts it, and S-game otherwise. Dominator wants to finish the game as fast as possible, while Staller wants to prolong it as much as possible. The game domination number $\gamma_g(G)$ of G is the number of moves played in D-game when both players play optimally. Similarly, $\gamma_g'(G)$ is the number of moves played in S-game.

Graphs G with $\gamma_g(G) = 2$, graphs with $\gamma_g'(G) = 2$, as well as graphs extremal with respect to the diameter among these graphs are characterized. In particular, $\gamma_g'(G) = 2$ and $\mathrm{diam}(G) = 3$ hold for a graph G if and only if G is a so-called gamburger. Graphs G with $\gamma_g(G) = 3$ and $\mathrm{diam}(G) = 6$, as well as graphs G with $\gamma_g'(G) = 3$ and $\mathrm{diam}(G) = 5$ are also characterized. The latter can be described as the so-called double-gamburgers.

Keywords: domination game; game domination number; diameter; gamburger

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1 Introduction

The domination game is played on an arbitrary graph G by *Dominator* and *Staller*. The two players are taking turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated. The game ends when no move is possible and the score of the game is the total number of vertices chosen. Dominator wants to minimize the score, while Staller wants to maximize it. By *D-game* we mean a game in which Dominator has the first move and by *S-game* a game started by Staller.

Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ of a graph G denotes the score of D-gameplayed on G. Similarly, the *Staller-start game domination number* $\gamma_g'(G)$ is defined as the score of optimal S-game.

The game was introduced in [4] and already received a considerable attention. One of the reasons for this interest is the so-called 3/5-conjecture from [15] asserting that $\gamma_g(G) \leq 3n/5$ holds for any isolate free graph of order n. Trees that attain this bound were investigated in [5], while recently Bujtas [7] made a breakthrough by proving that the conjecture holds for all graphs with the minimum degree at least 3. In order to achieve this result she further developed the proof technique introduced in [6] where the conjecture is verified for all trees in which no two leaves are at distance 4. Henning and Kinnersley [11] further established the truth of the 3/5-conjecture over the class of graphs with minimum degree at least 2, hence the 3/5-conjecture remains open only for graphs with pendant vertices. We point out that Bujtas' proof technique already turned out to be useful elsewhere [8].

Recently, two closely related games were introduced. The total version of the domination game was investigated in [12] and further studied in [13] where it was proved that for any graph of order n in which every component contains at least three vertices, the corresponding total invariant is bounded above by $4n/5$. The second related game, named the *disjoint domination game*, was studied in [9]. Among the additional investigations of the domination game we mention the complexity studies from [1], the behaviour of the game played on the disjoint union of graphs [10], and a characterization of trees T for which $\gamma_g(T) = \gamma(T)$ holds [17], where $\gamma(T)$ is the usual domination number of T.

Motivated in part by the characterization of graphs with $\gamma_g = 3$ and $\gamma_g' = 2$ from [2] and by the complexity studies from [1], we study in this paper graphs that have small game domination number. In the next section we introduce notations needed, recall some results, and bound the diameter of a graph from above in terms of the game domination number (see [3, Corollary 4.1] for a closely related result). In Section 3 we first characterize graphs G with $\gamma_g(G) = 2$ and graphs with $\gamma_g'(G) = 2$. We also characterize graphs extremal with respect to the diameter among these graphs. In particular, we introduce the concept of the so-called gamburger and prove that G has $\gamma_g'(G) = 2$ and $\mathrm{diam}(G) = 3$ if and only if G is a gamburger. In Section 4 we then characterize extremal graphs (w.r.t. the diameter) in the class of graphs with the game domination number equal 3. In particular, $\gamma_g'(G) = 3$ and $\mathrm{diam}(G) = 5$ hold for a graph G if and only if G is the so-called double-gamburger.

2 Preliminaries

If x is a vertex of a graph G, then $N_G(x)$ (resp. $N_G[x]$) denotes the neigborhood (resp. closed neighborhood) of x. Let $d_G(x, y)$ be the standard shortest-path distance between vertices x and y of G. The eccentricity $\mathrm{ecc}_G(x)$ of x is $\max\{d(x, y) : y \in V(G)\}$ and the diameter $\mathrm{diam}(G)$ of G is the maximum eccentricity of its vertices. $S_r^G(x) = \{y \in V(G) : d_G(x, y) = r\}$ is called the sphere with center x and radius r and

$B_r^G(x) = \{y \in V(G) : \ d_G(x,y) \le r\}$ is called the ball with center x and radius r. If G will be clear from the context, we will simplify the above notations to $N(x)$, $N[x]$, $d(x,y)$, $S_r(x)$, $B_r(x)$, and $\mathrm{ecc}(x)$.

We next collect some known results (or part of the folklore results) to be used later on. A fundamental result about the domination game is the following theorem for which the fact that $\gamma_g(G) \le \gamma_g'(G) + 1$ holds was proved in [4], while the inequality $\gamma_g'(G) \le \gamma_g(G) + 1$ was later established in [15].

Theorem 2.1 [4, 15] *For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \le 1$.*

If $(\gamma_g(G), \gamma_g'(G)) = (k, \ell)$ then we say that G *realizes* (k, ℓ). By Theorem 2.1, if G realizes the pair (k, ℓ), then $|k - \ell| \le 1$.

Vertices u and v of a graph G are called *twins* if $N[u] = N[v]$. Note that twins are necessarily adjacent. A graph is called *twin-free* if it contains no twins. The following result is not difficult to prove and was implicitly or explicitly (cf. [2]) used earlier and is also stated in [17].

Proposition 2.2 *If u and v are twins in a graph G, then $\gamma_g(G) = \gamma_g(G - u)$ and $\gamma_g'(G) = \gamma_g'(G - u)$.*

A subgraph H of a graph G is *guarded in* G if for any vertex x in G there exists a vertex $y \in V(H)$ such that $N[x] \cap V(H) \subseteq N[y] \cap V(H)$. The vertex y is called a *guard* of x in H. (If $x \in V(H)$, then x is a guard of itself.) The concept of a guarded subgraph was introduced in [3] where the following result was proved:

Theorem 2.3 *If H is guarded in G, then $\gamma_g(H) \le \gamma_g(G)$ and $\gamma_g'(H) \le \gamma_g'(G)$.*

To bound the diameter of a graph with its game domination number we recall the following result proved in [14], cf. also [16].

Proposition 2.4 *If $n \ge 1$, then*

(i) $$\gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \pmod 4, \\ \lceil \frac{n}{2} \rceil; & \textit{otherwise}. \end{cases}$$

(ii) $\gamma_g'(P_n) = \lceil \frac{n}{2} \rceil$.

We are now ready for the announced bound that was not earlier reported (at least explicitly) in the literature, hence we include its proof.

Proposition 2.5 *If G is a graph, then*

(i) $$\mathrm{diam}(G) \le \begin{cases} 2\gamma_g(G); & \gamma_g(G) \ \textit{odd}, \\ 2\gamma_g(G) - 1; & \textit{otherwise}. \end{cases}$$

(ii) $\mathrm{diam}(G) \le 2\gamma_g'(G) - 1$.

Moreover, the bounds are tight.

Proof. Let G be a graph and let P be a diametrical path of G. Then P is a shortest path isomorphic to $P_{\mathrm{diam}(G)+1}$ and by the proof of [3, Corollary 4.1], P is guarded in G. Hence by Theorem 2.3, $\gamma_g(G) \geq \gamma_g(P)$ and $\gamma_g'(G) \geq \gamma_g'(P)$. Using the latter inequality and Proposition 2.4 we get

$$\gamma_g'(G) \geq \gamma_g'\left(P_{\mathrm{diam}(G)+1}\right) = \left\lceil \frac{\mathrm{diam}(G)+1}{2} \right\rceil \geq \frac{\mathrm{diam}(G)+1}{2},$$

which proves the assertion (ii).

Let $\gamma_g(G)$ be odd. Assume by way of contradiction that $\mathrm{diam}(G) \geq 2\gamma_g(G)+1$. Then G contains a shortest path P' of order $2\gamma_g(G)+2$ and hence $\gamma_g(G) \geq \gamma_g(P') = \lceil (2\gamma_g(G)+2)/2 \rceil = \gamma_g(G)+1$, a contradiction. Hence $\mathrm{diam}(G) \leq 2\gamma_g(G)$ holds when $\gamma_g(G)$ is odd.

Let $\gamma_g(G)$ be even and suppose that $\mathrm{diam}(G) \geq 2\gamma_g(G)$ holds. Then G contains a shortest path P'' of order $2\gamma_g(G)+1$. Since $\gamma_g(G)$ is even, $2\gamma_g(G)+1 \equiv 1 \pmod 4$, and therefore, $\gamma_g(G) \geq \gamma_g(P'') = \lceil (2\gamma_g(G)+1)/2 \rceil = \gamma_g(G)+1$, a contradiction. We conclude that $\mathrm{diam}(G) \leq 2\gamma_g(G)-1$ holds when $\gamma_g(G)$ is even.

That the bounds are tight follows by considering the path graphs. More precisely, for any $k \geq 1$ we have

- $\gamma_g(P_{4k}) = 2k$ and $\mathrm{diam}(P_{4k}) = 4k-1 = 2\gamma_g(P_{4k})-1$,
- $\gamma_g(P_{4k+3}) = 2k+1$ and $\mathrm{diam}(P_{4k+3}) = 4k+2 = 2\gamma_g(P_{4k+3})$, and
- $\gamma_g'(P_{2k}) = k$ and $\mathrm{diam}(P_{2k}) = 2k-1 = 2\gamma_g(P_{2k})-1$.

□

3 Graphs G with $\gamma_g(G) = 2$ or $\gamma_g'(G) = 2$

In this section we study the structure of graphs G with $\gamma_g(G) = 2$ or $\gamma_g'(G) = 2$. Before doing it, we observe that

- $\gamma_g(G) = 1$ if and only if $\Delta(G) = n-1$, and
- $\gamma_g'(G) = 1$ if and only if G is a complete graph.

Note also that if $\gamma_g(G) = 2$ and G is not connected, then G consists of two components, one containing a universal vertex and the other being complete. Moreover, if $\gamma_g'(G) = 2$ and G is not connected, then G consists of two complete components. To characterize connected graphs with $\gamma_g(G) = 2$ we introduce the following concept. We say that a vertex u of a (connected) graph G is 2-*dense*, if

- $\mathrm{ecc}(u) = 2$,

- there is a join between $S_2(u)$ and the neighborhood of $S_2(u)$ in $S_1(u)$, and
- $S_2(u)$ induces a clique.

Let u be a 2-dense vertex of G and let $L(u)$ be the neighbors of u that are at distance 3 from $S_2(u)$, see Fig. 1. In other words, $L(u) = S_3(S_2(u))$. We point out that it is possible that $L(u) = \emptyset$.

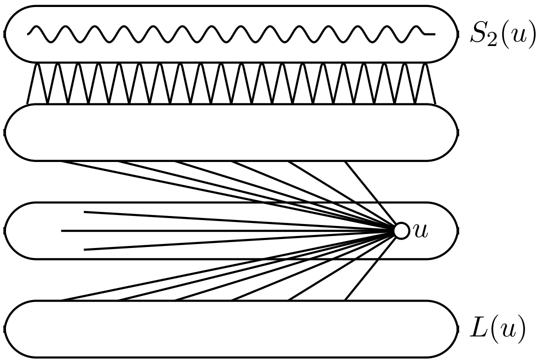

Figure 1: A 2-dense vertex u

Proposition 3.1 *If G is a connected graph, then the following hold.*

(i) $\gamma_g(G) = 2$ *if and only if* G *contains a* 2*-dense vertex.*

(ii) $\gamma'_g(G) = 2$ *if and only if* G *is not complete and every vertex lies in a dominating set of order* 2.

Proof. (i) Suppose first that $\gamma_g(G) = 2$ and let u be an optimal first move of Dominator. Then $\mathrm{ecc}(u) \geq 2$, for otherwise Dominator could play a universal vertex and finish the game in one move. Since Staller, by playing a vertex in $S_1(u)$, dominates no vertex in $S_3(u)$ it follows that $\mathrm{ecc}(u) = 2$. If there would exist two nonadjacent vertices in $S_2(u)$, then Staller could play any one of them in order to prolong the game for one more move. Hence $S_2(u)$ must induce a clique. Assume finally that there is a vertex $x \in S_1(u)$ that is adjacent to $y \in S_2(u)$ and not adjacent to $z \in S_2(u)$. Then Staller can play x (or z) in order to prolong the game for one more move. Therefore there is no such vertex z, that is, there is a join between $S_2(u)$ and the neighborhood of $S_2(u)$ in $S_1(u)$. We conclude that u is a 2-dense vertex.

Conversely, suppose that G contains a 2-dense vertex u. Then it is straightforward to see that Dominator forces Staller to finish the game in the next move by playing u. Hence $\gamma_g(G) \leq 2$. On the other hand, $\mathrm{ecc}(u) = 2$ implies that $\mathrm{diam}(G) \geq 2$ which in turn implies that $\gamma_g(G) \geq 2$.

(ii) Suppose that $\gamma'_g(G) = 2$. Then G is clearly not complete. Let next u be an arbitrary vertex of G. There is nothing to prove if u is a universal vertex (just add any vertex to u to form a dominating set). If u is not universal, then if Staller plays u as the first move, Dominator has a reply v such that the game is over after this move. But then $\{u, v\}$ is a dominating set.

Conversely, suppose that G is not complete and every vertex lies in a dominating set of order 2. Then $\gamma_g'(G) \geq 2$ because G is not complete. Moreover, $\gamma_g'(G) \leq 2$ because after an arbitrary first move u of Staller, Dominator can play v, where $\{u, v\}$ is a dominating set. $\square$

If $\gamma_g(G) = 2$ or $\gamma_g'(G) = 2$, then G realizes one of the pairs $(1, 2)$, $(2, 2)$, $(3, 2)$, or $(2, 3)$. Among the twin-free graphs, the classes of graphs that characterize the first three pairs can be described in the following simple way.

Observation 3.2 *Let G be a connected, twin-free graph with $\gamma_g'(G) = 2$. Then*

(i) G realizes $(1, 2)$ if and only if $\Delta(G) = n - 1$.

(ii) G realizes $(2, 2)$ if and only if $\Delta(G) = n - 2$.

(iii) G realizes $(3, 2)$ if and only if $\Delta(G) \leq n - 3$.

Proof. The statement (i) is clear, (iii) follows from [2, Corollary 3.2], and then (ii) characterizes the remaining connected, twin-free graphs G with $\gamma_g'(G) = 2$. $\square$

The last pair to consider is $(2, 3)$. It is straightforward to see that G realizes the pair $(2, 3)$ if and only if $\gamma(G) = 2$ and G contains a vertex that is in no minimum dominating set.

By Proposition 2.5, if $\gamma_g(G) = 2$ or $\gamma_g'(G) = 2$, then $\mathrm{diam}(G) \leq 3$. In the rest of the section we give an explicit description of the structure of related extremal classes of graphs, that is, those with the diameter equal to 3. For γ_g Proposition 3.1 immediately implies:

Corollary 3.3 *Let G be a connected graph. Then $\gamma_g(G) = 2$ and $\mathrm{diam}(G) = 3$ if and only if G contains a 2-dense vertex u such that $L(u) \neq \emptyset$.*

In order to characterize graphs with $\gamma_g'(G) = 2$ and $\mathrm{diam}(G) = 3$, we introduce the following concept. We say that a connected graph G is a *gamburger with a gamburger structure* Q_1, T_1, T_2, Q_2, if G is the disjoint union of non-empty subgraphs Q_1, T_1, T_2, and Q_2, where Q_1 and Q_2 induce cliques with no edges between them and, in addition, the following hold for any $i \in \{1, 2\}$ (see Fig. 2).

- There is a join between Q_i and T_i and there are no edges between Q_i and T_{3-i}.
- For any vertex $x \in T_i$ there exists a vertex $x' \in T_{3-i} \cup Q_{3-i}$ such that $T_1 \cup T_2 \subseteq N[x] \cup N[x']$.

We first show that every gamburger can be presented in a canonical form.

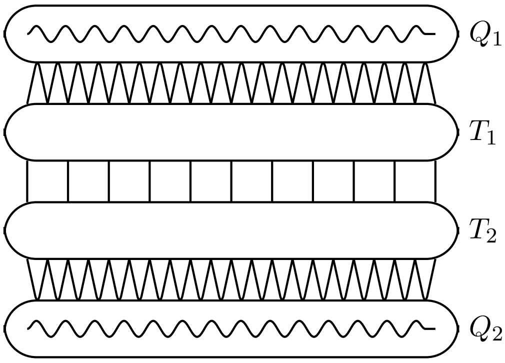

Figure 2: Gamburger (with cliques Q_1 and Q_2)

Lemma 3.4 *If G is a gamburger with a burger structure Q_1, T_1, T_2, Q_2, then there exists a gamburger structure Q'_1, T'_1, T'_2, Q'_2 for G such that any vertex from T'_i, $1 \leq i \leq 2$, has at least one neighbor in T'_{3-i}.*

Proof. Let Q_1, Q_2, T_1, T_2 be a gamburger structure for G. For $i \in \{1, 2\}$, we define W_i as the subset of vertices of T_i whose neighborhood is included in $T_i \cup Q_i$. We put $Q'_i = Q_i \cup W_i$ and $T'_i = T_i \setminus W_i$. Note that, since G is connected $W_i \neq T_i$ and then T'_i is not empty. To show that this defines a new gamburger structure for G, we only need to show that $N[w] = Q_i \cup T_i$ holds for all $w \in W_i$. Suppose, this is not the case. Since there is a join between Q_i and T_i, this means that there exists $x \in T_i$, which is not a neighbor of w. Since G is a gamburger, by the definition there exists a vertex $x' \in G \setminus (T_i \cup Q_i)$ such that w belongs to $N(x')$. This is a contradiction with the definition of W_i. In conclusion, $N[w] = Q_i \cup T_i$ and Q'_i induces a clique. $\square$

Lemma 3.5 *A gamburger has diameter* 3.

Proof. Since Q_1 and Q_2 are not empty, $\mathrm{diam}(G) \geq 3$. By Lemma 3.4 we can assume that every vertex in T_1 (resp. T_2) has a neighbor in T_2 (resp. T_1) which in turn implies $\mathrm{diam}(G) \leq 3$. $\square$

We can now prove that gamburgers are precisely the graphs G with $\gamma'_g(G) = 2$ and extremal diameter.

Theorem 3.6 *A graph G has $\gamma'_g(G) = 2$ and* $\mathrm{diam}(G) = 3$ *if and only if G is a gamburger.*

Proof. Assume first that G is a gamburger. By Lemma 3.5, $\mathrm{diam}(G) = 3$. We prove that $\gamma'_g(G)$ is 2. Since $\mathrm{diam}(G) = 3$, $\gamma'_g(G)$ is clearly greater or equal to 2. First, suppose Staller selects a vertex in one of the two cliques, says Q_1. Dominator ends the game at his turn, by playing any vertex in Q_2. Suppose next that Staller does not play in one of the two cliques, but says on $x \in T_1$. There is a vertex $x' \in T_2 \cup Q_2$ such that

$N[x] \cup N[x']$ contains $T_1 \cup T_2$. Since no vertex in Q_2 is dominated, playing x' is a legal move, which enables Dominator to end the game at his turn.

Conversely, suppose that $\gamma_g'(G) = 2$ and $\mathrm{diam}(G) = 3$. Let u be an arbitrary diametrical vertex of G. We set $Q_1 = \{u\}$, $T_1 = S_1(u)$, $T_2 = S_2(u)$ and $Q_2 = S_3(u)$. These four sets obviously form a partition of $V(G)$. We claim that Q_1, T_1, T_2, Q_2 is a gamburger structure for G. It is clear that there are no edges that would violate the gamburger structure, that is, there are no edges between Q_1 and Q_2 and no edges between Q_i and T_{3-i} for $1 \le i \le 2$. It is also clear that Q_1 induced a (one-vertex) clique and hence that there is a join between Q_1 and T_1. Whatever Staller plays as her first move, Dominator has to be able to end the game in the next turn. If Staller plays in Q_2, Dominator will not be able at his turn to simultaneously dominate vertices in Q_1 and Q_2. Hence, all the vertices of Q_2 must be dominated by such a Staller's first move. It follows that Q_2 induces a clique. Assume next that Staller plays a vertex $y \in T_2$. In order to dominate the vertex in Q_1, Dominator has to play in $T_1 \cup Q_1$. Hence, all the vertices of Q_2 are adjacent to y. Since y was an arbitrary vertex of T_2, it follows that there is a join between Q_2 and T_2. Consider now an optimal move y' of Dominator. As already observed, y' belongs to $T_1 \cup Q_1$. Since the game ends with this move, it follows that $T_1 \cup T_2 \subseteq N[y] \cup N[y']$. Similarly, for all $x \in T_1$, there is a vertex $x' \in T_2 \cup Q_2$ such that $T_1 \cup T_2 \subseteq N[x] \cup N[x']$. In conclusion, G is a gamburger and Q_1, T_1, T_2, Q_2 is a gamburger structure for G. $\square$

If a gamburger G with a gamburger structure Q_1, T_1, T_2, Q_2 contains a vertex $x \in T_1$ such that $T_1 \cup T_2 \subseteq N[x]$, then we say that G is a *full-gamburger*. We will also say that the vertex x is a *full vertex* of G.

With this definition we can specialize Theorem 3.6 to $(2,2)$-graphs as follows.

Corollary 3.7 *A graph G is a $(2,2)$-graph with $\mathrm{diam}(G) = 3$ if and only if G is a full-gamburger.*

Proof. Assume first that G is a full-gamburger. By Theorem 3.6, $\gamma_g'(G) = 2$ and $\mathrm{diam}(G) = 3$. Note that a full vertex x from T_1 is 2-dense. Hence $\gamma_g(G) = 2$ holds by Proposition 3.1.

Conversely, suppose G is a $(2,2)$-graph with diameter 3. By Theorem 3.6, G has a gamburger structure Q_1, T_1, T_2, Q_2. Moreover, by Lemma 3.4 we may without loss of generality assume that every vertex in T_i has a neighbor in T_{3-i}, $1 \le i \le 2$. Let x be an optimal first move of Dominator in D-game. Then x belongs to $T_1 \cup T_2$, for if x would belong to Q_i, Staller could play in T_i without ending the game. Assume without loss of generality that $x \in T_1$ and suppose that $N[x]$ does not contain $T_1 \cup T_2$. Let y be a vertex of $T_1 \cup T_2$ which is not in $N[x]$. If y belongs to T_1, Staller can play y and this will not end the game. If y belongs to T_2, we know that y has a neighbor in T_1, say y'. Clearly, y' is distinct from x, hence playing y' is a legal move because it newly dominates y. In both cases, we thus have a contradiction with $\gamma_g(G) = 2$. We conclude that $T_1 \cup T_2 \subseteq N[x]$ and hence x is a full vertex of the full-gamburger G. $\square$

4 On graphs with game domination number 3

The class of graphs G with $\gamma_g(G) = 3$ seems too rich to allow some nice characterization. In [2] the subclass of these graphs with the property that $\gamma'_g(G) = 2$ was characterized. For instance, the cycles C_5 and C_6 both belong to this class. In view of Proposition 2.5, in the first part of this section we characterize the subclass of $\gamma_g = 3$ graphs extremal with respect to the diameter, that is graphs with $\gamma_g(G) = 3$ and $\mathrm{diam}(G) = 6$. For instance, this class contains P_7 and is disjoint from the above class. That the two classes are disjoint follows from facts that $\gamma(G) = 2$ holds for all graphs G from the first class and $\gamma(G) \geq 3$ for all graphs G from the second class. In Fig. 3 the situation is resumed.

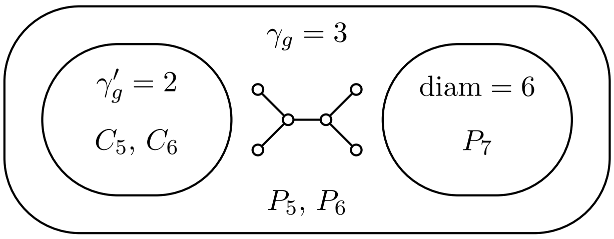

Figure 3: Subclasses of $\gamma_g = 3$ graphs

In the second part of the section we then characterize the graphs G with $\gamma'_g(G) = 3$ and $\mathrm{diam}(G) = 5$.

4.1 Graphs with $\gamma_g = 3$ and $\mathrm{diam} = 6$

Full-burgers turned out to be useful also for the first main result of this section. In addition to this concept, we also introduce the following. If G is a connected graph, then a vertex u of G is *nice* if the following conditions are fulfilled.

- There exists $v_1 \in S_1(u)$ such that $N[v_1] = B_2(u)$.
- There is a join between $N(S_3(u)) \cap S_2(u)$ and $S_3(u)$.
- $S_3(u)$, $S_4(u)$, $S_5(u)$, $S_6(u)$ is a full-gamburger structure with a full vertex in $S_5(u)$.

With these notations, the announced result reads as follows.

Theorem 4.1 *If G is a connected graph, then the following statements are equivalent.*

(i) $\gamma_g(G) = 3$ *and* $\mathrm{diam}(G) = 6$.

(ii) Any diametrical pair of vertices contains at least one nice vertex.

(iii) There exists a nice diametrical vertex.

Proof. We first prove that (i) implies (ii). Let u_1 and u_2 be vertices of G with $d(u_1, u_2) = 6$ and let P be a shortest u_1, u_2-path. Since $\gamma_g(G) = 3 = \gamma_g(P_7)$, and because the two neighbors of pendant vertices of P_7 are the only optimal start moves for Dominator when playing D-game on P_7, we infer that Dominator's first move must be either in $S_1(u_1)$ or in $S_1(u_2)$. Indeed, for otherwise Staller could guarantee (by considering the game restricted to P) the game to last at least four moves. Let $i \in \{1, 2\}$ be such such that we have an optimal move for Dominator in $S_1(u_i)$. For convenience, we will write S_j, $0 \le j \le 6$, instead of $S_j(u_i)$. Set also $u = u_i$.

Since the diameter of the graph is 6, the spheres S_j are non-empty and form a partition of $V(G)$. We are going to prove several claims that will establish the structure as described in (ii). Let $x \in S_1$ be an optimal first move of Dominator.

Claim 1. There exists a vertex $v_1 \in S_1$ such that $N[v_1] = B_2(u)$.
Staller's move could be in S_4. To finish the game in three turns, Dominator must then play in $S_5 \cup S_6$. Hence, his first move x must dominate the whole $B_2(u)$ and therefore $x = v_1$ is a required vertex.

Claim 2. There is a join between $N(S_3) \cap S_2$ and S_3.
Staller could play her first move anywhere in $N(S_3) \cap S_2$. Such moves are actually legal because they dominate at least one new vertex in S_3. In that case Dominator has to answer in S_5. Hence, all the vertices in S_3 must have been dominated by the Staller's move. Since this move of Staller in $N(S_3) \cap S_2$ was arbitrary, we have a join between $N(S_3) \cap S_2$ and S_3.

It remains to prove that S_3, S_4, S_5, S_6 form a full-gamburger structure with a full vertex in S_5.

Claim 3. S_3 and S_6 are cliques.
Assume Staller plays first in S_3 (resp. S_6). Dominator must play next in $S_5 \cup S_6$ (resp. $S_3 \cup S_4$). Since this move of Dominator has to end the game, all the vertices of S_3 (resp. S_6) must have been dominated by the move of Staller. This move could be arbitrary in the sphere, because x dominates no vertex in S_3. It follows that S_3 (resp. S_6) induces a clique.

Claim 4. There is a join between S_3 and S_4 and between S_5 and S_6. Moreover, for any vertex y in S_4 (resp. in S_5) there is a vertex y' in $S_5 \cup S_6$ (resp. in $S_3 \cup S_4$) such that $N[y] \cup N[y']$ contains $S_4 \cup S_5$.
Suppose Staller plays a vertex y in S_4. Dominator must play next in $S_5 \cup S_6$. Hence, all the vertices of S_3 must have been dominated by the move of Staller. Because y is an arbitrary vertex from S_4, there must be a join between S_3 and S_4. Let now y' in $S_5 \cup S_6$ be an optimal answer of Dominator. Vertices of $S_4 \cup S_5$ could only be dominated by the last two moves of the game, that is, by y and y'. Therefore, these two spheres are contained in $N[y] \cup N[y']$. Similarly, when Staller plays as her first move a vertex y from S_5, Dominator replies by playing a vertex $y' \in S_3 \cup S_4$ and we get the same conclusion.

Until now we have already seen that S_3, S_4, S_5, S_6 defines a gamburger structure. Hence it remains to prove the following.

Claim 5. The gamburger on S_3, S_4, S_5, S_6 has a full vertex in S_5.
As in Claim 2, assume that Staller plays in S_2. Dominator has to answer in S_5 and must dominate all the vertices in S_4, S_5 and S_6 with this move. Therefore, such a reply of Dominator in S_5 is a full vertex of the gamburger with the gamburger structure S_3, S_4, S_5, S_6.

In conclusion we have proved that u is a nice vertex.

Since (ii) trivially implies (iii), it remains to prove that (iii) implies (i). Assume that there exists a nice diametrical vertex u. As above, we simply write S_i instead of $S_i(u)$, for any $i \in \{0, \dots, 6\}$. Let v_1 be a vertex in S_1 such that $N[v_1] = B_2(u)$.

We prove first that $\text{diam}(G) = 6$. Since S_3, S_4, S_5, S_6 induce a gamburger structure, none of these spheres is empty. Hence $\text{ecc}(u) = 6$ and $\text{diam}(G) \geq 6$. By Lemma 3.5, the subgraph induced by S_3, S_4, S_5, S_6 has diameter 3. Moreover, using the vertex $v_1 \in S_1$, it is easy to show that all the vertices from $B_2(u)$ are at distance at most 3 from any vertex in S_3. In conclusion, G has diameter 6.

Since $\text{diam}(G) = 6$, we have $\gamma_g(G) \geq 3$. Hence, to conclude the proof we need to show that Dominator could ensure the game to end in three turns. His strategy is to play v_1 as the first move. After that, all the vertices in $B_2(u)$ are dominated and the remaining legal moves for Staller are in $G \backslash B_1(u)$. First assume Staller plays in $G \backslash B_2(u)$. Since these vertices induce a gamburger, Theorem 3.6 implies that Dominator has an answer in $G \setminus B_2(u)$, such that all the vertices of this subgraph are dominated after this move that clearly ends the game. Second, if Staller plays in S_2, then to be legal, this move has to belong to $N(S_3)$. Because of the join between $N(S_3) \cap S_2$ and S_3, all the vertices in S_3 are dominated. Dominator can then finish the game by playing a full vertex in S_5. $\square$

We point out that both characterizations from Theorem 4.1 of graphs with $\gamma_g(G) = 3$ and $\text{diam}(G) = 6$ are useful if we wish to give a fast recognition algorithm for these graphs. Indeed, we only need to select a pair of diametrical vertices and check if one of them leads to the structure as described in (iii). If not, then we know from (ii) that no other diametrical vertex can give us the desired structure.

4.2 Graphs with $\gamma_g' = 3$ and diam $= 5$

To characterize the graphs from the title, we introduce one more concept. We say that a connected graph G is a *double-gamburger*, if G is the disjoint union of non-empty subgraphs $Q_1, R_1, T_1, T_2, R_2, Q_2$, for which the following hold for any $i \in \{1, 2\}$.

- Q_i and T_i induce cliques.
- There are joins between Q_i and R_i, and between R_i and T_i.
- R_i induces a clique minus a matching M_i.
- If M_i is perfect, then M_{3-i} is empty and there is a join between T_1 and T_2.
- There exists a vertex $x_i \in T_i$, such that $T_{3-i} \subseteq N(x_i)$.

- There are no edges between Q_i and $G \setminus R_i$ and between R_i and $G \setminus (Q_i \cup T_i)$.

See Fig. 4. We say that $Q_1, R_1, T_1, T_2, R_2, Q_2$ is a *double-gamburger structure* for G.

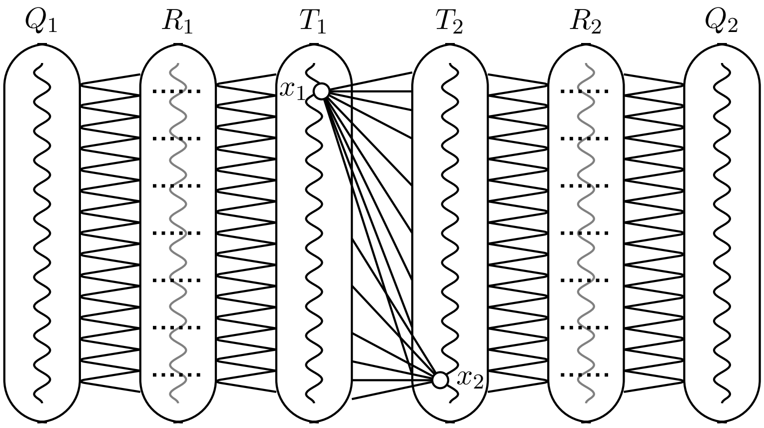

Figure 4: Double-gamburger

Lemma 4.2 *A double-gamburger has diameter* 5*.*

Proof. The graphs H_1 and H_2, respectively induced by Q_1, R_1, T_1 and T_2, R_2, Q_2 have diameter 2. For any $i \in \{1, 2\}$, we have a vertex $x \in T_i$, such that $T_{3-i} \subseteq N(x)$. Using these vertices, it is straightforward to see that the distance between a vertex in H_i and the subgraph H_{3-i} is 3. In conclusion, a double-gamburger has diameter 5. □

Theorem 4.3 *If G is a connected graph, then the following statements are equivalent.*

(i) $\gamma'_g(G) = 3$ *and* $\operatorname{diam}(G) = 5$*.*

(ii) For any diametrical vertex u, defining $Q_1 = S_0(u) \cup W_1$, $R_1 = S_1(u) \setminus W_1$, $T_1 = S_2(u)$, $T_2 = S_3(u)$, $R_2 = S_4(u)$ and $Q_2 = S_5(u)$, where $W_1 = S_1(u) \setminus N(S_2(u))$ is a double-gamburger structure for G.

(iii) G is a double-gamburger.

Proof. We first prove that (i) implies (ii). Let u be a vertex of G with $\operatorname{ecc}(u) = 5$. We are going to prove four claims which together establish that the subgraphs $Q_1, R_1, T_1, T_2, R_2, Q_2$ as defined in (*ii*) form a double-gamburger structure for G. Note first that these subgraphs are clearly not empty and there are no edges but the one permitted by the double-gamburger structure. For $i \in \{1, 2\}$, we denote by $\bar{i}$ the other element of $\{1, 2\}$.

Claim 1. For $i \in \{1, 2\}$, Q_i induces a clique, there is a join between Q_i and R_i and there exists $u \in T_{\bar{i}}$ such that $N(u)$ contains T_i.

Assume $x \in Q_i$ is the first move of Staller. If Dominator plays in $R_i \cup T_i$, Staller will be able to play her second move in $T_{\bar{i}}$. In that case, $Q_{\bar{i}}$ will remain undominated at the end of the third turn. If he plays in $Q_{\bar{i}}$, she could answer in $R_{\bar{i}}$ and T_i will not be dominated. Hence Dominator's optimal answer is either in $T_{\bar{i}}$ or in $R_{\bar{i}}$. In both cases, there is still a legal move for Staller in $T_{\bar{i}} \cup R_{\bar{i}}$. Since the game must end in three turns, this implies that $Q_i \cup R_i$ has been entirely dominated by x. Hence, Q_i is a clique and there is a join between Q_i and R_i. Now assume, that Dominator's move was in $T_{\bar{i}}$. Staller can play her last move in $R_{\bar{i}}$. So, the game could end in three turns only if Dominator's move dominates all T_i. On the other hand, if Dominator played in $R_{\bar{i}}$, Staller can play her last move in $T_{\bar{i}}$. Once more, this move must dominate all T_i. In conclusion, we have at least one vertex in $T_{\bar{i}}$ whose neighborhood contains T_i.

Claim 2. For $i \in \{1, 2\}$, T_i induces a clique and there is a join between T_i and R_i.
By way of contradiction, assume there exists $x \in T_i$, such that $N[x]$ does not contain $T_i \cup R_i$ and let y be a vertex in $(T_i \cup R_i) \setminus N[x]$. Staller can play the vertex x. If Dominator plays in $Q_i \cup R_i \cup T_i$, Staller will be able to play her second move in $T_{\bar{i}}$. If he plays in T_2, she could answer in Q_i. In both cases, the vertices in $Q_{\bar{i}}$ will not be dominated at the end of the third turn. Therefore, Dominator has to play in $R_{\bar{i}} \cup Q_{\bar{i}}$. Now, if $y \in T_i$, playing y is a legal move for Staller and after this move no vertex in Q_i is dominated. Otherwise, if $y \in R_i$, then this vertex has a neighbor $y' \in T_i$. Indeed, the way we define T_i and R_i ensures that $N(T_i) \setminus T_i = R_i$. Since playing y' will dominate the new vertex y, it is a legal move. By playing it, Staller ensures that no vertex in Q_i is already dominated. In both cases, the game is not over at the end of the third turn. This is a contradiction with the assumption $\gamma_g'(G) = 3$.

Claim 3. For $i \in \{1, 2\}$, R_i induces a clique from which a matching M_i has been removed.
By way of contradiction, suppose that R_i does not induced a clique minus a matching. In other words, there exist distinct vertices y, y' and x in R_i such that x is neither adjacent to y nor to y'. If Staller starts by playing y, Dominator's optimal answer is in $T_{\bar{i}} \cup R_{\bar{i}} \cup Q_{\bar{i}}$. Otherwise, Staller could play her second move in $T_{\bar{i}}$, and $Q_{\bar{i}}$ would remain undominated. After the move of Dominator, y' is still undominated. Hence Staller can play this vertex as her second move. Finally, after these three turns, x is not yet dominated, again contradicting $\gamma_g'(G) = 3$.

Claim 4. For $i \in \{1, 2\}$, if the matching M_i is perfect, then $M_{\bar{i}}$ is empty and there is a join between T_1 and T_2.
Assume first there is not a join between T_1 and T_2. Then Staller can choose a vertex in T_i, such that at least one vertex in $T_{\bar{i}}$ is not dominated. As in Claim 2, Dominator must play in $R_{\bar{i}} \cup Q_{\bar{i}}$. Since $T_{\bar{i}}$ is not completely dominated after Staller's first move, Dominator has no choice but to play in $R_{\bar{i}}$. Moreover, Staller could play her last move in $Q_i \cup R_i$. So all the vertices in $T_{\bar{i}} \cup R_{\bar{i}} \cup Q_{\bar{i}}$ must be dominated by Dominator's move. For we must have a vertex in $R_{\bar{i}}$ whose closed neighborhood contains at least $R_{\bar{i}}$. In conclusion, $M_{\bar{i}}$ is not a perfect matching.

Suppose now that $M_{\bar{i}}$ is not empty (i.e. $R_{\bar{i}}$ does not induce a clique). Staller can play a vertex in $R_{\bar{i}}$, whose closed neighborhood does not contain $R_{\bar{i}}$. As in Claim 3, Dominator has to play in $T_i \cup R_i \cup Q_i$. But, Staller would be in all cases able to make her last move in $R_{\bar{i}}$. Hence, the game could end in three turns only if Dominator can dominate all $T_i \cup R_i \cup Q_i$ in one move. As above, it implies that the matching M_i is not perfect.

Statement (ii) obviously implies (iii). It remains to prove that (iii) leads to (i). Assume that G has a double-gamburger structure. By Lemma 4.2, $\text{diam}(G) = 5$ which in turn implies that $\gamma'_g(G) \geq 3$. So, we only have to give a strategy for Dominator which ensures that the game ends in at most three turns. By symmetry, we have three cases.

Case 1. Staller's first move is in Q_1. Then Dominator chooses to play a vertex from T_2 whose neighborhood contains T_1. Clearly, after such a move, all the vertices of $G \setminus Q_2$ are dominated. Moreover, all the remaining legal moves for Staller are in $R_2 \cup Q_2$. Since there is a join between these two sets, whatever she plays, all the vertices in Q_2 will be dominated and the game will be over in three turns.

Case 2. Staller's first move is in R_1. They are two possibilities. First, the matching M_2 removed from a clique to get R_2 is perfect. Therefore R_1 is a clique. Dominator selects a vertex in R_2. All the vertices of G, except one in R_2, are dominated after this move. Staller has no choice but to end the game by selecting a vertex in the closed neighborhood of this last undominated vertex. Second, the matching M_2 is not perfect. Hence, there is a vertex $x \in R_2$ whose neighborhood contains R_2. Dominator chooses this vertex. Now, there is at most one undominated vertex in G, which is actually in R_1. Staller has to dominated this vertex and the game is over after her move.

Case 3. Staller's first move is in T_1. Suppose first that we have a join between T_1 and T_2. Dominator selects any vertex in Q_2. On the other hand, if there is no join between T_1 and T_2, then M_2 is not a perfect matching and we have a vertex $x \in R_2$ such that $N[x]$ contains $T_2 \cup R_2 \cup Q_2$. Dominator plays this vertex. In both situations, all the vertices in $G \setminus Q_1$ are now dominated. The legal moves for Staller are only in $Q_1 \cup R_1$. Since there is a join between these two sets, whatever she chooses the game will end after her move. □

As for Theorem 4.1, both characterizations of graphs with $\gamma'_g(G) = 3$ and $\text{diam}(G) = 5$ are useful for a fast recognition algorithm.

To conclude the paper we show that a double-gamburger$G$ always has $\gamma_g(G) = 3$. In other words, the following is true.

Corollary 4.4 *A graph G is a $(3,3)$-graph of diameter 5 if and only if G is a double-gamburger. In particular, there is no $(4,3)$-graph with diameter 5.*

Proof. By Theorem 4.3 we only need to prove that if G is a double-gamburger, then $\gamma'_g(G) = 3$. Since $\text{diam}(G) = 5$, [3, Corollary 4.1] implies that $\gamma_g(G) \geq 3$. Hence we only have to prove that Dominator can end the game in at most three turns.

Let M_1 and M_2 be the two matchings defined by the double-gamburger structure of G. Assume first that both are not perfect. Then Dominator can start by playing a vertex in R_1 which dominates all R_1. As his second move, he selects a vertex in R_2 which dominates all R_2. Such a vertex is still available, because Staller plays optimally and hence she did not select such a vertex. Playing this way, all the vertices will be dominated at the end of the third turn. On the other hand, if one of the matching, say M_1 is perfect, by the double-gamburger structure of G, R_2 is a clique and we have a join between T_1 and T_2. Dominator starts the game with any vertex of Q_1. If Staller answers in $R_1 \cup T_1 \cup T_2$, Dominator plays his second move in R_2. If Staller plays in $R_2 \cup Q_2$, then he plays in T_2. Note that these two cases are exhaustive, because the vertices in Q_1 are not legal moves for Staller. Finally, it is straightforward to show that in both cases all the vertices are dominated at the end of the third turn. $\square$

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