On graphs with largest possible game domination number

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Abstract

Let $\gamma(G)$ and $\gamma_g(G)$ be the domination number and the game domination number of a graph $G$, respectively. In this paper $\gamma_g$-maximal graphs are introduced as the graphs $G$ for which $\gamma_g(G) = 2\gamma(G) - 1$ holds. Large families of $\gamma_g$-maximal graphs are constructed among the graphs in which their supportive sets are minimum dominating sets. $\gamma_g$-maximal graphs are also characterized among the starlike trees, that is, trees which have exactly one vertex of degree at least 3.

Keywords: domination game; $\gamma_g$-maximal graph; supportive dominating set; starlike tree

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1 Introduction

If $G = (V(G), E(G))$ is a graph, then a vertex $u \in V(G)$ dominates a vertex $v \in V(G)$ if $u = v$ or $u$ is adjacent to $v$. $S \subseteq V(G)$ is a dominating set of $G$ if every vertex in $G$ is dominated by a vertex in $S$. The size of a smallest dominating set of $G$ is the domination number $\gamma(G)$ of $G$. A smallest dominating set will be briefly called a $\gamma$-set.

The domination game is played on a graph $G$ by two players that are usually called Dominator and Staller. They take turns choosing a vertex from $G$ such that at least one previously undominated vertex becomes dominated until no move is
possible. The score of the game is the total number of vertices chosen by them in this game. The players have opposite goals: Dominator wants to minimize the score and Staller wants to maximize it. A game is called a D-game (resp. S-game) if Dominator (resp. Staller) has the first move. The game domination number $\gamma_d(G)$ of $G$ is the score of a D-game played on $G$ assuming that both players play optimally, the Staller-start game domination number $\gamma'_s(G)$ is the score of an optimal S-game.

This game was introduced in [3] and investigated by now in about 30 papers. One of the reasons for this large interest is the $3/5$-conjecture due to Kinnersley, West and Zamani asserting that $\gamma_d(G) \leq 3|V(G)|/5$ holds for any isolate-free graph $G$ [13, Conjecture 6.2]. (Related conjectures were stated also for the S-game, as well as for both games played on forests.) Bujtás [5, 6] developed an innovative discharging-like method to attack this conjecture. Using the method, the conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [10]. Along these lines Schmidt [19] determined a largest known class of trees for which the conjecture holds. Moreover, Marcus and Peleg reported in arXiv [18] that the conjecture holds on all isolate-free forests. Among the other aspects of the domination game we list here: domination game critical graphs [7]; the somehow peculiar behaviour of the game on unions of graphs [9]; graphs with small game domination number [14]; different realizations of the game domination number [15]; a characterization of forests with the game domination number equal to the domination number [17]; bluffing aspects of the domination game [1]; and the PSPACE-completeness of the game domination number [2]. We also mention two related games that were introduced based on the domination game: the total domination game [11] and the disjoint domination game [8].

It was shown in [3, Theorem 1] that $\gamma(G) \leq \gamma_d(G) \leq 2\gamma(G) - 1$ holds for any graph $G$. Moreover, all possible values for $\gamma_d$ are eventually realizable [3, Theorem 10]. It is hence natural to ask for which graphs $G$ the equalities $\gamma_d(G) = \gamma(G)$ and $\gamma_d(G) = 2\gamma(G) - 1$ hold, respectively. The former problem was solved for the case of trees in [17], where it was also conjectured that if $G$ is a connected graph with $\gamma_d(G) = \gamma(G)$, then $G$ is either a tree or has girth at most 7. The general problem to characterize the graphs $G$ with $\gamma_d(G) = \gamma(G)$ seems highly difficult thought. In this paper we consider the other extreme case, that is, which graphs $G$ have the largest possible game domination number $2\gamma(G) - 1$. We will call such graphs $\gamma_d$-maximal.

In the next section additional concepts needed are introduced, several known results to be used later recalled, and a couple of useful facts deduced. In Section 3 large families of $\gamma_d$-maximal graphs are constructed among the graphs in which their supportive sets (vertices adjacent to leaves) are $\gamma$-sets. In the last two sections we consider trees which have exactly one vertex of degree at least 3, called starlike trees. In Section 4 we characterize $\gamma_d$-maximal starlike trees among the starlike trees with at least one 1-arm, while in Section 5 we characterize $\gamma_d$-maximal starlike trees among the other starlike trees.
2 Preliminaries

We will use the notation \([k] = \{1, \ldots, k\}\) for a positive integer \(k\). The maximum degree and the minimum degree in a graph \(G\) are denoted by \(\Delta(G)\) and \(\delta(G)\), respectively. A vertex \(v\) of \(G\) with \(\text{deg}_G(v) = 1\) is called a pendant vertex (alias leaf), the vertex adjacent to \(v\) is a support vertex (to \(v\)). Let \(L(G)\) and \(\text{Supp}(G)\) denote the set of pendant and support vertices of \(G\), respectively. For a vertex \(v\) of \(G\) let \(L(v) = L(G) \cap N(v)\), where \(N(v)\) is the open neighborhood of \(v\). Clearly, \(L(v) \neq \emptyset\) if and only if \(v \in \text{Supp}(G)\). Note also that \(L(K_2) = \text{Supp}(K_2) = V(K_2)\). On the other hand, if \(G\) is connected and of order at least 3, then a support vertex is of degree at least 2, so that \(L(G) \cap \text{Supp}(G) = \emptyset\).

Suppose that a D-game is played. Then we will denote the sequence of vertices selected by Dominator with \(d_1,d_2,\ldots\), and with \(s_1,s_2,\ldots\) the sequence chosen by Staller. A partially-dominated graph is a graph \(G\) together with a declaration that some vertices \(S \subseteq V(G)\) are already dominated in the sense that they need not be dominated in the rest of the game. It is denoted with \(G|S\).

We next recall the following fundamental results to be used later.

**Lemma 2.1** (Continuation Principle, [13]) Let \(G\) be a graph with \(A, B \subseteq V(G)\). If \(B \subseteq A\), then \(\gamma_g(G|A) \leq \gamma_g(G|B)\) and \(\gamma'_g(G|A) \leq \gamma'_g(G|B)\).

**Theorem 2.2** ([13]) If \(F\) is a forest with \(S \subseteq V(F)\), then \(\gamma_g(F|S) \leq \gamma'_g(F|S)\).

**Theorem 2.3** ([3, 13]) If \(G\) is any graph, then \(|\gamma_g(G) - \gamma'_g(G)| \leq 1\).

Setting \(S = \emptyset\) in Theorem 2.2 and specializing to trees we get:

**Corollary 2.4** Let \(T\) be a tree of order \(n \geq 2\). Then \(\gamma_g(T) \leq \gamma'_g(T)\).

Denoting with \(G \cup H\) the union of graphs \(G\) and \(H\) we have the following result that will be utmost useful to us.

**Lemma 2.5** ([13, Lemma 5.4]) If \(F_1\) and \(F_2\) are partially dominated forests, then

\[
\gamma_g(F_1 \cup F_2) \leq \gamma_g(F_1) + \gamma'_g(F_2) \quad \text{and} \quad \gamma'_g(F_1 \cup F_2) \leq \gamma'_g(F_1) + \gamma'_g(F_2).
\]

The next result was first proved in the unpublished manuscript [12]. Five years later the first published proof appeared in [16].

**Theorem 2.6** ([12, 16]) If \(n \geq 1\), then

(i) \(\gamma_g(P_n) = \left\lfloor \frac{n}{2} \right\rfloor - 1; \quad n \equiv 3 \text{ (mod 4)},\)

(ii) \(\gamma'_g(P_n) = \left\lceil \frac{n}{4} \right\rceil\) otherwise.
Following the notation from [16], let $P'_n$ denote the partially dominated path of order $n+1$ with one of its leaves dominated. Then from the Košmrlj’s proof of [16, Theorem 2.6] we extract the following information useful to us.

**Lemma 2.7** If $S$-game is played on $P'_n$ or a union of some $P'_n$s, then there exists a dominated leaf as an optimal move for Staller. Moreover, if $n \geq 1$, then

(i) $\gamma_g(P'_n) = \gamma_g(P'_n)$, and

(ii) $\gamma_g(P'_{n+3}) = 1 + \gamma_g'(P'_n)$.

**Proof.** The first assertion can be found in [16, p. 132]. The assertion (i) follows from the facts that in a $D$-game, a vertex adjacent to a leaf in $P_n$ is an optimal first move for Dominator, and that choosing a vertex at distance 2 from a dominated leaf in $P'_n$ is always optimal for Dominator, see [16, p. 132] again. The first of these two facts also implies the last assertion of the lemma. □

### 3 Graphs with supportive dominating sets

It this section we consider graphs $G$ such that $\text{Supp}(G)$ forms a dominating set of $G$. In such a case $\text{Supp}(G)$ is called a *supportive dominating set*. Clearly, a supportive dominating set of a graph $G$ must be a $\gamma_g$-set of $G$.

Ideally we wish to determine $\gamma_g(G)$ for any graph $G$ that contains a supportive dominating set. But this task seems to be quite demanding. For instance, for combs (a $k$-comb is obtained from $P_k$ by attaching a separate pendant vertex to each of the vertices of $P_k$), which form a simple class of graphs with supportive dominating sets, the task to determine $\gamma_g$ turned out to be quite tricky, see [15, Theorem 4.1].

We first establish a large class of graphs with supportive dominating sets that are $\gamma_g$-maximal.

**Theorem 3.1** Let $G$ be a connected graph of order at least 3 and $\delta(G) = 1$. If $G$ has a supportive dominating set and there are at least $\lceil \log_2 \gamma(G) \rceil + 1$ pendant vertices adjacent to each vertex of $\text{Supp}(G)$, then $G$ is $\gamma_g$-maximal.

**Proof.** Let $\gamma(G) = t$ and let $\text{Supp}(G) = \{x_1, \ldots, x_t\}$. If $t = 1$, then $G$ has a universal vertex and consequently $\gamma_g(G) = 2\gamma(G) - 1 = 1$ holds. Hence assume in the rest that $t \geq 2$. Set $V_i = \{x_i\} \cup L(x_i)$, $i \in [t]$. By Continuation Principle (Lemma 2.1) we may assume that during the game in which Dominator is playing optimally, he never chooses a pendant vertex of $G$. Indeed, if a pendant vertex is a legal move for Dominator, then the support vertex to it is also legal.

We only need to prove that $\gamma_g(G) \geq 2t - 1$. For this sake consider the following strategy of Staller:

**Rule A:** When it is Staller’s turn, she plays a legal vertex from $L(x_i)$, where $i$ is selected such that the number of vertices played so far in $L(x_i)$ is as small as possible.
Suppose first that $t = 2^k$ for some positive integer $k$. Recall that Dominator will never play a vertex of $L(x_i)$ ($i \in [t]$). Hence after the first $2^{k-1}$ moves of Dominator, Rule A implies that Staller will play one vertex from different sets $L(x_i)$. By the same argument, after the next $2^{k-2}$ moves of Dominator, Staller will play $2^{k-2}$ vertices from different sets $L(x_i)$. In this way after $2^{k-1} + 2^{k-2}$ moves of Dominator, Staller will play precisely two vertices in each of these $2^{k-2}$ sets $L(x_i)$. Then the game continues along the above process. There is still a legal pendant vertex for Staller after Dominator has played $2^k - 1$ vertices, hence Staller plays such a vertex then. At that point of the game there is at least one vertex $x_i \in \text{Supp}(G)$ not yet played by Dominator (neither by Staller), such that in $L(x_i)$ at most $\log_2 t$ vertices were played by Staller. Since $|L(x_i)| \geq \lceil \log_2 t \rceil + 1$, Dominator will be forced to play one more move. Then we have $\gamma_s(G) \geq 2(2^k - 1) + 1 = 2t - 1$ as desired.

Assume next that $t = 2^k + t_0$, where $0 < t_0 < 2^k$. By a similar argument as above, we find that there is still one pendant vertex which can be played by Staller after Dominator has played $t - 1$ vertices. Moreover, at least one $x_i \in \text{Supp}(G)$ has not yet been played by Dominator, and in $L(x_i)$ at most $\lceil \log_2 t \rceil = k + 1$ vertices were played by Staller. Hence at least one more move is needed which implies that $\gamma_s(G) \geq 2t - 1$.

Hence in any case Rule A guarantees that at least $2\gamma(G) - 1$ moves will be played which completes the argument. \hfill \Box

Denoting by $c(n)$ the number of connected graphs of order $n$, Theorem 3.1 yields the following consequence.

**Corollary 3.2** If $n \geq 2$ and $k \geq 1$, then there exist $c(n)$ graphs $G$ of order $n(k + \lfloor \log_2 n \rfloor)$ that are $\gamma_s$-maximal.

**Proof.** Let $n \geq 2$, $k \geq 1$, and let $G$ be a connected graph of order $n$. Let $G'$ be the graph obtained from $G$ by attaching $\lceil \log_2 n \rceil + k$ pendant vertices to each of the vertices of $G$, respectively. Then $\text{Supp}(G') = V(G)$, so that $\gamma(G') = n = |\text{Supp}(G')|$. The assertion now follows directly from Theorem 3.1. \hfill \Box

A special case of the construction from Corollary 3.2 (attaching $n + 1$ leaves to each vertex of the complete graph $K_n$) was earlier applied in [13] in order to show that all the values between $\gamma$ and $2\gamma - 1$ are possible values for $\gamma_g$.

In the rest of the section we restrict ourself to trees and first strengthen Theorem 3.1 in a special case as follows.

**Theorem 3.3** Let $T$ be a tree with $\gamma(T) = t \geq 2$ in which $\text{Supp}(T) = \{x_1, \ldots, x_t\}$ forms a $\gamma$-set. If the vertices other than the support vertices and their attached leaves induce a subtree of order $a$, and $|L(x_i)| \geq \lfloor \log_2 (t - \lceil \frac{a}{2} \rceil) \rfloor + 1, i \in [t]$, then $\gamma_s(T) = 2t - 1$.

**Proof.** Let $V_i = \{x_i\} \cup L(x_i)$, $i \in [t]$. Then by a theorem’s assumption the vertices $V(T) \setminus \bigcup_{i=1}^{t} V_i$ induce a subtree of $T$, denote this subtree with $T'$. It again suffices to prove that $\gamma_s(T') \geq 2t - 1$. \hfill \Box
Theorem 3.1. In this way at least \(2^{\lceil \frac{1}{2}\log_2 n\rceil - 1}\) vertices from Supp\((T)\) have at most one neighbor in \(T'\). Moreover, since Supp\((T)\) forms a \(\gamma\)-set of \(T\), we also infer that

\[
N(u) \cap \text{Supp}(T) \neq \emptyset, \quad \forall u \in V(T').
\] (2)

Let \(V(T') = \{w_1, \ldots, w_n\}\). By (1) and (2) there exists a matching between \(V(T')\) and Supp\((T)\) that covers all the vertices of \(T'\). We may without loss of generality assume that \(w_i x_i \in E(T), i \in [n]\). Since \(T'\) is a tree, the vertices \(x_1, \ldots, x_n\) induce an independent set. In addition, from the same reason, a vertex \(x_i, i > a\), is adjacent to at most one vertex among the vertices \(x_1, \ldots, x_a\).

The starting strategy of Staller is to reply to the first \(\lfloor \frac{n}{2}\rfloor\) moves of Dominator with a vertex in Supp\((T)\). Moreover, since Supp\((T)\) induces an independent set, in the second part of the game, Staller now applies the same strategy as she used in the proof of Theorem 3.1. In other words, Staller is the following. If Dominator plays a vertex from Supp\((T)\), she replies with a vertex \(w_i\), such that \(x_i\) has not yet been dominated. And if Dominator plays a vertex from \(T'\), then Staller plays a vertex \(x_i\) that has not yet been dominated.

After the first part of the game is finished, at most \(\lfloor \frac{n}{2}\rfloor\) moves will be played all together. In this way at least \(2(t - \lfloor \frac{n}{2}\rfloor) - 1\) additional moves will be played. Hence at least \(2(t - \lfloor \frac{n}{2}\rfloor) - 1 + 2\lfloor \frac{n}{2}\rfloor = 2t - 1\) moves will be played together. \(\square\)

Consider the following examples. Let \(k \geq 2\) and let \(a_i > 0\) for \(i \in [k]\). A comb-like tree \(Co(a_1, \ldots, a_k)\) is a tree obtained from the path \(P_k = v_1 \ldots v_k\) by attaching \(a_i\) pendant vertices to \(v_i\), \(i \in [k]\). In particular, if \(a_1 = a_2 = \cdots = a_k = 1\), then \(Co(a_1, \ldots, a_k)\) is exactly an ordinary comb. A 1-generalized comb-like tree \(Co^{(1)}(a_1, \ldots, a_k)\) is a tree obtained by attaching \(a_i\) pendant vertices to the \(i\)-th leaf, \(i \in [k]\), of \(Co(1, \ldots, 1)\). In Fig. 1 the 1-generalized comb-like tree \(Co^{(1)}(2, 1, 3, 1, 1, 2, 1, 3)\) is shown. If \(a_1 = \cdots = a_k = a\), then \(Co^{(1)}(a_1, \ldots, a_k)\) will be called a balanced 1-generalized comb-like tree and shortly denoted with \(Co^{(1)}(a^{(d)}(a))\).

If \(a_i \geq \lceil \frac{\log_2 a}{2} \rceil + 1, i \in [k]\), then Theorem 3.1 implies that \(\gamma_g(\text{Co}(a_1, \ldots, a_k)) = 2\gamma(\text{Co}(a_1, \ldots, a_k))\) - 1. Similarly, if \(a_i \geq \lceil \frac{\log_2 a}{2} \rceil + 1, i \in [k]\), then Theorem 3.3 implies that \(\gamma_g(\text{Co}^{(1)}(a_1, \ldots, a_k)) = 2\gamma(\text{Co}^{(1)}(a_1, \ldots, a_k))\) - 1.

It was proved in [4] that if \(T\) is a tree of order \(n\), then

\[
\gamma_g(T) \geq \left\lceil \frac{2n}{\Delta(T) + 3} \right\rceil - 1.
\] (3)
Proof. Combining the well known fact that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ ($n \geq 1$) with Theorem 2.6 it follows that $\gamma_g(P_n) = 2\gamma(P_n) - 1$ if and only if

$$2 \left\lfloor \frac{n}{3} \right\rfloor - 1 = \begin{cases} \left\lceil \frac{n}{3} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor; & \text{otherwise}. \end{cases}$$

This equality can hold only for $n \leq 10$, hence the result follows by checking these small values. \qed
Lemma 4.2 Let \( T = T(k_1, \ldots, k_s) \) (\( s \geq 3 \)), where \( 1 \in \{k_1, \ldots, k_s\} \). If \( T \) contains precisely \( t \) arms of respective lengths \( k_i > 1 \), \( i \in [t] \), then \( \gamma(T) = 1 + \sum_{i=1}^{t} \left\lceil \frac{k_i - 1}{3} \right\rceil \).

**Proof.** Let \( v \) be the unique vertex in \( T \) of maximum degree. Since \( T \) is a starlike tree with at least one 1-arm, there exists a \( \gamma \)-set \( D \) of \( T \) with \( v \in D \). Since \( T \setminus \{v\} \) consists of a disjoint union of paths \( P_{k_i - 1}, i \in [t] \), the result follows by the fact that \( \gamma(P_{k_i - 1}) = \left\lceil \frac{k_i - 1}{3} \right\rceil \). \( \square \)

Lemma 4.3 Let \( T = T(k_1, \ldots, k_s) \) (\( s \geq 3 \)), be a starlike tree with at least one 1-arm. If \( \gamma_g(T) = 2\gamma(T) - 1 \), then \( k_i \in \{3, 4, 7\} \) as long as \( k_i \neq 1 \).

**Proof.** Suppose that \( T \) has \( t \) arms of length \( k_i \geq 2 \). We may assume without loss of generality that \( i = 1, \ldots, t \). Suppose that \( k_1 \notin \{3, 4, 7\} \). Then by Lemma 4.1,

\[ \gamma_g(P_{k_1 + 2}) \leq 2\gamma(P_{k_1 + 2}) - 2. \]  \( (5) \)

Moreover, from Lemma 4.2 we infer that \( \gamma(T) = \sum_{i=1}^{t} \gamma(P_{k_i + 2}) - (t - 1) \) and consequently

\[ 2 \sum_{i=1}^{t} \gamma(P_{k_i + 2}) - 2t + 1 = 2\gamma(T) - 1. \]  \( (6) \)

Let \( d_1 \) be the vertex of \( T \) of degree at least 3. Then

\[
\gamma_g(T) \leq 1 + \gamma'_g\left( \bigcup_{i=1}^{t} P'_{k_i - 1} \right)
\]

\[
\leq 1 + \sum_{i=1}^{t} \gamma'_g(P'_{k_i - 1}) \quad \text{(by Lemma 2.5)}
\]

\[
= \sum_{i=2}^{t} \gamma_g(P_{k_i + 2}) + \gamma_g(P_{k_1 + 2}) - (t - 1) \quad \text{(by Lemma 2.7(ii))}
\]

\[
\leq \sum_{i=2}^{t} \left[ 2\gamma(P_{k_i + 2}) - 1 \right] + \left[ 2\gamma(P_{k_1 + 2}) - 2 \right] - (t - 1) \quad \text{(by (5))}
\]

\[
= \sum_{i=1}^{t} \left[ 2\gamma(P_{k_i + 2}) - 1 \right] - t
\]

\[
= 2 \sum_{i=1}^{t} \gamma(P_{k_i + 2}) - 2t
\]

\[
< 2\gamma(T) - 1 \quad \text{(by (6))}.
\]
This contradiction proves the lemma. □

Lemma 4.3 thus asserts that a $γ_g$-maximal starlike tree $T$ with at least one 1-arm has at most 4 different lengths of arms. To simplify the notation we will write $x^{(j)}$ to briefly denote that there are $j$ arms of length $x$. For instance, using this convention $T(1, 2, 2, 2, 3, 3)$ is shortly denoted with $T(1, 2(4), 3(2))$. For convenience, the star $S_n = T(1(α-1))$ will also be written as $T(1(α-2), 1)$. To formulate the main result of this section we set:

$$
\begin{align*}
ST^1 &= \{T(1^{(1)}, k) : l_1 \geq 1, k \in \{1, 3, 4, 7\}\}, \\
ST^2 &= \{T(1^{(1)}, 3, 4), T(1^{(1)}, 4^{(2)}), T(1^{(1)}, 4^{(3)}), T(1^{(1)}, 4, 7) : l_1 \geq 1\}, \text{ and} \\
ST^* &= ST^1 \cup ST^2.
\end{align*}
$$

We can now formulate the main result of this section.

**Theorem 4.4** A starlike tree $T$ with at least one 1-arm is $γ_g$-maximal if and only if $T \in ST^*$.

**Proof.** Assume that $T$ has $t$ arms of respective lengths $k_1, \ldots, k_t$, where $k_i \geq 2$ for $i \in [t]$. If $t = 0$, then $T$ is a star and hence $γ(T) = 1 = 2γ(T) − 1$. So in the following we always assume that $t \geq 1$. Let $u$ be the vertex of $T$ of degree at least 3, and let $v_{ij}$ be the vertex in the $k_i$-arm ($i \in [t]$) in $T$ at distance $j$ from $u$.

Using Lemma 4.2 we deduce that $γ(T(1^{(1)}, 3)) = γ(T(1^{(1)}, 4)) = 2$, $γ(T(1^{(1)}, 7)) = γ(T(1^{(1)}, 3, 4)) = γ(T(1^{(1)}, 4^{(2)})) = 3$, and $γ(T(1^{(1)}, 4^{(3)})) = γ(T(1^{(1)}, 4, 7)) = 4$. By a direct case analysis we get that $γ(T) = 2γ(T) − 1$ for any tree $T \in ST^*$. (We have checked this also by computer.) It thus remains to prove the “only if” part of the statement.

Suppose thus that $T$ is a $γ_g$-maximal starlike tree with at least one 1-arm (and at least one arm longer than 1). By Lemma 4.3 we know that $k_i \in \{3, 4, 7\}$ for $i \in [t]$. If $t = 1$, then using Lemma 4.1 we infer that $k_1 \in \{3, 4, 7\}$. Hence assume in the rest that $t \geq 2$.

**Claim 1.** $4 \in \{k_1, \ldots, k_t\}$.

**Proof (of Claim 1).** Suppose on the contrary that $4 \notin \{k_1, \ldots, k_t\}$. Let $d_1 = u$. Then, having in mind Lemma 2.5,

$$γ_g(T) ≤ 1 + γ_g′(\bigcup_{i=1}^{t} P_{k_i-1}) ≤ 1 + \sum_{i=1}^{t} γ_g′(P_{k_i-1}).$$

By Lemma 2.7, we may without loss of generality assume (by re-indexing the arms if necessary) that $s_1 = v_{11}$ is an optimal reply of Staller to $d_1 = u$. Setting $d_2 = v_{23}$
and applying Lemma 2.7 (ii) we can then estimate as follows:

\[
\gamma_g(T) \leq 3 + \sum_{i=3}^{t} \gamma_g'(P_{k_i-1}^r) + \gamma_g'(P_{k_i-2}^r) + \gamma_g'(P_{k_i-4}^r) \\
= 3 + \sum_{i=3}^{t} \gamma_g(P_{k_i+2}) - (t - 2) + \gamma_g(P_{k_i+1}) - 1 + \gamma_g'(P_{k_i-4}^r) \\
= 3 + \sum_{i=3}^{t} [2\gamma(P_{k_i+2}) - 1] - (t - 2) + \gamma_g(P_{k_i+1}) - 1 + \gamma_g'(P_{k_i-4}^r) \\
= 2 \sum_{i=3}^{t} \gamma(P_{k_i+2}) - 2t + \gamma_g(P_{k_i+1}) + \gamma_g'(P_{k_i-4}^r) + 6.
\]

Since \(2\gamma(T) - 1 = 2 \sum_{i=1}^{t} \gamma(P_{k_i+2}) - 2t + 1\) and \(\gamma_g(T) < 2\gamma(T) - 1\) holds by Theorem 2.6 for each pair \((k_1, k_2) \in \{(3, 3), (3, 7), (7, 3), (7, 7)\}\), we conclude that the set \([k_1]\) must contain 4.

\(\square\) (Claim 1)

By the final part of the proof of Claim 1, we have \(k_1 = 4\) or \(k_2 = 4\) when \(d_1 = u\), \(s_1 = v_{11}\) and \(d_2 = v_{23}\). Moreover, if \(k_1 \neq 4\), then \(k_2 = 4\). But now by a similar reasoning as that in the proof of Claim 1 we get \(\gamma_g(T) \leq 2 \sum_{i=3}^{t} \gamma(P_{k_i+2}) - 2t + \gamma_g(P_{k_i+1}) + \gamma_g'(P_{k_i-4}^r) + 6 < 2\gamma(T) - 1\), where \(\gamma_g'(P_{k_i}^r) = 0\). This is a contradiction. Thus, by Lemma 2.7, \(s_1\) must be on a 4-arm in \(T\) as an optimal move. Without loss of generality, assume that \(s_1 = v_{11}\) where \(v_{11}\) is in a \(k_1\)-arm with \(k_1 = 4\) in the following.

If \(t = 2\), by Claim 1 and the proof of the “if” part, we deduce that \(T\) must belong to the following set \(\{(1^{(l_1)}, 3, 4), T(1^{(l_1)}, 4^{(2)}), T(1^{(l_1)}, 4, 7) : l_1 \geq 1\} \subseteq ST^*\). So we only need to consider the case \(t \geq 3\) in the following. Let \(h \geq 1\) be the number of 4-arms in \(T\). Now we can check that \(\gamma_g(T) = 2\gamma(T) - 1\) if \(T \cong T_l(1^{(l_1)}, 4^{(3)})\) and \(\gamma_g(T) = 2t - 2 < 2t - 1 = 2\gamma(T) - 1\) if \(T \cong T_l(1^{(l_1)}, 4^{(4)})\) with \(t \geq 4\). So we only need to prove that \(\gamma_g(T) < 2\gamma(T) - 1\) for \(t > h \geq 1\) and \(t \geq 3\).

First we prove the result for \(h = 1\). Note that \(s_1 = v_{11}\). Let \(d_2 = v_{13}\). Hence we have \(\gamma_g(T) \leq 3 + \gamma_g'(\bigcup_{i=2}^{t} P_{k_i-1}^r)\). Denote by \(T_1\) the subtree of \(T\) obtained by deleting all vertices except \(u\) of the \(k_1\)-arm of \(T\). Since \(h = 1\), we infer that \(T_1\) is a starlike tree without 4-arms. By the reasoning from the proof of Claim 1 we get \(\gamma_g(T_1) \leq 1 + \gamma_g'(\bigcup_{i=2}^{t} P_{k_i-1}^r) < 2\gamma(T_1) - 1\). Then \(\gamma_g(T) < 2 + 2\gamma(T_1) - 1 = 2\gamma(T) - 1\).

Assume that \(t > h \geq 2\). Recall that \(d_1 = u\) and \(s_1 = v_{11}\). Also let \(d_2 = v_{13}\). Denote by \(T_2\) the subtree obtained by deleting all vertices but \(u\) of the \(k_1\)-arm of \(T\). Then \(T_2\) is a starlike tree with \(h - 1\) 4-arms. By a similar reasoning iteratively on the number
Let $T$ be a starlike tree without $1$-arms. Then $T$ is $\gamma_g'$-maximal if and only if $T$ is one of the trees $T(4^{(3)})$ and $T(2, 3^{(2)})$.

**Proof.** One can check directly (or by computer) that $\gamma(T(4^{(3)})) = 4$ and $\gamma_g(T(4^{(3)})) = 7$, as well as that $\gamma(T(2, 3^{(2)})) = 3$ and $\gamma_g(T(2, 3^{(2)})) = 5$. Hence $T(4^{(3)})$ and $T(2, 3^{(2)})$ are $\gamma_g'$-maximal. In the rest we thus need to prove that among the starlike trees without $1$-arms there is no additional $\gamma_g'$-maximal tree.

Assume thus in the rest that $T$ is a starlike tree without $1$-arms and with $\gamma_g(T) = 2\gamma(T) - 1$. Let $u$ be the maximum degree vertex of $T$. We divide our argument into the following two cases.

**Case 1.** $u$ lies in some $\gamma$-set of $T$.
Let $T$ has $t \geq 3$ arms of respective lengths $k_1, \ldots, k_t, k_i \geq 2$, with $i \in [t]$. Let $T'$ be a tree obtained from $T$ by attaching a pendant vertex to $u$. Then, by the case assumption, $\gamma(T) = \gamma(T')$. Since $T$ has $t \geq 3$ arms of length at least 2, $T'$ can only be isomorphic to $T(1, 4^{(3)})$ among the trees from $\mathcal{ST}^*$. If $T' \neq T(1, 4^{(3)})$, then setting $d_1 = u$ in a game played on $T'$ we have, using a similar reasoning as that in the proof of Theorem 4.4, that $\gamma_g(T') \leq 1 + \gamma_g'\left(\bigcup_{i=1}^{t} P_{k_i-1}'\right) < 2\gamma(T') - 1$. Hence if $T \neq T(4^{(3)})$, then by setting $d_1 = u$ in the game played on $T$, we have $\gamma_g(T) \leq 1 + \gamma_g'\left(\bigcup_{i=1}^{t} P_{k_i-1}'\right) < 2\gamma(T') - 1 = 2\gamma(T) - 1$.

By the proof of the “if” part we conclude that $T \equiv T(4^{(3)})$.

**Case 2.** No $\gamma$-set of $T$ contains $u$.
In this case we first assert that there is no arm of length $k$ in $T$ with $k \equiv 1 \pmod{3}$. Indeed, suppose on the contrary that $T$ contains an arm $P$ of length $3k + 1$. If $D$ is a
\(\gamma\)-set of \(T\), then \(|D \cap P| = k + 1\). But then \(D\) can be modified to a \(\gamma\)-set \(D'\) of \(T\) such that \(u \in D'\), a contradiction.

Consider now an arbitrary \(\gamma\)-set of \(T\). By the case assumption, there is a neighbor \(v\) of \(u\) such that \(v \in D\). It then follows that the arm on which \(v\) lies is of length \(m\), where \(m \equiv 2 \pmod{3}\).

Assume that \(T\) has \(a\) arms of respective lengths \(3t_1 + 2, \ldots, 3t_a + 2\) and \(b\) arms of respective lengths \(3l_1, \ldots, 3l_b\), where \(a \geq 1, b \geq 0\). Thus we have \(\gamma(T) = \sum_{i=1}^{a} (t_i + 1) + \sum_{j=1}^{a} l_j\), that is,

\[
2\gamma(T) - 1 = 2 \sum_{i=1}^{a} t_i + 2 \sum_{j=1}^{b} l_j + 2a - 1. \tag{7}
\]

Below we prove three claims.

**Claim 1.** \(b \neq 0\).

**Proof (of Claim 1.)** If not, we have \(b = 0\). Then \(2\gamma(T) - 1 = 2 \sum_{i=1}^{a} t_i + 2a - 1\) and \(a \geq 3\), since the maximum degree of \(T\) is at least 3. Let the first move of Dominator is just \(u\). By Lemma 2.7, without loss of generality, we may assume that \(s_1\) is a neighbor of \(u\) on the \((3t_1 + 2)\)-arm in \(T\). By Lemma 2.7 (i) we have \(\gamma_g(P_{3k_1}) = \gamma_g(P_{3k_1})\). Then, by Lemma 2.5, we have

\[
\gamma_g(T) \leq 2 + \gamma_g \left( \bigcup_{i=2}^{a} P'_{3t_i+1} \right) + \gamma_g \left( \bigcup_{i=2}^{a} P'_{3t_i} \right)
\]

\[
\leq 2 + \gamma_g(P_{3t_1}) + \sum_{i=2}^{a} \gamma_g(P'_{3t_i+1})
\]

\[
= 2 + \gamma_g(P_{3t_1}) + \sum_{i=2}^{a} \gamma_g(P_{3t_i+4}) - (a - 1)
\]

\[
< 2 + 2\gamma(P_{3t_1}) - 1 + 2 \sum_{i=2}^{a} \gamma(P_{3t_i+4}) - 2(a - 1)
\]

\[
(\text{as } \gamma_g(P_{3t_i+4}) < 2\gamma(P_{3t_i+4}) - 1 \text{ with } t_i \geq 0 \text{ by Lemma 4.1})
\]

\[
= 2 \sum_{i=2}^{a} (t_i + 2) + 2t_1 - 2a + 3
\]

\[
= 2 \sum_{i=1}^{a} t_i + 2a - 1
\]

\[
= 2\gamma(T) - 1 \quad \text{(by (7)).}
\]

This is a contradiction. So \(b \geq 1\) holds. \(\square\) (Claim 1)

From Claim 1, we have \(b > 0\). If \(d_1 = u\), then similarly as above we conclude that \(s_1\) must be on a neighbor of \(u\) on a \((3l_i)\)-arm in \(T\).
Claim 2. \( t_i = 0 \) for \( i \in [a] \).

**Proof (of Claim 2)** Otherwise assume without loss of generality that \( t_1 \geq 1 \). Let \( d_1 = u \). Then we may without loss of generality assume that \( s_1 \) is a neighbor of \( u \) on the \((3l_1)\)-arm in \( T \). Let \( d_2 \) be the vertex of the \((3l_1 + 2)\)-arm at distance 4 to \( u \) in \( T \). Note that \( \gamma'_g(P'_i) = 1 \). Then, by Lemmas 2.5 and 2.7(ii), we have

\[
\gamma'_g(T) \leq 3 + \gamma'_g(\bigcup_{i=2}^{a} P'_{3l_i+1} + \bigcup_{j=2}^{b} P'_{3l_j-1} + \bigcup_{i=2}^{a} P'_{3l_i-1} + \bigcup_{i=2}^{a} P'_{3l_i-2} + \bigcup_{i=2}^{a} P'_{1})
\]

\[
\leq 3 + \sum_{i=2}^{a} \gamma'_g(P'_{3l_i+1}) + \sum_{j=2}^{b} \gamma'_g(P'_{3l_j-1}) + \gamma'_g(P'_{3l_i-1}) + \gamma'_g(P'_{3l_i-2}) + \gamma'_g(P'_{1})
\]

\[
= 4 + \sum_{i=2}^{a} \gamma_g(P'_{3l_i+4}) - (a - 1) + \sum_{j=2}^{b} \gamma_g(P'_{3l_j+2}) - (b - 1)
\]

\[
+ \gamma_g(P'_{3l_i-2}) - 1 + \gamma_g(P'_{3l_i+1}) - 1
\]

\[
\leq 2 + 2 \sum_{i=2}^{a} \gamma_g(P'_{3l_i+4}) - 2(a - 1) + 2 \sum_{j=2}^{b} \gamma_g(P'_{3l_j+2}) - 2(b - 1)
\]

\[
+ 2 \gamma_g(P'_{3l_i-2}) - 1 + 2 \gamma_g(P'_{3l_i+1}) - 1
\]

\[
= 2 \sum_{i=2}^{a} (t_i + 2) - 2a + 2 \sum_{j=2}^{b} (l_j + 1) - 2b + 2t_1 + 2k_1 + 4
\]

\[
= 2 \sum_{i=1}^{a} t_i + 2 \sum_{j=1}^{b} l_j + 2a - 2
\]

\[
< 2 \gamma(T) - 1.
\]

A contradiction occurs again. \( \square \) (Claim 2)

By Claims 1 and 2 we have \( 2 \gamma(T) - 1 = 2 \sum_{j=1}^{b} \gamma_g(P'_{3l_j+1}) + 2 \sum_{j=1}^{b} \gamma_g(P'_{3l_j+2}) \) with \( a \geq 1 \), \( b \geq 1 \) and \( l_j \geq 1 \) for \( j \in [b] \). Moreover, if \( d_1 = u \), then one optimal vertex of \( s_1 \), which is adjacent to \( u \), will be a vertex in a \((3l_i)\)-arm not a 2-arm as an optimal move of Staller, since she always want to prolong the game.

Claim 3. \( l_j = 1 \) for \( j \in [b] \).

**Proof (of Claim 3)** Otherwise, we have \( l_j \geq 2 \) for some \( j \in [b] \). Similarly as above, let \( d_1 = u \). Then we can assume that \( s_1 \) lies on a \((3l_1)\)-arm as a neighbor of \( u \) in \( T \). Next we distinguish the following two subcases.

**Subcase 1.** \( j = 1 \).

In this subcase, if \( b = 1 \), then \( a \geq 2 \) since \( T \) has maximum degree at least 3. Now \( T \) has only one \((3l_1)\)-arm and \( a \) 2-arms with \( \gamma(T) = a + l_1 \). Note that \( \gamma'_g(P'_1) = 1 \)
and $\gamma'_g(P'_{3j-2}) \leq \gamma'_g(P'_{3l_1-2})$ from Corollary 2.4. Then, considering that $a \geq 2$ and by Lemma 2.5, we have

$$
\gamma'_g(T) \leq 2 + \gamma'_g\left( \bigcup_{i=1}^{a} P'_i \bigcup P'_{3l_1-2} \right)
$$

$$
\leq 2 + \sum_{i=1}^{a} \gamma'_g(P'_i) + \gamma'_g(P'_{3l_1-2})
$$

$$
\leq 2 + \sum_{i=1}^{a} \gamma'_g(P'_i) + \gamma'_g(P'_{3l_1-2})
$$

$$
= a + 2 + \gamma'_g(P'_{3l_1-2})
$$

$$
= a + 3 + \gamma'_g(P'_{3l_1-3})
$$

(since $\gamma'_g(P'_{3l_1-2}) = 1 + \gamma'_g(P_{3l_1-3})$ by Lemma 2.7)

$$
\leq a + 3 + 2\gamma(P_{3l_1-3}) - 1
$$

$$
= 2l_1 + a
$$

$$
< 2\gamma(T) - 1.
$$

If $b \geq 2$, then let $d_2$ be the vertex of the $(3l_2)$-arm at distance 2 from $u$. Note that $\gamma'_g(P'_{3l_1-2}) = 1 + \gamma'_g(P_{3l_1-3})$. Similarly as above we now have

$$
\gamma'_g(T) \leq 3 + \gamma'_g\left( \bigcup_{i=1}^{a} P'_i \bigcup \bigcup_{j=3}^{b} P'_{3j-1} \bigcup P'_{3l_1-2} \bigcup P'_{3l_2-3} \right)
$$

$$
\leq 3 + \sum_{i=1}^{a} \gamma'_g(P'_i) + \sum_{j=3}^{b} \gamma'_g(P'_{3j-1}) + \gamma'_g(P'_{3l_1-2}) + \gamma'_g(P'_{3l_2-3})
$$

$$
= a + 3 + \sum_{j=3}^{b} \gamma'_g(P_{3l_1+2}) - (b - 2) + \gamma'_g(P_{3l_1-3}) + 1 + \gamma'_g(P_{3l_2}) - 1
$$

$$
\leq a + 3 + 2 \sum_{j=3}^{b} \gamma(P_{3l_1+2}) - 2(b - 2) + 2\gamma(P_{3l_1-3}) - 1 + 2\gamma(P_{3l_2}) - 1
$$

$$
= 2 \sum_{j=1}^{b} l_j + a - 1
$$

$$
< 2\gamma(T) - 1.
$$

This finishing the proof of Subcase 1.

Subcase 2. $j \neq 1$.

Without loss of generality assume that $j = 2$. Let $d_2$ be the vertex on the $(3l_2)$-arm in $T$ at distance 4 from $u$. Note that $\gamma'_g(P'_{1''}) = 1 = \gamma'_g(P'_{1})$, $\gamma'_g(P'_{3l_1-2}) = 1 + \gamma'_g(P_{3l_1-3})$
and $\gamma'_g(P'_{3l_2-3}) = 1 + \gamma'_g(P_{3l_2-6})$. Then, by Lemma 2.5, we have

$$
\gamma'_g(T) \leq 3 + \gamma'_g \left( \bigcup_{i=1}^{a} P'_1 \bigcup_{j=3}^{b} P'_{3l_1-1} \bigcup P'_{3l_2-2} \bigcup P'_{3l_2-5} \bigcup P''_1 \right)
$$

$$
\leq 3 + \sum_{i=1}^{a} \gamma'_g(P'_1) + \sum_{j=3}^{b} \gamma'_g(P'_{3l_1-1}) + \gamma'_g(P'_{3l_2-2}) + \gamma'_g(P'_{3l_2-5}) + \gamma'_g(P''_1)
$$

$$
= a + 4 + \sum_{j=3}^{b} \gamma_g(P_{3l_2-3}) - (b - 2) + 1 + \gamma_g(P_{3l_2-3} + 1 + \gamma_g(P_{3l_2-6})
$$

$$
\leq a + 6 + 2 \sum_{j=3}^{b} \gamma(P_{3l_2-3}) - 2(b - 2) + 2\gamma(P_{3l_2-3} - 1 + 2\gamma(P_{3l_2-6}) - 1
$$

$$
= a + 4 + 2 \sum_{j=3}^{b} l_j + 2(l_1 - 1) + 2(l_2 - 2)
$$

$$
= 2 \sum_{j=1}^{b} l_j + a - 2
$$

$$
< 2\gamma(T) - 1.
$$

This is also a contradiction. \hfill \Box \text{(Claim 3)}

By Claims 1, 2 and 3, we conclude that $T \equiv T(2(6), 3(6))$ with $n = 2a + 3b + 1$. Note that $a \geq 1$, $b \geq 1$, and $a + b \geq 3$. From the structure of $T$, we have $\gamma(T) = a + b$, that is, $2\gamma(T) - 1 = 2a + 2b - 1$. Assume that the $D$-game is played on $T$. Let $d_1$ be the vertex with maximum degree in $T$. Afterwards Dominator can guarantee that the game will be finished in the total of $\lceil \frac{b}{2} \rceil + a + b + 1$ moves. The move $d_1$ might not be optimal for Dominator, but in any case we have $\gamma'_g(T) \leq \left\lceil \frac{b}{2} \right\rceil + a + b + 1$. Thus $T$ can only be $\gamma'_g$-maximal if $\gamma'_g(T) = \left\lceil \frac{b}{2} \right\rceil + a + b + 1 = 2a + 2b - 1$. From the second equality, we have $(a, b) \in \{(1, 2), (2, 1)\}$. But it can be verified that $\gamma_g(T(2(2), 3)) = 4 < 2\gamma(T) - 1$. Therefore, $T$ is $\gamma'_g$-maximal if and only if $T \equiv T(2, 3(2))$. This completes the proof of the theorem. \hfill \Box

Set

$$
ST = ST^* \bigcup \{T(4(3)), T(2, 3(2))\}.
$$

Then combining Theorems 4.2 and 5.1 we arrive at the following result.

**Theorem 5.2** Let $T$ be a starlike tree. Then $T$ is $\gamma'_g$-maximal if and only if $T \in ST$. 

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References


