On the domination number and the 2-packing number of Fibonacci cubes and Lucas cubes

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Abstract
Let $\Gamma_n$ and $\Lambda_n$ be the $n$-dimensional Fibonacci cube and Lucas cube, respectively. The domination number $\gamma$ of Fibonacci cubes and Lucas cubes is studied. In particular it is proved that $\gamma(\Lambda_n)$ is bounded below by $\left\lceil \frac{L_n-2n}{n-1} \right\rceil$, where $L_n$ is the $n$-th Lucas number. The 2-packing number $\rho$ of these cubes is also studied. It is proved that $\rho(\Gamma_n)$ is bounded below by $2^{\frac{1+\sqrt{5}}{2}n-1}$ and the exact values of $\rho(\Gamma_n)$ and $\rho(\Lambda_n)$ are obtained for $n \leq 10$. It is also shown that $\text{Aut}(\Gamma_n) \approx \mathbb{Z}_2$.

Key words: Fibonacci cubes; Lucas cubes; domination number; 2-packing number; automorphism group


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1 Introduction

Fibonacci cubes form a class of graphs introduced because of their properties applicable for interconnection networks [5]. Lucas cubes [10] are subgraphs of Fibonacci cubes in which certain “non-symmetric” vertices are removed. In this way we get graphs with more symmetries, a fact that will be further justified in this paper. Both classes of cubes have been considered from various points of view, see [1, 2, 3, 8, 11, 13].

In this paper we study Fibonacci cubes and Lucas cubes from the viewpoint of domination and packing. While searching for (vertex) subsets of a graph (like dominating sets) it is useful to know symmetries of the graph, hence we first describe autormorphism groups of these graphs in Section 2.

In Section 3 we study the domination number of Fibonacci cubes as initiated in [12], and also investigate that of Lucas cubes. We first give some connections between the domination number of Fibonacci cubes and Lucas cubes and construct dominating sets for 9-dimensional cubes. Then we obtain a lower bound on the domination number of Lucas cubes.

A graph invariant closely related to the domination number is the 2-packing number, which is the topic of Section 4. We first obtain an exponential (in terms of the dimension) lower bound on the 2-packing number of the Lucas cubes which is a natural lower bound for the Fibonacci cubes. Combining computer search with some arguments the exact values for the 2-packing number of both classes of cubes up to and including dimension 10 are obtained.

In the rest of this section we define the concepts needed in this paper. For a connected graph $G$, the distance $d_G(u, v)$ (or $d(u, v)$ for short) between vertices $u$ and $v$ is the usual shortest path distance.

Let $n \geq 1$ and $Q_n$ be the $n$-dimensional hypercube. A Fibonacci string of length $n$ is a binary string $b_1b_2 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, Fibonacci strings are binary strings that contain no consecutive 1’s. The Fibonacci cube $\Gamma_n$, for $n \geq 1$ is the subgraph of $Q_n$ induced by the Fibonacci strings of length $n$. A Fibonacci string $b_1b_2 \ldots b_n$ is a Lucas string if $b_1 \cdot b_n = 0$. The Lucas cube $\Lambda_n$, for $n \geq 1$ is the subgraph of $Q_n$ induced by the Lucas strings of length $n$.

It is well-known (cf. [5]) that $|V(\Gamma_n)| = F_{n+2}$, where $F_n$ are the Fibonacci numbers: $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Similarly, $|V(\Lambda_n)| = L_n$ for $n \geq 1$, see [10], where $L_n$ are the Lucas numbers: $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

For $n \geq 1$ and $0 \leq k \leq n$, let $\Gamma_{n,k}$ be the set of vertices of $\Gamma_n$ that contain $k$ 1’s. Hence $\Gamma_{n,k}$ is the set of vertices of $\Gamma_n$ at distance $k$ from $0^n$. $\Lambda_{n,k}$ is defined analogously. In particular, $\Gamma_{n,0} = \Lambda_{n,0} = \{0^n\}$ and $\Gamma_{n,1} = \Lambda_{n,1} = \{0^{n-1}, 010^{n-2}, \ldots, 0^{n-1}1\}$. If $uv \in E(\Gamma_n)$, where $u \in \Gamma_{n,k}$ and $v \in \Gamma_{n,k-1}$ ($k \geq 1$), then we say that $v$ is a down-neighbor of $u$ and that $u$ is an up-neighbor of $v$. The same terminology again applies to Lucas cubes.

For a binary string $b = b_1b_2 \ldots b_n$, let $\bar{b}$ be the binary complement of $b$ and let $b^R = b_n b_{n-1} \ldots b_1$ be the reverse of $b$. For binary strings $b$ and $c$ of equal length, let $b + c$ denote their sum computed bitwise modulo 2. For $1 \leq i \leq n$, let $c_i$ be the binary string of length $n$ with 1 in the $i$-th position and 0 elsewhere. According to this notation, $\Gamma_{n,1} = \Lambda_{n,1} = \{e_1, e_2, \ldots, e_n\}$.

Let $G$ be a graph. Then $D \subseteq V(G)$ is a dominating set if every vertex from $V(G) \setminus D$ is
adjacent to some vertex from $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of $G$. It is well known that for any graph $G$, $\gamma(G) \geq \rho(G)$, cf. [6].

Finally, the automorphism group of a graph $G$ is denoted by $\text{Aut}(G)$. For instance, $\text{Aut}(C_n) = D_{2n}$, where $C_n$ is the $n$-cycle and $D_{2n}$ is the dihedral group on $n$ elements. Recall that $D_{2n}$ can be represented as $\langle x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1 \rangle$.

## 2 Automorphism groups

In this section we determine the automorphism groups of Fibonacci cubes and Lucas cubes.

Let $n \geq 1$ and define the reverse map $r : \Gamma_n \rightarrow \Gamma_n$ with:

$$r(b_1 b_2 \ldots b_n) = b_n b_{n-1} \ldots b_1. \quad (1)$$

It is easy to observe that $r$ is an automorphism of $\Gamma_n$. We are going to prove that $r$ is the only nontrivial automorphism of $\Gamma_n$. For this sake, the following lemma is useful.

**Lemma 2.1** Let $n \geq 3$ and $k \geq 2$. Then any different $u, v \in \Gamma_{n,k}$ have different sets of down-neighbors.

**Proof.** Since $u, v \in \Gamma_{n,k}$, $d(u, v) \geq 2$. We distinguish two cases.

Suppose first $d(u, v) = 2$ and let $u$ and $v$ differ in positions $i$ and $j$. Since $u, v \in \Gamma_{n,k}$, we may assume without loss of generality that $u_i = v_j = 1$ and $u_j = v_i = 0$. Moreover, $u$ and $v$ agree in all the other positions. Since $k \geq 2$, there exists an index $\ell \neq i, j$ such that $u_\ell = v_\ell = 1$. Then $u + e_\ell$ is a down-neighbor of $u$ but not a down-neighbor of $v$.

Assume now $d(u, v) \geq 3$. Let $i$ be an arbitrary index such that $u_i = v_i$. We may assume that $u_i = 1$. Then $u + e_i$ is a down-neighbor of $u$ but not of $v$. \hfill \Box

**Theorem 2.2** For any $n \geq 1$, $\text{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.

**Proof.** The assertion is clear for $n \leq 2$, hence assume in the rest that $n \geq 3$. Let $\alpha \in \text{Aut}(\Gamma_n)$. Since $0^n$ is the only vertex of degree $n$, $\alpha(0^n) = 0^n$. Therefore, $\alpha$ maps $\Gamma_{n,1}$ onto $\Gamma_{n,1}$. Let $\Gamma'_{n,1} = \{10^{n-1}, 0^{n-1}\}$ and $\Gamma''_{n,1} = \Gamma_{n,1} \setminus \Gamma'_{n,1}$. Since $10^{n-1}$ and $0^{n-1}$ are the only vertices of degree $n - 1$, $\alpha$ maps $\Gamma'_{n,1}$ and $\Gamma''_{n,1}$ onto $\Gamma'_{n,1}$ and $\Gamma''_{n,1}$, respectively. We distinguish two cases.

**Case 1:** $\alpha(10^{n-1}) = 10^{n-1}$. Then, because $\alpha$ maps $\Gamma'_{n,1}$ onto $\Gamma'_{n,1}$, we have $\alpha(0^{n-1}1) = 0^{n-1}1$. Among the vertices of $\Gamma''_{n,1}$, only $010^{n-2}$ has no common up-neighbor with $10^{n-1}$. Therefore, $\alpha(010^{n-2}) = 010^{n-2}$. In turn, among the remaining vertices of $\Gamma''_{n,1}$, only $0010^{n-3}$ has no common up-neighbor with $010^{n-2}$. Therefore $\alpha(0010^{n-3}) = 0010^{n-3}$. By proceeding with the same argument, $\alpha$ fixes $\Gamma''_{n,1}$ pointwise and hence fixes $\Gamma_{n,1}$ pointwise. Now apply Lemma 2.1
and induction on \( k \) to conclude that \( \alpha \) fixes \( \Gamma_{n,k} \) pointwise for all \( k \). Therefore \( \alpha = \text{id} \) in this case.

**Case 2:** \( \alpha(10^{n-1}) = 0^{n-1}1 \).

Now \( \alpha(0^{n-1}) = 10^{n-1} \). Among the vertices of \( \Gamma_{n,1}'' \), only \( 010^{n-2}10 \) has no common up-neighbor with \( 10^{n-1} \). Thus \( \alpha(010^{n-2}) = 0^{n-2}10 \), which is the only element of \( \Gamma_{n,1}'' \) with no common up-neighbor together with \( \alpha(10^{n-1}) = 0^{n-1}1 \). By proceeding with the same argument, \( \alpha \) reverses all the elements of \( \Gamma_{n,1}'' \), that is, \( \alpha = r \) on \( \Gamma_{n,1}'' \) and consecutively \( \alpha = r \) on \( \Gamma_{n,1} \). By Lemma 2.1 and induction on \( k \), the same holds for any \( \Gamma_{n,k} \), \( k \geq 2 \).

Therefore \( \alpha = r \) in this case. \( \Box \)

Let \( n \geq 1 \). An equivalent way to define \( \Lambda_n \) is that it is the subgraph of \( Q_n \) induced on all the binary strings of length \( n \) that have no two consecutive 1's in circular manner. This definition is more symmetric than the definition of the Fibonacci strings, so it is reasonable to expect that \( \text{Aut}(\Lambda_n) \) is richer than \( \text{Aut}(\Gamma_n) \). This is indeed the case. Define \( \varphi : \Lambda_n \to \Lambda_n \) by

\[
\varphi(b_1 b_2 \ldots b_n) = b_n b_1 \ldots b_{n-1}.
\]

By the above remark it is clear that \( \varphi \in \text{Aut}(\Lambda_n) \). Zagaglia Salvi [14] proved that the automorphism groups of the Lucas semilattices are the dihedral groups. The arguments that determine the automorphism group of the Lucas cubes are in a way parallel to the arguments from [14], hence we next give just a sketch of them.

Note first that Lemma 2.1 (with the same proof) applies to Lucas cubes as well. Let \( \alpha \in \text{Aut}(\Lambda_n) \). Suppose that for some \( a, b \in \{0, 1, \ldots, n-1\} \), \( \alpha(10^{n-1}) = 0^a10^{n-a-1} \) and \( \alpha(0^{n-1}) = 0^b10^{n-b-1} \), where computations are mod \( n \). Then either \( b = a - 1 \) or \( b = a + 1 \) because \( \alpha(10^{n-1}) \) and \( \alpha(0^{n-1}) \) cannot have a common up-neighbor. When \( b = a - 1 \) we get \( \alpha = \varphi^a \) and in the other case \( \alpha = \varphi^{a+1} \circ r \). We conclude that \( \text{Aut}(\Lambda_n) \) is generated by \( r \) and \( \varphi^a \) for \( 0 \leq a \leq n - 1 \), and hence:

**Theorem 2.3** For any \( n \geq 3 \), \( \text{Aut}(\Lambda_n) \simeq D_{2n} \).

## 3 The domination number

In this section we consider the domination number of Fibonacci and Lucas cubes. We first interrelate their domination numbers. Then we discuss exact domination numbers for small dimensions. The section is concluded by establishing a general lower bound on the domination number of Lucas cubes.

**Proposition 3.1** Let \( n \geq 4 \), then

\[
(i) \quad \gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3}), \\
(ii) \quad \gamma(\Lambda_n) \leq \gamma(\Gamma_n) \leq \gamma(\Lambda_n) + \gamma(\Gamma_{n-4}).
\]

**Proof.** (i) \( V(\Lambda_n) \) can be partitioned into vertices that start with 0 and vertices that start with 1. The latter vertices are of the form 100...0 and hence can be dominated by \( \{10b0 \mid b \in U\} \) where \( U \) is a minimum dominating set of \( \Gamma_{n-3} \) with \( \gamma(\Gamma_{n-3}) \) vertices.
While the former vertices can be dominated by $\gamma(\Gamma_{n-1})$ vertices. (ii) Let $D$ be a minimum dominating set of $\Gamma_n$ and set

$$D' = \{ u \mid u \text{ is a Lucas string from } D \} \cup \{ 0b_2 \ldots b_{n-1}1 \mid 1b_2 \ldots b_{n-1}0 \in D \}.$$  

A vertex $1b_2 \ldots b_{n-1}1$ dominates two Lucas vertices, namely $0b_2 \ldots b_{n-1}1$ and $1b_2 \ldots b_{n-1}0$. Since these two vertices are dominated by $0b_2 \ldots b_{n-1}0$, we infer that $D'$ is a dominating set of $\Lambda_n$. It follows that $\gamma(\Lambda_n) \leq \gamma(\Gamma_n)$.

A dominating set of $\Lambda_n$ dominates all vertices of $\Gamma_n$ but the vertices of the form $10b_3 \ldots b_{n-2}01$. These vertices can be dominated by $\gamma(\Gamma_{n-4})$ vertices. □

Pike and Zou [12] obtained exact values of $\gamma(\Gamma_n)$ for $n \leq 8$, see Table 2. By computer search they found 509 minimum dominating sets of $\Gamma_8$. Following their approach we have computed the domination numbers of $\Lambda_n$, $n \leq 8$, see Table 2 again.

Hence the smallest Fibonacci cube and Lucas cube for which the domination numbers are not known are $\Gamma_9$ and $\Lambda_9$. Since $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$, it follows that $\gamma(\Gamma_9) \leq 20$, cf. [12, Lemma 3.1]. In order to find a smaller dominating set we have used a local search procedure, that is, to get a new dominating set we have replaced one or more vertices with another vertex. In this way we were able to construct a dominating set of $\Gamma_9$ of size 17 given on the left-hand side of Table 1. Similarly we have found a dominating set of $\Lambda_9$ of order 16 given on the right-hand side of Table 1. Hence:

**Proposition 3.2** $\gamma(\Gamma_9) \leq 17$ and $\gamma(\Lambda_9) \leq 16$.

| 010000000  | 000000000  |
| 100100000  | 000010000  |
| 010100000  | 000000100  |
| 001000100  | 000100100  |
| 000010010  | 000100010  |
| 000001010  | 010000010  |
| 101001000  | 100101000  |
| 101000100  | 010100010  |
| 100010100  | 010001010  |
| 100001010  | 001001001  |
| 001010010  | 101010010  |
| 000101001  | 101001010  |
| 000101010  | 010101001  |
| 001001010  | 010010101  |
| 101010001  | 001010101  |
| 010010101  |  |

Table 1: A dominating set of $\Gamma_9$ and a dominating set of $\Lambda_9$
We conjecture that $\gamma(\Gamma_9) = 17$ and $\gamma(\Lambda_9) = 16$ hold.

Pike and Zou [12] also proved that for any $n \geq 4$,

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 3}{n - 2} \right\rceil.$$  

We next prove a parallel lower bound for the domination number of Lucas cubes. For this sake we first consider degrees of some specific vertices in Lucas cubes.

Let $n \geq 1$. Recall that $\Lambda_{n,1}$ is the set of all the vertices with exactly one 1. In addition, set

$$\Lambda'_{n,2} = \{0^a1010^n-a-3 \mid 0 \leq a \leq n - 1\},$$

where we again compute by modulo $n$. Hence $\Lambda'_{n,2}$ is the subset of $\Lambda_{n,2}$ consisting of the Lucas strings containing (in circular manner) 101 as a substring.

**Lemma 3.3** Let $n \geq 7$. Then for the Lucas cube $\Lambda_n$ the followings hold.

(i) The vertex $0^n$ is the only vertex of the maximum degree $n$.

(ii) The vertices of $\Lambda_{n,1}$ have degree $n - 2$.

(iii) Among the vertices with at least two 1’s, only the vertices of $\Lambda'_{n,2}$ have degree $n - 3$ and all the other vertices have degree at most $n - 4$.

**Proof.** (i) and (ii) are clear.

(iii) Let $u \in \Lambda_{n,k}$ for some $k \geq 2$. Then $u$ has $k$ down-neighbors. The up-neighbors of $u$ are obtained by switching a bit 0 into 1. Let $i_1 < i_2 < \cdots < i_k$ be the positions in which $u$ contains 1. Throughout the proof, the indices of $i$’s will be considered by modulo $k$ and $i_j$ by modulo $n$. As no consecutive bits of 1’s are allowed, $i_{j+1} - i_j \geq 2$ for all $1 \leq j \leq k$. Let $I_j = \{i_j - 1, i_j + 1\}$ be the set of the positions which are adjacent to $i_j$ for each $1 \leq j \leq k$ and let $I = \bigcup_{1 \leq j \leq k} I_j$. Then any bit which is not in $I$ can be switched to 1 and hence the number of up-neighbors of $u$ is $n - k - |I|$. Therefore, $\deg(u) = n - |I|$.

Note that $I_j \cap I_{j'} = \emptyset$ if $|j - j'| \geq 2$, therefore by pigeon-hole principle, $|I| \geq k$. The equality holds if and only if $I_j \cap I_{j+1} \neq \emptyset$ for all $1 \leq j \leq k$, which occurs if and only if $i_{j+1} = i_j + 2$ for all $1 \leq j \leq k$, which is in turn if and only if $n$ is even and $k = \frac{n}{2}$. But in this case, $\deg(u) = \frac{n}{2} \leq n - 4$ as $n \geq 8$. In the other cases, $|I| \geq k + 1$ and hence $\deg(u) \leq n - k - 1$. If $k \geq 3$, then $\deg(u) \leq n - 4$. Assume $k = 2$. Then $\deg(u) \leq n - 3$, where the equality holds exactly when $|I| = 3$ and $I_1 \cap I_2 \neq \emptyset$ which means that $u \in \Lambda'_{n,2}$.

**Lemma 3.4** Let $n \geq 1$. Then any $\ell$ vertices from $\Lambda'_{n,2}$ has at least $\ell$ down-neighbors, that is, at least $\ell$ neighbors in $\Lambda_{n,1}$.

**Proof.** For $1 \leq i \leq \ell$, let $A_i$ be the set of down-neighbors of some $v_i \in \Lambda'_{n,2}$. Then $|A_i| = 2$ for each $i$. Considering bits by modulo $n$, each vertex $0^a1010^n-a-1$ in $\Lambda_{n,1}$ can be a down-neighbor of at most two vertices $0^a1010^n-a-3$ and $0^a21010^n-a-1$, and hence at most two of $v_1, \ldots, v_i$. By pigeon-hole principle, the assertion is true.

\[\square\]
To establish the announced lower bound, we will apply the natural concept of over-domination, just as it is done in [12]. It is defined as follows. Let \( D \) be a dominating set of a graph \( G \). Then the over-domination of \( G \) with respect to \( D \) is:

\[
OD_G(D) = \sum_{v \in D} (\deg_G(v) + 1) - |V(G)|. 
\]

(3)

Note that \( OD_G(D) = 0 \) if and only if \( D \) is a perfect dominating set [9, 4], that is, a dominating set such that each vertex is dominated exactly once.

**Theorem 3.5** For any \( n \geq 7 \), \( \gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil \).

**Proof.** Let \( D \) be a minimum dominating set of \( \Lambda_n \). Set \( D_1 = D \cap \Lambda_{n,1} \) and \( D_2 = D \cap \Lambda_{n,2} \), and let \( k = |D_1| \) and \( l = |D_2| \). Then clearly \( 0 \leq k, l \leq n \). Note that the over-domination of \( G \) with respect to \( D \) can be rewritten as

\[
OD(G) = \sum_{u \in V(\Lambda_n)} (|\{v \in D \mid d(u, v) \leq 1\}| - 1). 
\]

(4)

For a vertex \( u \) of \( \Lambda_n \), set \( t(u) = |\{v \in D \mid d(u, v) \leq 1\}| - 1 \). As \( D \) is a dominating set, \( t(u) \geq 0 \) for all \( u \in V(\Lambda_n) \). We now distinguish two cases.

**Case 1:** \( 0^n \in D \).

Combining Lemma 3.3 with Equation (3) we get

\[
OD(D) \leq (n + 1) + k(n - 1) + l(n - 2) + (\gamma(\Lambda_n) - k - l - 1)(n - 3) - L_n = \gamma(\Lambda_n)(n - 3) + 2k + l + 4 - L_n.
\]

Also as \( t(u) \geq 0 \) for all \( u \in V \), Equation (4) implies

\[
OD(D) \geq t(0^n) + \sum_{v \in D_1} t(v) \geq 2k.
\]

Therefore \( \gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil \).

**Case 2:** \( 0^n \notin D \).

Again, combining Lemma 3.3 with Equation (3) we infer

\[
OD(D) \leq k(n + 1) + l(n - 2) + (\gamma(\Lambda_n) - k - l)(n - 3) - L_n = \gamma(\Lambda_n)(n - 3) + 2k + l - L_n.
\]

Let \( A \) be the set of down-neighbors of \( D_2 \). Then for \( u \in D_1 \cap A \), \( t(u) \geq 1 \). By Lemma 3.4, \( |A| \geq l \) and hence \( |D_1 \cap A| \geq k + l - n \). Therefore by Equation (4),

\[
OD(D) \geq \sum_{v \in D_1 \cap A} t(v) \geq k + l - n.
\]

Thus \( \gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil \).

By Case 1 and Case 2, \( \gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil \). □
4 The 2-packing number

We now turn to the 2-packing number and first prove the following asymptotical lower bound.

**Theorem 4.1** For any \( n \geq 8 \), \( \rho(\Gamma_n) \geq \rho(\Lambda_n) \geq 2^{\lfloor \lg n \rfloor - 1} \).

**Proof.** Since for any \( n \geq 1 \), \( \Lambda_n \) is an isometric subgraph of \( \Gamma_n \), cf. [7], a 2-packing of \( \Lambda_n \) is also a 2-packing of \( \Gamma_n \). Therefore \( \rho(\Gamma_n) \geq \rho(\Lambda_n) \).

Let \( r, s \geq 1 \) and let \( X \) and \( Y \) be maximum 2-packings of \( \Lambda_r \) and \( \Lambda_s \), respectively. Then \( \{x0y \mid x \in X, y \in Y\} \) is a 2-packing of \( \Lambda_{r+s+1} \) of size \( \rho(\Lambda_s)\rho(\Lambda_r) \). It follows that

\[
\rho(\Lambda_{r+s+1}) \geq \rho(\Lambda_r)\rho(\Lambda_s).
\]

Set now \( k = \lfloor \lg n \rfloor \). Then \( \rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^k-1}) \geq \rho(\Lambda_{2^k-2})^2 \). By repeatedly applying this argument we get

\[
\rho(\Lambda_n) \geq \rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^k-2})^2.
\]

When \( k \) is even, take \( l = \frac{k-2}{2} \) to get \( \rho(\Lambda_n) \geq \rho(\Lambda_4)^{\frac{k-2}{2}} = 2^{2^{\frac{k-2}{2}}} \). When \( k \) is odd, take \( l = \frac{k-3}{2} \) to get \( \rho(\Lambda_n) \geq \rho(\Lambda_8)^{\frac{k-3}{2}} \geq 8^{2^{\frac{k-3}{2}}} = 2^{3 \times 2^{\frac{k-3}{2}}} \geq 2^{2^{\frac{k-3}{2}}} \).

□

Using computer we obtained the 2-packing numbers of \( \Gamma_n \) and \( \Lambda_n \) for \( n \leq 10 \) given in Table 2.

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</tbody>
</table>

Table 2: Domination numbers and 2-packing numbers of small cubes

Table 2 needs several comments.

- The computer search found exactly ten 2-packings of size 20 in \( \Gamma_{10} \). This already implies that \( \rho(\Gamma_{10}) = 20 \). Indeed, if \( \Gamma_{10} \) would contain a 2-packing of size 21, then it would contain at least twenty-one 2-packings of size 20.

- By exhaustive search with computer no 2-packing of size 19 but eighty 2-packing of size 18 in \( \Lambda_{10} \) were found, hence \( \rho(\Lambda_{10}) = 18 \).

- There is only one (up to isomorphisms of the graphs considered) maximum 2-packing of \( \Lambda_5, \Lambda_6, \Lambda_7, \Lambda_9 \), as well as \( \Gamma_6 \). There are two non-isomorphic 2-packings of maximum cardinality of \( \Gamma_9 \), which are presented in Table 3.
Since the reverse map given in (1) is an automorphism of Fibonacci cubes, the reverse of a 2-packing is also a 2-packing. Interestingly, the maximum 2-packing of $\Gamma_9$ shown on the left-hand side of Table 3, denoted $X$, is also invariant under the reverse map. That is, $r(X) = X$.

Similarly, the shifts $\varphi^i$, where $\varphi$ is given in (2) and are automorphisms of Lucas cubes, hence they map 2-packings into 2-packings. Now consider the 2-packing of $\Lambda_9$ shown in Table 4, denote it $Y$. Then it can be checked that $\varphi^3(Y) = Y$. As a consequence, $\varphi^6(Y) = Y$.

$$
\begin{array}{cccccccc}
000001010 & 000001000 \\
010100000 & 000100100 \\
000100101 & 001000010 \\
101001000 & 001010001 \\
001000001 & 010000101 \\
100000100 & 010010000 \\
010001001 & 010100010 \\
100100010 & 010101001 \\
010010101 & 100010100 \\
101010010 & 100101010 \\
001010100 & 100100001 \\
010010010 & 100101010 \\
100010001 & 101000100 \\
100101001 & 101001001 \\
\end{array}
$$

Table 3: Maximum 2-packings of $\Gamma_9$

$$
\begin{array}{cccccccc}
100100100 \\
000010001 \\
000101001 \\
001000010 \\
001000101 \\
010001000 \\
010010100 \\
010100010 \\
010100101 \\
100010010 \\
100101010 \\
101001000 \\
101010100 \\
\end{array}
$$

Table 4: Maximum 2-packing of $\Lambda_9$
5 Concluding remarks

Based on the data from Table 2 we ask whether some of the followings are true.

**Problem 5.1** Is it true that

(i) $\gamma(\Gamma_n) - \rho(\Gamma_n) \geq \gamma(\Lambda_n) - \rho(\Lambda_n)$ for $n \geq 1$?
(ii) $\gamma(\Lambda_n) \geq \rho(\Gamma_n)$ for $n \geq 4$?
(iii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3}) - 1$ for $n \geq 6$?

Note that the last question, if it has an affirmative answer, reduces the bound of $\gamma(\Lambda_n)$ in Proposition 3.1 (i) by 1. Moreover, if (iii) is true, then one can also ask whether $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-4})$ holds for $n \geq 6$.

References


