On the Mostar index of trees and product graphs

Yaser Alizadeh\textsuperscript{a}, Kexiang Xu\textsuperscript{b}, Sandi Klavžar\textsuperscript{c,d,e}

\textsuperscript{a}Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran
\textsuperscript{b}College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, PR China
\textsuperscript{c}Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
\textsuperscript{d}Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
\textsuperscript{e}Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

Abstract. If $G$ is a graph, and if for $e = uv \in E(G)$ the number of vertices closer to $u$ than to $v$ is denoted by $n_u$, then $\text{Mo}(G) = \sum_{e \in E(G)} |n_u - n_v|$ is the Mostar index of $G$. In this paper, the Mostar index is studied on trees and graph products. Lower and upper bounds are given on the difference between the Mostar indices of a tree and a tree obtained by contraction one of its edges and the corresponding extremal trees are characterized. An upper bound on the Mostar index for the class of all trees but the stars is proved. Extremal results are also determined on the $(k+1)$-th largest/smallest Mostar index. The index is also studied on Cartesian and corona products.

1. Introduction

If $G = (V(G), E(G))$ is a graph and $uv \in E(G)$, then the number of vertices that are closer (w.r.t. the standard shortest path metric) to $u$ than to $v$ is denoted by $n_u$; analogously $n_v$ is defined. With this notation in hand, the Mostar index of $G$ is

$$\text{Mo}(G) = \sum_{e \in E(G)} |n_u - n_v|.$$ 

Denoting by $\phi_G(e) = |n_u - n_v|$ the contribution of the edge $e = uv$ to $\text{Mo}(G)$, we can write the Mostar index of $G$ in an even more compact form as follows:

$$\text{Mo}(G) = \sum_{e \in E(G)} \phi_G(e).$$

The Mostar index received a lot of attention right away after its introduction in 2018 by Došlić et al. [12]. First, it was considered on several classes of chemically important graphs [4, 9, 14, 19]. The difference between the Mostar index and the irregularity of graphs was studied in [13]. Cacti and and extremal

\textit{2010 Mathematics Subject Classification.} Primary: 05C09; Secondary: 05C12, 05C35

\textit{Keywords.} Mostar index; Transmission of vertex; Tree; Graph product

Received: 2, November, 2020; Revised: 28, February, 2021.

Communicated by Paola Bonacini

K.X. is supported by NNSF of China (No. 11671202). S.K. acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109, J1-1693, N1-0095).

Corresponding author: Kexiang Xu.

\textit{Email addresses:} y.alizadeh.s@gmail.com (Yaser Alizadeh), kexxu1221@126.com (Kexiang Xu), sandi.klavzar@fmf.uni-lj.si (Sandi Klavžar)
bicyclic graphs with respect to the Mostar index were studied in [17, 26], respectively. In [8], the Mostar index of diameter 2 graphs and of some graph operations was clarified. The paper [14] studies the Mostar index with respect to graph symmetries and describes the structures of graphs whose Mostar index is 1. Papers [10, 18] and [11] deal with the extremal values of the Mostar index of trees with different parameters and chemical trees, respectively. Finally, maximal Mostar indices in hexagonal chains were investigated in [27].

The Mostar index is naturally related to several established graph theory concepts. Let $G$ be a graph. $G$ is distance-balanced [20] if $\phi_G(e) = 0$ holds for each edge $e$ of $G$, so that $\varphi(G) = 0$ if and only if $G$ is distance-balanced. The irregularity [3] of $G$ is $\varphi(G) = \sum_{uv \in E(G)} |\deg(u) - \deg(v)|$, which was extended [1] to its total version $\varphi(G) = \sum_{uv \in V(G)} |\deg(u) - \deg(v)|$ called total irregularity. Some related results can be found in [28].

In this paper, some further results are obtained on the Mostar index of graphs. In Section 2, additional concepts and notation needed are given, and results to be used in later sections recalled. In Section 3 we consider the Mostar index of trees. We first give sharp lower and upper bounds on the distance-balanced nature of edge contraction in trees on the Mostar index of trees, and describe the structures of graphs whose Mostar index is 1.

3. On the Mostar index of trees

In this section, we first consider the effect of edge contraction in trees on the Mostar index of trees, and follow with several extremal results on the Mostar index of trees. We begin with the following result that clearly holds by the structure of trees.

2. Preliminaries

Graphs considered in this paper are connected, unless stated otherwise. We use the notation $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. If $G = (V(G), E(G))$ is a graph, then we denote by $N_G(v)$ the neighborhood of the vertex $v$ of $G$. The size of $N(v)$ is called the degree of $v$ and denoted by $\deg_G(v)$ or simply $\deg(v)$. The order and the size of $G$ will be denoted by $n(G)$ and $m(G)$, respectively. A vertex with $\deg(v) = 1$ is a pendant vertex (or leaf if $G$ is a tree) in a graph $G$. If $e \in E(G)$, then $G.e$ denotes the graph obtained from $G$ by contracting the edge $e$. The transmission $\text{Tr}_G(v)$ or simply $\text{Tr}(v)$ of a vertex $v \in V(G)$ is the sum of the distances from $v$ to all other vertices of $G$. The relevance of the transmission for the Mostar index follows from the following basic, important result.

**Lemma 2.1.** [6] If $uv \in E(G)$, then $\text{Tr}(u) - \text{Tr}(v) = n_v - n_u$.

Lemma 2.1 immediately yields:

**Corollary 2.2.** If $G$ is a graph, then $\text{Mo}(G) = \sum_{uv \in E(G)} [\text{Tr}(u) - \text{Tr}(v)]$.

We note in passing that the right-hand side of the equality in Corollary 2.2 was very recently introduced as a graph invariant under the name transmission irregularity of a graph $G$ [25]. Corollary 2.2 however says that there is no need for this new name.

Recall that $\phi_G(e) = |n_u - n_v|$ is the contribution of the edge $e = uv$ to $\text{Mo}(G)$. Hence it makes sense to say that $e$ is equi-effective if $\phi_G(e) = 0$. We can then state that $G$ is distance-balanced if and only if every edge of $G$ is equi-effective.

To conclude the preliminaries, we recall the following result.

**Proposition 2.3.** ([12]) Let $G$ be a graph of order $n > 2$ with $uv \in E(G)$. Then $|n_v - n_u| \leq n - 2$ with equality if and only if one of $u$ and $v$ is a pendant vertex.
Lemma 3.1. Let $T$ be a tree of order $n \geq 2$. Then $T$ has at most one equi-effective edge. Moreover, if $T$ has one equi-effective edge, then $n$ is even.

If $T$ is a tree and $v \in V(T)$, then let $\ell_T(v)$ denote the number of leaves adjacent to $v$. Our first main result now reads as follows.

Theorem 3.2. Let $T$ be a tree with $n(T) = n \geq 3$, and let $e = uv \in E(T)$ with $s = \deg_T(u) \geq \deg_T(v)$. Then

$$n - 2 \leq Mo(T) - Mo(T,e) \leq 2(n - 2).$$

The left equality holds if and only if $e$ is the equi-effective edge of $T$. The right equality is achieved if $v$ is a leaf in $T$ with

$$p = \ell_T(u) \geq \begin{cases} \max\{ \max_{\{i,j\} \subseteq [s-p]} |n_u^{(0)} - n_u^{(0)}|, 1\}; & s - p \geq 2, \\ \max|n_u^{(0)}|; & s - p = 1, \end{cases}$$

such that $T - u = \bigcup_{i=1}^{s-p} T_u^{(i)} \cup pK_1$ where $T_u^{(i)}$ is a non-trivial subtree of $T - u$ of order $n_u^{(i)}$ for $i \in [s - p]$.

Proof. Let $T - e = T_u \cup T_v$ where $T_u$ and $T_v$ are components containing $u$ and $v$, respectively. Assume that $n(T_u) \geq n(T_v)$. For any edge $f \in E(T_u)$, we have $\phi_T(f) = \phi_T(v) + 1$. Now we consider any edge $f = xy \in E(T_u)$. Let $T - f = T_x \cup T_y$ with $d(x, u) < d(y, u)$. If $n(T_x) < n(T_y)$, we have $\phi_T(f) = \phi_T(x)$ and while $n(T_y) \geq n(T_x)$, then $\phi_T(f) = \phi_T(x) - 1$ holds possibly. Assume that there are $q$ edges $f$ such that $\phi_T(f) - \phi_{T_x}(f) = 1$. Setting $A = Mo(T) - Mo(T,e)$, we have

$$A = \phi_T(e) + \sum_{f \in E(T_x)} (\phi_T(f) - \phi_T(x)) + \sum_{f \in E(T_u)} (\phi_T(f) - \phi_T(x))$$

$$= \left[n(T_u) - n(T_v)\right] + n(T_u) - 1 - q + \left[n(T_u) - 1 - q\right]$$

$$= 2n(T_u) - 2 - 2q \leq 2(n - 2)$$

with equality holding if and only if $n(T_u) = n - 1$ and $q = 0$, that is, $\deg_T(v) = 1$ with $q = 0$. Next we characterize the structure of $T$ with $q = 0$. First assume that $\deg_T(u) = s \geq p + 2$ and $u$ has $s - p$ non-leaf neighbors $u_i \in V(T_u^{(i)})$ in $T$ for $i \in [s - p]$. Without loss of generality, we can assume that $\max_{\{i,j\} \subseteq [s-p]} |n_u^{(0)} - n_u^{(0)}| = n_u^{(1)} - n_u^{(2)}$. Note that $n = \sum_{i=1}^{s-p} n_u^{(i)} + p + 1$. In view of $p \geq \max_{\{i,j\} \subseteq [s-p]} |n_u^{(0)} - n_u^{(0)}|$, we have $n_u^{(2)} \leq n_u^{(1)} \leq \frac{n - 2}{2}$. Considering the contribution of $uu_i$ to $Mo(T)$ and $Mo(T,e)$, respectively, we have

$$\phi_T(uu_1) = |n_u^{(1)} - (n - n_u^{(1)})|$$

$$= n - n_u^{(1)}$$

$$= |n_u^{(1)} - (n - 1 - n_u^{(1)})| + 1$$

$$= \phi_{T_x}(xy) + 1.$$ 

Analogously, $\phi_T(uu_i) = \phi_{T_x}(uu_i) + 1$ for any $i \in [s - p] \setminus \{1\}$. Now we choose any edge $xy \in E(T_u^{(1)})$ with $d(x, u) < d(y, u)$. Recall that $n_u^{(1)} \leq \frac{n - 2}{2}$, we have

$$\phi_T(xy) = |n_x - (n - n_x)|$$

$$= n - 2n_x$$

$$= |n_x - (n - 1 - n_x)| + 1$$

$$= \phi_{T_x}(xy) + 1.$$
Moreover, we obtain \( \phi_T(xy) = \phi_T(x) + 1 \) in a parallel way for any edge \( xy \in E(T_u^{(i)}) \) with \( i \in [s-p] \setminus \{1\} \). By Proposition 2.3, for any pendant edge \( f \neq e \) incident with \( u \), we have \( \phi_{T,e}(f) = n - 3 = n - 2 - 1 = \phi_T(f) - 1 \). Recall that \( e = uv \) is a pendant edge in \( T \). Then it follows that \( \text{Mo}(T) = \text{Mo}(T,e) + 2(n-2) \) as desired. For the case \( \deg_T(u) = s = p + 1 \), the result can be similarly proved and here we omit its proof.

Note that there are at most \( \left\lfloor \frac{n(T_u) - n(T_v)}{2} \right\rfloor \) edges \( f = xy \) such that \( n(T_y) \geq n(T_x) \), that is, \( q \leq \left\lfloor \frac{n(T_u) - n(T_v)}{2} \right\rfloor \). Then, from \( n(T_u) + n(T_v) = n \), it follows that

\[
\text{Mo}(T) - \text{Mo}(T,e) = 2n(T_u) - 2 - 2q \\
\geq 2n(T_u) - 2 - 2\left\lfloor \frac{n(T_u) - n(T_v)}{2} \right\rfloor \\
= n - 2
\]

with equality holding if and only if \( n(T_u) = n(T_v) \), that is, \( e \) is the equi-effective edge of \( T \). Moreover, by Lemma 3.1, we conclude that \( n \) is even. This finishes the proof of the theorem. \( \square \)

For the star \( S_n \) with \( n \geq 2 \), we have \( \text{Mo}(S_n) - \text{Mo}(S_n,e) = 2(n-2) \) for any edge \( e \in E(S_n) \). This is just the case \( s = p \) in the proof of Theorem 3.2.

**Corollary 3.3.** If \( T \) is a tree with \( n(T) = n \geq 3 \) and \( H \) is a proper subtree of \( T \), then \( \text{Mo}(T) > \text{Mo}(H) + n - 2 \).

**Proof.** We prove the theorem by induction on the order of \( T \). The statement holds for \( n = 3 \). Let \( e = uv \) be a pendant edge of \( T \) such that \( e \notin E(H) \). By Theorem 3.2, we have \( \text{Mo}(T) > \text{Mo}(T,e) + n - 2 \), since \( e \) is never an equi-effective edge in \( T \). Note that \( H \) is a subtree of \( T.e \). Then, by induction hypothesis, we get \( \text{Mo}(H) \leq \text{Mo}(T,e) \). This completes the proof of the corollary. \( \square \)

In [12] it was proved that \( \text{Mo}(T) \leq (n-1)(n-2) \) for any tree \( T \) with \( n(T) = n \geq 4 \) with equality holding if and only if \( T \) is a star. We next improve this result by giving an exact upper bound for all non-star trees.

To prove the result, we need the following lemma.

**Lemma 3.4.** Let \( uv \) be a non-pendant edge in a tree of order \( n \geq 4 \). Then \( |n_u - n_v| \leq n - 4 \) with equality holding if and only if either \( u \) or \( v \) is adjacent to a leaf of \( T \).

**Proof.** Without loss of generality, we assume that \( \deg_T(u) \geq \deg_T(v) \). Since \( uv \) is a non-pendant edge of \( T \), we have \( \deg_T(v) \geq 2 \). By Proposition 2.3, \( |n_u - n_v| < n - 2 \) for \( uv \) being non-pendant in \( T \). Note that \( n_u + n_v = n \) in \( T \). Then we have \( |n_u - n_v| \leq n - 4 \) with equality holding if and only if \( \max(n_u, n_v) = n - 2 \) and \( \min(n_u, n_v) = 2 \). Equivalently, \( \deg_T(v) = 2 \) with \( v \) being adjacent to a leaf of \( T \), completing the proof of the lemma. \( \square \)

A double star \( S_n(p,q) \) is a tree obtained from attaching \( p \geq 1 \) vertices to an end-vertex of \( K_2 \) and attaching \( n - p - 2 \) vertices to the other vertex of it where \( 0 \leq p \leq \left\lfloor \frac{n-2}{2} \right\rfloor \). Note that \( S_n(0,n-2) \) is the star graph \( S_n \).

**Theorem 3.5.** Let \( T \) be a tree of order \( n \geq 3 \) different from \( S_n \). Then we have \( \text{Mo}(T) \leq (n-1)(n-2) - 2 \) with equality holding if and only if \( T \equiv S_n(1,n-3) \).

**Proof.** Let \( T \neq S_n \) be a tree of order \( n \geq 3 \) with the maximum Mostar index. Since \( T \neq S_n \), there is a non-pendant edge \( uv \) in \( T \). Assume that \( \deg_T(v) \leq \deg_T(u) \). From Lemma 3.4 and Proposition 2.3, we have

\[
\text{Mo}(T) = \phi_T(uv) + \sum_{e \in E(T) \setminus \{uv\}} \phi_T(e) \\
\leq n - 4 + (n - 2)^2 \\
= (n - 1)(n - 2) - 2
\]

with equality holding if and only if \( \deg_T(v) = 2 \) with \( v \) being adjacent to a leaf in \( T \) and other \( n - 2 \) edges are pendant in \( T \), that is, \( T \equiv S_n(1,n-3) \), finishing the proof of the theorem. \( \square \)

Let \( T_n(1,k,\ell) \) be a tree of order \( n = k + \ell + 2 \) obtained by attaching a pendant vertex at the vertex \( v_k \) of a path \( P_{k+1} = v_0v_1v_2 \ldots v_{k+\ell}v_{k+1} \) with natural adjacency relation with \( k \in [\ell] \). To obtain the final result in this section, we first list a lemma and prove a preliminary result.
Lemma 3.6. ([12]) If $T$ is a tree of order $n \geq 4$, then

$$\left\lfloor \frac{(n-1)^2}{2} \right\rfloor \leq \text{Mo}(T) \leq (n-1)(n-2)$$

with left equality if and only if $T \cong P_n$ and right equality if and only if $T \cong S_n$.

Lemma 3.7. Let $T$ be a tree. Then $\text{Mo}(T)$ is an even number.

Proof. Let $n \geq 2$ be the order of $T$ with $e = uv \in E(T)$. If $n_u = m$, then $|n_u - n_v| = |n - 2m|$. It means that the contribution of all edges to $\text{Mo}(T)$ have the same parity. If $n$ is even (odd, resp.), then $\text{Mo}(T)$ is the sum of $n - 1$ even (odd, resp.) numbers. This implies that $\text{Mo}(T)$ is an even number. \end{proof}

Theorem 3.8. Among the trees of order $n \geq 4$, the $(k+1)$-th largest Mostar index is attained at $S_n(k, n-2-k)$ with $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and the $(k+1)$-th smallest Mostar index is attained at $T_n(1, k, n-2-k)$ with $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. From the definitions of the Mostar index and the double star $S_n(k, n-2-k)$, we have $\text{Mo}(S_n(k, n-2-k)) = (n-2)^2 + n - 2(k + 1)$. Note that $\text{Mo}(S_n(k, n-2-k)) - \text{Mo}(S_n(k+1, n-3-k)) = 2$. Combining Theorem 3.5 with Lemmas 3.7 and 3.6, we conclude that the $(k+1)$-th largest Mostar index is attained at $S_n(k, n-2-k)$ with $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

From the structure of $T_n(1, k, n-2-k)$ and some elementary calculation, we have $\text{Mo}(T_n(1, k, n-2-k)) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor + 2k$ for every $k$ with $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. By Lemmas 3.7 and 3.6, we have the result with the $(k+1)$-th smallest Mostar index for $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. \end{proof}

4. On the Mostar index of graph products

In [8] the Mostar index was considered on the join of graphs, the disjunction of graphs, and the symmetric difference of graphs. Graphs obtained by each of these operations have diameter at most 2, and for the latter graphs it was proved in [8] that their Mostar index equals the irregularity. We restate here this result and show that Corollary 2.2 enables a short simple proof of it.

Theorem 4.1. If $G$ is a graph of diameter at most 2, then $\text{Mo}(G) = \text{irr}(G)$.

Proof. The result is clear for complete graphs, hence let $G$ be a graph with diameter 2. Then $\text{Tr}(u) = \deg_G(u) + 2\left|n - 1 - \deg_G(u)\right| = 2n - 2 - \deg_G(u)$ for any vertex $u \in V(G)$. Thus $\phi_G(uv) = |\deg_G(u) - \deg_G(v)|$ for every edge $uv \in E(G)$. The result now follows from Corollary 2.2. \end{proof}

The Mostar index of graph products was further investigated in [2], where corona products, Cartesian products, joins of graphs, lexicographic products, and the so-called Indu-Bala products are treated. In this section we give some further insight in this direction. We first give a short proof of the formula for the Mostar index of Cartesian products and then give an exact result for the corona product.

4.1. Cartesian product

The Cartesian product $G_1 \square G_2$ of graphs $G_1$ and $G_2$ is the graph with the vertex set $V(G_1) \times V(G_2)$ and the edge set $\{(u_1, u_2)(v_1, v_2) : u_1 = v_1 \text{ and } u_2v_2 \in E(G_2), \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)\}$. In [12] formulas were proved for the Mostar index of Cartesian products in which both factors are paths or both factors are partial cubes. (See [5, 7, 23, 24] for recent investigations of partial cubes.) A formula for the Mostar index of arbitrary Cartesian products was then independently given in [2, Theorem 1.5] and in [21, Theorem 3.1]. Our contribution in this section is a short proof of the formula. The short proof reveals how Corollary 2.2 is extremely useful.

Theorem 4.2. If $G_1$ and $G_2$ are graphs, then

$$\text{Mo}(G_1 \square G_2) = n(G_1)^2\text{Mo}(G_2) + n(G_2)^2\text{Mo}(G_1).$$
Proof. Set \( n_1 = n(G_1) \) and \( n_2 = n(G_2) \). Recall the folklore result that the distance function is additive in Cartesian products, that is, \( d_G((u_1, u_2), (v_1, v_2)) = d_G(u_1, v_1) + d_G(u_2, v_2) \) holds, see \cite{16}. Therefore \( \text{Tr}_G((u, v)) = n_2 \text{Tr}_{G_2}(u) + n_1 \text{Tr}_{G_2}(v) \) for any vertex \((u, v) \in V(G)\). Then, by Corollary 2.2,

\[
\text{Mo}(G) = \sum_{x \in V(G_1), \text{red}(G_2)} |\text{Tr}_G((u, v)) - \text{Tr}((u, w))| + \sum_{x \in V(G_2), \text{red}(G_1)} |\text{Tr}_G((v, u)) - \text{Tr}(w, u))|
\]

\[
= \sum_{x \in V(G_1), \text{red}(G_2)} [n_1 \text{Tr}_{G_2}(v) - n_1 \text{Tr}_{G_2}(w)] + \sum_{x \in V(G_2), \text{red}(G_1)} [n_2 \text{Tr}_{G_2}(v) - n_2 \text{Tr}_{G_2}(w)]
\]

\[
= n_1^2 \text{Mo}(G_2) + n_2^2 \text{Mo}(G_1),
\]

completing the proof of the theorem. \( \square \)

Using simple induction, Theorem 4.2 can be extended to Cartesian products with an arbitrary number of factor graphs. Here we give a formula for the special case of \( G^k \), the Cartesian product of \( k \) copies of \( G \).

**Corollary 4.3.** If \( G \) is a graph with \( n(G) \geq 2 \), then

\[
\text{Mo}(G^k) = kn(G)^{2k-2} \text{Mo}(G).
\]

4.2. Corona

The corona product \( G \circ H \) of graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) and \( n(G) \) copies of \( H \) and joining each vertex of the \( i \)-th copy of \( H \) with the \( i \)-th vertex of \( G \). In \cite[Theorem 1.1]{2} an upper bound on the Mostar index of corona product is given. We next give an exact result.

**Theorem 4.4.** If \( G_1 \) and \( G_2 \) are graphs, then

\[
\text{Mo}(G_1 \circ G_2) = (n(G_2) + 1)\text{Mo}(G_1) + n(G_1)^2 n(G_2) (n(G_2) + 1) - n(G_1)(2m(G_2) + n(G_2)) + n(G_1)\text{irr}(G_2).
\]

Proof. Let \( G = G_1 \circ G_2 \). For \( i \in [2] \) set \( V_i = V(G_i), E_i = E(G_i), n_i = n(G_i) \), and \( m_i = m(G_i) \). Let further \( V(G_1) = \{v_1, \ldots, v_{n_1}\} \), and let \( G_{2,i} \) be the copy of \( G_2 \) associated with \( v_i, i \in [n_1] \). Then

\[
\text{Mo}(G) = \sum_{v, v_j \in E_1} |n_G(v) - n_G(v_j)| + \sum_{i=1}^{n_1} \sum_{x \in V(G_{2,i})} |n_G(v) - n_G(x)|
\]

\[
+ \sum_{i=1}^{n_1} \sum_{x \in V(G_{2,i})} |n_G(x) - n_G(y)|.
\]

If \( v_i v_j \in E_1 \) and if \( v_k \in V(G_1) \) is closer to \( v_i \) than to \( v_j \) in \( G_1 \), then all vertices of \( G_{2,k} \) are closer to \( v_i \) than to \( v_j \) in \( G \). Then it follows that \( |n_G(v) - n_G(v_j)| = (n_2 + 1)|n_G(v) - n_G(v_j)| \). If \( e = xv_i \in E(G) \), where \( x \in V(G_2) \), then all vertices of \( G \), except the vertices adjacent to \( x \), are closer to \( v_i \) than to \( v_j \). Therefore we have \( \phi(e) = |V(G)| - \deg_{G_2}(x) - 1 = |n_1(n_2 + 1) - n_1 - \deg_{G_2}(x) - 1| \). Finally consider the edge \( f = xy \in E(G_{2,j}) \). Since all vertices of \( G_{2,k} \) are adjacent to \( v_i \), all the vertices in \( V(G) \setminus (N_{G_{2,i}}(x) \cup N_{G_{2,i}}(y)) \) have the same distance to \( x \) and to \( y \). Thus \( \phi(f) = |\deg_{G_2}(x) - \deg_{G_2}(y)| \). Therefore,

\[
\text{Mo}(G) = \sum_{v, v_j \in E_1} (n_2 + 1)|n_G(v) - n_G(v_j)|
\]

\[
+ \sum_{i=1}^{n_1} \sum_{x \in V(G_{2,i})} [n_1(n_2 + 1) - (\deg_{G_{2,i}}(x) + 1)]
\]

\[
+ \sum_{i=1}^{n_1} \sum_{x \in V(G_{2,i})} |\deg_{G_2}(x) - \deg_{G_2}(y)|
\]

\[
= (n_2 + 1)\text{Mo}(G_1) + n_1^2 n_2 (n_2 + 1) - n_1(2m_2 + n_2) + n_1\text{irr}(G_2)
\]
completing the proof of the theorem. □

The thorny graph $G^*(p_1, \ldots, p_n(G))$ of a graph $G$ with parameters $p_1, \ldots, p_n(G)$ is obtained from $G$ by attaching $p_i$ pendant vertices to the $i$-th vertex of $G$ with $i \in [n(G)]$ [15]. (For additional properties of thorny graphs see [22].) If $p_1 = \cdots = p_n = p$, then we simplify the notation to $G^*(p^n)$. Since $G^*(p^n) \cong G \odot pK_1$, Theorem 4.4 gives:

**Corollary 4.5.** If $G$ is a graph, then

$$\text{Mo}(G^*(p^n)) = (p + 1)\text{Mo}(G) + n(G)^2p(p + 1) - n(G)p.$$ 

**References**


