The general position number of the Cartesian product of two trees

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Abstract

The general position number of a connected graph is the cardinality of a largest set of vertices such that no three pairwise-distinct vertices from the set lie on a common shortest path. In this paper it is proved that the general position number is additive on the Cartesian product of two trees.

Keywords: general position set; general position number; Cartesian product; trees

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1 Introduction

Let $d_G(x, y)$ denote, as usual, the number of edges on a shortest $x, y$-path in $G$. A set $S$ of vertices of a connected graph $G$ is a general position set if $d_G(x, y) \neq d_G(x, z) + d_G(z, y)$ holds for every $\{x, y, z\} \in \binom{S}{3}$. The general position number $\text{gp}(G)$ of $G$ is the cardinality of a largest general position set in $G$. Such a set is briefly called a gp-set of $G$. 
Before the general position number was introduced in [9], an equivalent concept was proposed in [14]. Much earlier, however, the general position problem has been studied by Körner [8] in the special case of hypercubes. Following [9], the graph theory general position problem has been investigated in [1, 3, 5, 6, 10, 11, 13].

The Cartesian product $G \square H$ of vertex-disjoint graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, vertices $(g, h)$ and $(g', h')$ being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. In this paper we are interested in $gp(G \square H)$, a problem earlier studied in [3, 6, 10, 13]. More precisely, we are interested in Cartesian products of two (finite) trees. (For some of the other investigations of the Cartesian product of trees see [2, 12, 15].) An important reason for this interest is the fact that the general position number of products of paths is far from being trivial. First, denoting with $P_\infty$ the two-way infinite path, one of the main results from [10] asserts that $gp(P_\infty \square P_\infty) = 4$. Denoting further with $G^n$ the $n$-fold Cartesian product of $G$, it was demonstrated in the same paper that $10 \leq gp(P_\infty^3) \leq 16$. The lower bound 10 was improved to 14 in [6]. Very recently, these results were superseded in [7] by proving that if $n$ is an arbitrary positive integer, then $gp(P_\infty^n) = 2^{2n-1}$. Denoting with $n(G)$ the order of a graph $G$, in this paper we prove:

**Theorem 1.** If $T$ and $T^*$ are trees with $\min\{n(T), n(T^*)\} \geq 3$, then

$$gp(T \square T^*) = gp(T) + gp(T^*)$$

Theorem 1 widely extends the above mentioned result $gp(P_\infty \square P_\infty) = 4$. Further, the equality $gp(P_\infty^n) = 2^{2n-1}$ shows that Theorem 1 has no obvious (inductive) extension to Cartesian products of more than two trees. Hence, to determine the general position number of such products remains a challenging problem.

In the next section we give further definitions, recall known results needed, and prove several auxiliary new results. Then, in Section 3, we prove Theorem 1.

## 2 Preliminaries

Let $T$ be a tree. The set of leaves of $T$ will be denoted by $L(T)$, and let $\ell(T) = |L(T)|$. If $u$ and $v$ are vertices of $T$ with $\deg(u) \geq 2$ and $\deg(v) = 1$, then the unique $u, v$-path is a branching path of $T$. If $u$ is not a leaf of $T$, then there are exactly $\ell(T)$ branching paths starting from $u$; we say that the $u$ is the root of these branching paths and that the degree 1 vertex of a branching path $P$ is the leaf of $P$.

**Lemma 1.** ([9]) If $T$ is a tree, then $gp(T) = \ell(T)$.

We next describe which vertices of a tree lie in some $gp$-set of the tree.
Lemma 2. A non-leaf vertex \( u \) in a tree \( T \) belongs to a gp-set of \( T \) if and only if \( T - u \) has exactly two components and at least one of them is a path.

Proof. First, let \( R \) be a gp-set of \( T \) containing the non-leaf vertex \( u \). Suppose that \( T - u \) has at least three components, say \( T_1, T_2 \) and \( T_3 \). Since \( R \) is a gp-set containing \( u \), \( R \) intersects with at most one of \( T_1, T_2 \) and \( T_3 \). Assume without loss of generality that \( R \cap V(T_2) = \emptyset \) and \( R \cap V(T_3) = \emptyset \). Choose vertices \( v \) and \( w \) in \( T \) such that \( v \in V(T_2) \) and \( w \in V(T_3) \). Then \( (R - \{u\}) \cup \{v, w\} \) is a larger gp-set than \( R \) in \( T \), a contradiction.

Hence \( T - u \) has exactly two components, say \( T_1 \) and \( T_2 \). Now suppose that neither \( T_1 \) nor \( T_2 \) is a path. Then as above, we have \( R \cap V(T_1) = \emptyset \) or \( R \cap V(T_2) = \emptyset \). By symmetry, we assume that \( R \cap V(T_2) = \emptyset \). Since \( T_2 \) is not a path, there are at least two leaves \( x_1 \) and \( x_2 \) in \( T_2 \). Then the set \( (R - \{u\}) \cup \{x_1, x_2\} \) is a larger gp-set than \( R \), again, in \( T \). Therefore, at least one of \( T_1 \) and \( T_2 \) is a path.

Conversely, we observe that \( u \) is a non-leaf vertex on a pendant path in \( T \). Then \( u \) belongs to a gp-set in \( T \).

In \( G \sqcap H \), if \( h \in V(H) \), then the subgraph of \( G \sqcap H \) induced by the vertices \((g, h)\), \( g \in V(G) \), is a \( G \)-layer, denoted with \( G^h \). Analogously \( H \)-layers \( ^hH \) are defined. \( G \)-layers and \( H \)-layers are isomorphic to \( G \) and to \( H \), respectively. The distance function in Cartesian products is additive, that is, if \((g_1, h_1), (g_2, h_2) \in V(G \sqcap H)\), then

\[
d_{G \sqcap H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2).
\]

If \( u, v \in V(G) \), then the interval \( I_G(u, v) \) between \( u \) and \( v \) in \( G \) is the set of all vertices lying on shortest \( u, v \)-paths, that is,

\[
I_G(u, v) = \{w : d_G(u, v) = d_G(u, w) + d_G(w, u)\}.
\]

In what follows, the notations \( d_G(u, v) \) and \( I_G(u, v) \) may be simplified to \( d(u, v) \) and \( I(u, v) \) if \( G \) will be clear from the context. Equality (1) implies that intervals in Cartesian products have the following nice structure, cf. [4, Proposition 12.4].

Lemma 3. If \( G \) and \( H \) are connected graphs and \((g_1, h_1), (g_2, h_2) \in V(G \sqcap H)\), then

\[
I_{G \sqcap H}((g_1, h_1), (g_2, h_2)) = I_G(g_1, g_2) \times I_H(h_1, h_2).
\]

Equality (1) also easily implies the following fact (also proved in [13]).

Lemma 4. Let \( G \) and \( H \) be connected graphs and \( R \) a general position set of \( G \sqcap H \). If \( u = (g, h) \in R \), then \( V(^hH) \cap R = \{u\} \) or \( V(G^h) \cap R = \{u\} \).

For finite paths the already mentioned result \( \text{gp}(P_{\infty} \sqcap P_{\infty}) = 4 \) reduces to:
Lemma 5. ([10]) If \( n_1, n_2 \geq 2 \), then

\[
gp(P_{n_1} \square P_{n_2}) = \begin{cases} 
4; & \min\{n_1, n_2\} \geq 3, \\
3; & \text{otherwise}.
\end{cases}
\]

To conclude the preliminaries we construct special maximal (with respect to inclusion) general position sets in products of trees.

**Lemma 6.** Let \( T \) and \( T^* \) be two trees with \( \min\{n(T), n(T^*)\} \geq 3 \), \( v_i \in V(T) \setminus L(T) \), and \( v_j^* \in V(T^*) \setminus L(T^*) \). Then \( (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) is a maximal general position set of \( T \square T^* \).

**Proof.** Set \( R = (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) and let \( V_0 = \{u, v, w\} \subseteq R \). We first consider the case when \( V_0 \subseteq L(T) \times \{v_j^*\} \) or \( V_0 \subseteq \{v_i\} \times L(T^*) \). By symmetry, assume that \( V_0 \subseteq L(T) \times \{v_j^*\} \). Then each vertex of \( V_0 \) is corresponding to a leaf of \( L(T) \) in the layer \( T^*_j \cong T \). Therefore \( u, v, w \) do not lie on a common geodesic in \( T \square T^* \).

In the following, without loss of generality, we can assume that \( u, w \in L(T) \times \{v_j^*\} \) with \( u = (v_k, v_j^*), w = (v_s, v_i^*) \) and \( v = (v_i, v_i^*) \in \{v_i\} \times L(T^*) \). By Equality (1), we have \( d(u, v) = d_T(v_k, v_i) + d_{T^*}(v_j^*, v_s) \) and \( d(u, w) = d_T(v_k, v_s) + d_{T^*}(v_j^*, v_i^*) \). Note that \( v_k, v_s \) are two distinct vertices in \( L(T) \) of \( T \) and \( v_i \in V(T) \setminus L(T) \).

Then \( d_T(v_k, v_i) < d_T(v_k, v_s) + d_T(v_s, v_i) \) whenever \( v_i \) lies on the \( v_k, v_s \)-geodesic or outside \( v_k, v_s \)-geodesic of \( T \). This implies that \( d(u, v) < d(u, w) + d(w, v) \) in \( T \square T^* \). Therefore \( w \) does not lie on the \( u, v \)-geodesic in \( T \square T^* \). Analogously, neither \( u \) lies on the \( v, w \)-geodesic nor \( v \) lies on the \( u, w \)-geodesic of \( T \square T^* \). Thus \( u, v, w \) do not lie on a common geodesic in \( T \square T^* \), which implies that \( R \) is a general position set in \( T \square T^* \).

Next we prove the maximality of \( (L(T) \times \{v_j^*\}) \cup (\{v_i\} \times L(T^*)) \) as a general position set in \( T \square T^* \). Otherwise, there is a general position set \( R' \) in \( T \square T^* \) of order greater than \( \ell(T) + \ell(T^*) \) such that \( R \subset R' \). Then there exists a vertex \( z \in R' \setminus R \), say \( z = (v_p, v_q^*) \). If \( p = i \), then there exist two vertices \( (v_i, v_s^*), (v_i, v_i^*) \in \{v_i\} \times L(T^*) \) such that \( z \in I_{T \square T^*}((v_i, v_s^*), (v_i, v_i^*)) \) (since \( n^*T^* \cong T^* \)). This is a contradiction showing that \( p \neq i \). Similarly, we have \( q \neq j \). Now we consider the positions of \( v_p \) in \( T \) and \( v_q^* \) in \( T^* \).

Suppose first that \( v_p \in L(T) \), \( v_q^* \in L(T^*) \). Then there are two vertices \( (v_p, v_j^*), (v_i, v_i^*) \) in \( R \) such that \( z \in I_{T \square T^*}((v_p, v_j^*), (v_i, v_i^*)) \), contradicting that \( R \cup \{z\} \) is a general position set of \( T \square T^* \). If \( v_p \in L(T) \) and \( v_q^* \notin L(T^*) \), then we select a vertex \( v_q^* \in L(T^*) \) such that \( v_q^* \) is closer to the leaf of the corresponding branching path than \( v_q^* \) in \( T^* \). Then \( z \in I_{T \square T^*}((v_p, v_j^*), (v_i, v_q^*)) \), a contradiction. Similarly, \( v_p \notin L(T) \) and \( v_q^* \in L(T^*) \) cannot occur. Finally we assume that \( v_p \notin L(T) \), \( v_q^* \notin L(T^*) \). Now we select two vertices \( v_p \in L(T) \) and \( v_q^* \in L(T^*) \) such that \( v_p \) is closer to the leaf of the branching path than \( v_p \) in \( T \) and \( v_q^* \) is closer to the leaf of the branching path than \( v_q^* \) in \( T^* \). But then \( (v_p, v_q^*) \in I_{T \square T^*}((v_p, v_j^*), (v_i, v_q^*)) \), a final contradiction.

\[ \Box \]
3 Proof of Theorem 1

If $T$ and $T^*$ are both paths, then Theorem 1 holds by Lemma 5. In the following we may thus without loss of generality assume that $T^*$ is not a path. Lemma 6 implies that $\text{gp}(T \sqcap T^*) \geq \text{gp}(T) + \text{gp}(T^*)$, hence it remains to prove that $\text{gp}(T \sqcap T^*) \leq \text{gp}(T) + \text{gp}(T^*)$. Set $n = n(T)$, $n^* = n(T^*)$, $V(T) = \{v_1, \ldots, v_n\}$, and $V(T^*) = \{v_1^*, \ldots, v_n^*\}$.

Assume on the contrary that there exists a general position set $R$ of $T$ such that $|R| > \text{gp}(T) + \text{gp}(T^*)$. Since the restriction of $R$ to a $T$-layer of $T \sqcap T^*$ is a general position set of the layer (which is in turn isomorphic to $T$), the restriction contains at most $\text{gp}(T) = \ell(T)$ elements. Similarly, the restriction of $R$ to a $T^*$-layer contains at most $\text{gp}(T^*) = \ell(T^*)$ elements. We may now distinguish the following cases.

Case 1. There exists a $T$-layer $Tv_1^*$ with $|V(Tv_1^*) \cap R| = \text{gp}(T)$, or a $T^*$-layer $Tv_1$ with $|V(Tv_1) \cap R| = \text{gp}(T^*)$.

By the commutativity of the Cartesian product, we may without loss of generality assume that there is a layer $Tv_1^*$ with $|R \cap V(Tv_1^*)| = \text{gp}(T^*)$. Let $R = R_1 \cup R_2$, where $R_1 = R \cap V(Tv_1^*)$ and $R_2 = R \setminus R_1$, that is, $R_2 = \bigcup_{t \in [n] \setminus \{i\}} (V(Tv_1^*) \cap R)$. Let further $S^*$ be the projection of $R \cap V(Tv_1^*)$ on $T^*$, that is, $S^* = \{v_1^*: (v_i, v_1^*) \in R_1\}$. Since $|R_1| = \text{gp}(T^*)$, our assumption implies $|R_2| \geq \text{gp}(T) + 1$. Then, as $\text{gp}(T) = \ell(T)$, there exist two different vertices $w = (v_p, v_q^*)$ and $w' = (v_{p'}, v_{q'}^*)$ from $R_2$ such that $v_p$ and $v_{p'}$ lie on a same branching path $P$ of $T$. (Note that it is possible that $v_p = v_{p'}$.) We may assume that $d_T(v_{p'}, x) \leq d_T(v_p, x)$, where $x$ is the leaf of $P$. We proceed by distinguishing two subcases based on the position of $v_q^*$ and $v_{q'}^*$ in $T^*$.

Case 1.1. There exists a branching path $P^*$ of $T^*$ that contains both $v_q^*$ and $v_{q'}^*$.

Recall that $T^*$ is not a path. Lemma 2 implies that a vertex of a tree belongs to a gp-set if and only if it lies on a pendant path and has degree 1 or 2. Therefore, we can select $P^*$ with the root of degree at least 3. Assume that $d_{T^*}(v_q^*, y) \leq d_{T^*}(v_{q'}^*, y)$, where $y$ is the leaf of $P^*$. (The reverse case can be treated analogously.) Since $S^*$ is a gp-set of $T^*$ which is not isomorphic to a path, there is a vertex $v_k^* \in S^*$ lying on $P^*$. So we may consider that $P^*$ is a branching path that contains $v_q^*$, $v_{q'}^*$, and a vertex $v_k^* \in S^*$. (It is possible that some of these vertices are the same.) Let $z = (v_i, v_k^*)$. Then $z \in R_1$. We proceed by distinguishing the following subcases based on the position of $v_p$, $v_{p'}$, and $v_i$ in $T$.

Subcase 1.1.1. $v_{p'} \in I(v_i, v_p)$.

In this subcase, if $v_k^*$ is closer than $v_q^*$, $v_{q'}^*$ to the leaf $y$ of $P^*$, then, by Lemma 3, $w' \in I_{T \sqcap T^*}(w, z)$, a contradiction.

If $v_k^* \in I(v_q^*, v_{q'}^*)$, then since $\ell(T^*) \geq 3$, there exists $z' = (v_i, v_{k'}^*) \in \{v_i\} \times S^*$ such
that \( v_k^*, v_q^* \in I(v_q^*, v_k^*) \) in \( T^* \). Then we have
\[
d(w', z') = d_T(v_{p'}, v_i) + d_T(v_q^*, v_{k'}^*) = d_T(v_p, v_i) + d_T(v_q^*, v_k^*) + d_T(v_k^*, v_{k'}^*) = d(w', z) + d(z, z'),
\]
which implies that \( z \in I_{T \square T^*}(w', z') \), a contradiction.

**Subcase 1.1.2.** \( v_i \in I(v_p, v_{p'}) \).

In this subcase, if \( v_k^* \in I(v_q^*, v_{k'}^*) \) in \( P^* \), then \( z \in I_{T \square T^*}(w, w') \) by Lemma 3, a contradiction.

Assume that \( v_k^* \) is closer than \( v_q^*, v_{k'}^* \) to the leaf of \( P^* \). Since \( |S^*| = \ell(T^*) \geq 3 \), there is a vertex \( z' = (v_i, v_k^*) \in \{v_i\} \times S^* \) such that \( v_i^*, v_{k'}^* \in I(v_k^*, v_{k'}^*) \) in \( T^* \). Let \( v_i^* \) be on a branching path \( P''\) in \( T^* \) where \( P'' \neq P^* \). Note that \( \ell(T) + 1 \geq 3 \). There exists at least one vertex \( a = (v_x, v_y^*) \in R_2 \setminus \{w, w'\} \). Next we consider the positions of \( v_x, v_y^* \) in \( T, T^* \), respectively.

Suppose first that \( v_y^* \in V(P^* \cup P'') \). If \( v_x, v_p, v_{p'} \) and \( v_i \) lie on a path in \( T \), then there are five vertices \( w, w', z, z' \) and \( a \) in \( R_2 \), three of which lie on a common geodesic in \( T \square T^* \), a contradiction. Note that if \( T \) is a path, then we are done as above. Therefore, assume that \( T \) is not isomorphic to a path in the following and the root of \( P \) has degree at least 3. Otherwise, \( v_x \notin P \) and \( v_x, v_p \) lie on a common branching path in \( T \). Let \( V_s \) be the set of vertices of \( T \) but not contained in \( T_{ip} \) where \( T_{ip} \) is the subtree of \( T - v_p \) containing \( v_i \) and \( v_{p'} \). If there is a vertex \( a' = (v_s, v_i^*) \in R_2 \) with \( v_s \in V_s \), then \( R_2 \) contains \( w, w', z, z' \) and \( a' \), three of which are on a common geodesic, a contradiction. Therefore, the first coordinate of any vertex in \( R_2 \) cannot be in \( V_s \). Assume that \( P' \neq P \) is any branching path containing \( v_p \) and a leaf both in \( T_{ip} \) and \( T \). Then, besides \( w, P' \square T^* \) contains at most one vertex in \( R_2 \) of \( T \square T^* \). Otherwise, \( P' \square T^* \) contain two vertices \( h, h' \) in \( R_2 \). Then there exist two vertices \( h_0, h_0' \in \{v_i\} \times S^* \) such that three vertices from \( \{h, h', h_0, h_0', w\} \) lie on some geodesic in \( T \square T^* \), a contradiction. (Here \( h_0 \) may be equal to \( h_0' \).) Note that \( V_s \) contains at least two leaves of \( T \) since the root of \( P \) (just in \( V_s \)) has degree at least 3. Then \( T_{ip'} \) has at most \( \ell(T) - 2 \) leaves in \( T \). Since \( P \square T^* \) contains two vertices \( w \) and \( w' \) in \( R_2 \), we have \( |R_2| \leq \ell(T) - 2 + 1 < \ell(T) = \text{gp}(T) \), a contradiction with the assumption.

Assume now that \( v_y^* \notin V(P^* \cup P'') \). Then there exists a vertex \( z'' = (v_i, v_{k''}) \in \{v_i\} \times S^* \) such that \( v_y^*, v_{k''} \) lie on a common branching path in \( T^* \). If \( v_y^* \) is closer to the leaf of the branching path than \( v_{k''} \) in \( T^* \), then \( v_i \in I(v_x, v_i) \) and \( v_{k''} \in I(v_y^*, v_{k''}) \). Therefore, by Lemma 3, we get \( z'' \in I_{T \square T^*}(a, z) \), a contradiction. In the case that \( v_{k''} \) is closer to the leaf of the branching path than \( v_y^* \) in \( T^* \), we consider the positions of \( v_x, v_p, v_{p'} \) and \( v_i \) in \( T \). Let \( V_1 = \{z, z', w, w', a, z''\} \). Then \( V_1 \subseteq R_2 \). If \( v_x, v_p, v_{p'} \) and \( v_i \) lie on a path in \( T \), then there exist three vertices in \( V_1 \) lying on a common geodesic in
In this subcase, if \( v \in I(v_i, v_p) \).

In this subcase, since \( \ell(T^*) \geq 3 \), there exists a vertex \( z' = (v_i, v_{k'}) \in \{v_i\} \times S^* \) such that \( v_{k'} \notin P^* \) and \( v_q^* \in I(v_i, v_{k'}) \) in \( T^* \). Since

\[
d(z', w') = d_T(v_i, v_{k'}) + d_{T^*}(v_{k'}, v_q^*) = d_T(v_i, v_p) + d_{T^*}(v_{k'}, v_q^*) + d_T(v_p, v_{k'}) + d_{T^*}(v_{k'}, v_q^*) = d(z', w) + d(w, w'),
\]

we have \( w \in I_{T \square T^*}(z', w') \), a contradiction.

**Subcase 1.1.4.** \( v_i \notin V(P) \) such that \( v_i, v_p \) lie on a same branching path in \( T \).

In this subcase, since \( \ell(T^*) \geq 3 \), there is a vertex \( z' = (v_i, v_{k'}) \in \{v_i\} \times S^* \) such that \( v^*_q \in I(v_{k'}, v_{k'}) \) in \( T^* \). If \( v_{k'} \in I(v_q^*, v_{k'}) \), then obviously \( v_{k'} \in I(v_q^*, v_{k'}) \) and therefore,

\[
d(w', z') = d_T(v_p, v_i) + d_{T^*}(v_{k'}, v_{k'}) = d_T(v_p, v_i) + d_{T^*}(v_{q}, v_{k'}) + d_{T^*}(v_{k'}, v_{k'}) = d(w', z) + d(z, z')).
\]

We conclude that \( z \in I_{T \square T^*}(w', z') \), a contradiction.

If \( v_{k'}^* \) is closer to the leaf of \( P^* \) than \( v_{q}^*, v_{q'}^* \), then we get a contradiction similarly as in Subcase 1.1.2.

**Case 1.2.** \( v_q^* \) and \( v_{q'}^* \) do not lie on a same branching path in \( T^* \).

In this subcase, we may assume that \( v_q^* \) and \( v_{q'}^* \) lie on distinct branching paths \( P^* \) and \( P'^* \) in \( T^* \), respectively. Since \( \ell(T^*) \geq 3 \) and \( T^* \) is not isomorphic to a path, there exist two vertices \( z = (v_i, v_{k'}) \) and \( z' = (v_i, v_{k'}) \) from \( \{v_i\} \times S^* \), such that \( v_{k'} \in P^* \) and \( v_{k'}^* \in P'^* \). We consider the following subcases based on the positions of \( v_p, v_{p'} \) and \( v_i \) in \( T \).

**Subcase 1.2.1.** \( v_{p'} \in I(v_i, v_{p}) \).

In this subcase, if \( v_{k'}^* \) is closer than \( v_{q'}^* \) to the leaf of \( P'^* \), then \( v_{p'}^* \in I(v_{p}, v_{i}) \) and \( v_{q'}^* \in I(v_{p}, v_{k'}) \). Lemma 3 gives \( w' \in I_{T \square T^*}(w, z') \), a contradiction. On the other hand, if \( v_{q'}^* \) is closer than \( v_{k'}^* \) to the leaf of \( P'^* \), then \( v_i \in I(v_{i}, v_{p'}) \) and \( v_{k'}^* \in I(v_{k'}, v_{q'}) \), hence Lemma 3 gives \( z' \in I_{T \square T^*}(w', z) \), a contradiction again.

**Subcase 1.2.2.** \( v_i \in I(v_{p}, v_{p'}) \).

In this subcase, we first assume that \( v_{q'}^* \) is closer than \( v_{k'}^* \) to the leaf of \( P'^* \). Then \( v_i \in I(v_{i}, v_{p'}) \) and \( v_{k'}^* \in I(v_{k'}, v_{q'}) \). Therefore, by Lemma 3, we get \( z' \in I_{T \square T^*}(z, w') \) as a contradiction. Otherwise we suppose that \( v_{k'}^* \) is closer than \( v_{q'}^* \) to the leaf of \( P'^* \). If \( v_{q'}^* \)
is closer than \( v_k^* \) to the leaf of \( P^* \), then \( v_i \in I(v_p, v_i) \) and \( v_k^* \in I(v_q^*, v_k^*) \). Therefore, by Lemma 3, we get \( z \in I_{T \sqcap T^*}(w, z') \), a contradiction. In the case that \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), we find a contradiction similarly as the proof of Subcase 1.1.2.

**Subcase 1.2.3.** \( v_p \in I(v_i, v_p') \).

In this subcase, if \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), then \( v_p \in I(v_i, v_p') \) and \( v_k^* \in I(v_q^*, v_k^*) \). So Lemma 3 gives \( w \in I_{T \sqcap T^*}(z, w') \), a contradiction. And if \( v_q^* \) is closer than \( v_k^* \) to the leaf of \( P^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_k^*, v_q^*) \), hence we get \( z \in I_{T \sqcap T^*}(z', w) \).

**Subcase 1.2.4.** \( v_i \notin V(P) \) such that \( v_i, v_p \) lie on a same branching path in \( T \).

First suppose that \( v_q^* \) is closer to the leaf than \( v_k^* \) in \( P^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_q^*, v_k^*) \). Thus, by Lemma 3, we get \( z \in I_{T \sqcap T^*}(w, z') \).

Assume that \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \). If \( v_q^* \) is closer to the leaf than \( v_k^* \), then \( v_i \in I(v_i, v_p) \) and \( v_k^* \in I(v_q^*, v_k^*) \), which gives \( z' \in I_{T \sqcap T^*}(z, w') \). If \( v_k^* \) is closer than \( v_q^* \) to the leaf of \( P^* \), we can proceed similarly as in Subcase 1.1.4.

Now we turn to the second case.

**Case 2.** \( |R \cap V(v_k T^*)| < \ell(T^*) \) for any \( k \in [n] \), and \( |R \cap V(v_q T^*)| < \ell(T) \) for any \( t \in [n^*] \).

In this case, let \( v_k T^* \) be a layer with \( |R \cap V(v_k T^*)| = \max\{|R \cap V(v_k T^*)| : k \in [n]\} \). Let \( R = R_1 \cup R_2 \) where \( R_1 = R \cap V(v_k T^*) \) and \( R_2 = R \setminus R_1 \), that is, \( R_2 = \bigcup_{k \in [n] \setminus \{i\}} (V(v_k T^*) \cap R) \). Set further \( S^* = \{v_j^*: (v_i, v_j^*) \in R_1\} \). Then \( 1 \leq |S^*| \leq \ell(T^*) - 1 \).

Assume first \( |S^*| = 1 \). Therefore \( |R \cap V(v_k T^*)| \leq 1 \) for any \( k \in [n] \). Next we only need to consider \( |R \cap V(v_q T^*)| \leq 1 \) for any \( j \in [n^*] \). (If \( |R \cap V(v_q T^*)| \geq 2 \) for some \( j \in [n^*] \), by commutativity of \( T \sqcap T^* \), the proof is similar to the subcase in which \( 2 \leq |S^*| \leq \ell(T^*) - 1 \). Therefore, suppose that \( |R \cap V(v_q T^*)| \leq 1 \) for any \( j \in [n^*] \). Then \( |R| \leq \min\{n, n^*\} \). We now claim that \( |R| \leq \ell(T) + \ell(T^*) \). If not, then since \( |R| \geq \ell(T) + \ell(T^*) + 1 \geq 6 \), there exist three vertices \( u = (v_p, v_j^*), v = (v_p, v_q^*) \) and \( w = (v_s, v_t^*) \) from \( R \) such that \( v_p, v_p \) lie on a same branching path in \( T \), and \( v_j^*, v_q^* \) lie on a common branching path in \( T^* \). Note that there may be \( p' = s, q = t \). But we can always select a vertex \( h \in R \setminus \{u, v, w\} \) such that \( u, v, h \) or \( u, w, h \) lie on a same geodesic in \( T \sqcap T^* \), which is a contradiction. So our result holds when \( |S^*| = 1 \).

Suppose second that \( 2 \leq |S^*| \leq \ell(T^*) - 1 \). As \( |R_1| = |S^*| \), we need to prove that \( |R_2| \leq \ell(T) + \ell(T^*) - |S^*| \). Assume on the contrary that \( |R_2| \geq \ell(T) + \ell(T^*) - |S^*| + 1 \). Since \( |S^*| \geq 2 \), there are two distinct vertices \( w = (v_i, v_j^*) \) and \( w' = (v_i, v_j^*) \) from \( \{v_i\} \times S^* \). We distinguish the following cases based on the positions of \( v_j^*, v_j^* \) in \( T^* \).

**Case 2.1.** \( v_j^* \) and \( v_j^* \) lie on a same branching path \( P^* \) of \( T^* \).

In this subcase, we may without loss of generality assume that \( v_j^* \) is closer than \( v_j^* \).
to the leaf of $P^*$. Let $T_{v_j\ell}^*$ be the maximal subtree of $T^* - v_j'$ containing $v_j'$ and let $V_{s^*} = V(T^*) \setminus V(T_{v_j\ell}^*)$. Let further $S_1^* = \{v_q^* : v_q^* \in I(v_j', v_j^*), v_j^* \in S^* \cap V(T_{v_j\ell}^*)\}$. Now we prove the following claim.

**Claim 1.** If $z = (v_p, v_p^*) \in R_2$, then $v_j^* \in S_1^*$. 

**Proof of Claim 1.** If not, suppose first that $v_j^* \in V(P^*)$ is closer than $v_j'$ to the leaf of $P^*$. Then $v_i \in I(v_j, v_j')$ and $v_j^* \in I(v_j^*, v_j')$. Hence, $w' \in I_{T \cap T^*}(w, z)$. And if $v_j^* \in V_{s^*}$, then $v_j^* \in I(v_j^*, v_j')$. Combining this fact with $v_i \in I(v_i, v_p)$, we have $w \in I_{T \cap T^*}(w', z)$. This proves Claim 1.

By Claim 1, we have $|\bigcup_{v_j^* \in S_1^*} (V(T_{v_j\ell}^*) \cap R)| \geq \ell(T) + \ell(T^*) - |S^*| + 1 \geq \ell(T) + 1$.

Then there exist two vertices $z = (v_p, v_p^*)$ and $z' = (v_p', v_p'^*)$ from $\bigcup_{v_j^* \in S_1^*} (V(T_{v_j\ell}^*) \cap R)$ such that $v_j^*, v_{j'}^* \in S_1^*$ and $v_p, v_p'$ lie on a same branching path $P$ in $T$. Without loss of generality, let $v_p'$ be closer than $v_p$ to the leaf of $P$, and let $v_j^*, v_{j'}^* \in I(v_j^*, v_j')$ (by the definition of $S_1^*$). We consider the following subcases according to the positions of $v_i, v_p, v_p'$ in $T$.

**Subcase 2.1.1.** $v_p' \in I(v_i, v_p)$.

If $v_{j'}^*$ is closer than $v_j^*$ to $v_{j'}^*$ in $P^*$, then we have $v_p' \in I(v_i, v_p)$ and $v_{j'}^* \in I(v_{j'}^*, v_{j'}^*)$. Therefore, $z' \in I_{T \cap T^*}(z, w')$. And if $v_j^*$ is closer than $v_p'$ to $v_p$ in $P^*$, then we have $v_p' \in I(v_i, v_p)$ and $v_p' \in I(v_p^*, v_p'^*)$ and so $z' \in I_{T \cap T^*}(z, w)$.

**Subcase 2.1.2.** $v_i \in I(v_p, v_p')$.

Note that $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$. Then there exists at least a vertex $a = (v_j, v_j') \in \bigcup_{v_j^* \in S_1^*} (V(T_{v_j\ell}^*) \cap R)$ different from $z$ and $z'$. Based on the position of $v_j^* (v_j^* \in P^*$ or $v_j^* \notin P^*)$ in $T^*$, and the positions of $v_i, v_j, v_p$ and $v_p'$ in $T$, we get contradictions using a similar proof as in Subcase 1.1.2.

**Subcase 2.1.3.** $v_p \in I(v_i, v_p')$.

If $v_{j'}^*$ is closer than $v_j^*$ to $v_{j'}^*$ in $P^*$, then $v_j \in I(v_i, v_p')$ and $v_{j'}^* \in I(v_{j'}^*, v_{j'}^*)$, therefore $z \in I_{T \cap T^*}(w, z')$. And if $v_j^*$ is closer than $v_p'$ to $v_p^*$ in $P^*$, then $v_j \in I(v_i, v_p')$ and $v_j^* \in I(v_j^*, v_j'^*)$, hence $z \in I_{T \cap T^*}(w, z')$.

**Subcase 2.1.4.** $v_i \notin V(P)$ such that $v_i, v_p$ lie on a same branching path in $T$.

Since $\ell(T) + \ell(T^*) - |S^*| + 1 \geq 4$, there exists a vertex $(v_j, v_j') \in \bigcup_{v_j^* \in S_1^*} (V(T_{v_j\ell}^*) \cap R)$. Proceeding similarly as in Subcase 1.1.4, we get required contradictions. But then $|\bigcup_{v_j^* \in S_1^*} (V(T_{v_j\ell}^*) \cap R)| \leq \ell(T) + \ell(T^*) - |S^*|$, a contradiction with the assumption.

**Case 2.2.** $v_j^*, v_j'^*$ lie on different branching paths $P^*, P^*$ in $T^*$, respectively.

In this subcase, let $S_2^*$ be a set of vertices of $v_i T^*$ closer to the leaf of a branching path than $v_j^*$ for any $v_j^* \in S^*$. Note that $S^* \cap S_2^* = \emptyset$. We prove the following claim.

**Claim 2.** If $(v_p, v_p^*) \in R_2$, then $v_j^* \in V(T^*) \setminus (S^* \cup S_2^*)$. 

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Proof of Claim 2. Lemma 4 implies $v^*_i \notin S^*$. Assume that $v^*_i \in S^*_2$ lies on a same branching path for some $v^*_g$ in $T^*$. Note that $|S^*| \geq 2$. Then there exists another vertex $v^*_g$ such that $v^*_g \in I(v^*_i, v^*_g)$. Combining this fact with $v_i \in I(v_i, v_p)$, we arrive at a contradiction $w \in I_T(z, w')$. This proves Claim 2.

Let now $S^*_r = \{v^*_q : v^*_q \in I(v^*_g, v^*_g), v^*_g, v^*_g \in S^*\}$. By a parallel reasoning as in Subcase 2.1 and with Claim 2 in hands we infer that $| \bigcup_{v^*_i \in S^*_r} (V(T^*_{v^*_i}) \cap R) | \leq \ell(T)$.

Let $S = \{v_k : (v_k, v^*_i) \in \bigcup_{v^*_i \in S^*_r} (V(T^*_{v^*_i}) \cap R) \}$ and set $S^{**} = V(T^*) \setminus (S \cup S^*_r)$. From the assumption we have $| \bigcup_{v^*_i \in S^{**}} (V(T^*_{v^*_i}) \cap R) | \geq \ell(T) + \ell(T^*) - |S| - |S^*| + 1$. So there exists a vertex $z = (v_p, v^*_i) \in \bigcup_{v^*_i \in S^{**}} (V(T^*_{v^*_i}) \cap R)$, and we can always select two distinct vertices $u = (v_h, v^*_g)$ and $v = (v_{h'}, v^*_g')$ from $R$ such that $v_p$ and $v_h$ lie on a same branching path in $T$, while $v^*_i$ and $v^*_g$ lie on a common branching path in $T^*$. But we can choose another vertex $w \in R$ such that either $u, w, z$ or $u, v, z$ lie on a same geodesic in $T \Box T^*$ as a contradiction. Therefore,

$$| \bigcup_{v^*_i \in S^{**}} (V(T^*_{v^*_i}) \cap R) | \leq \ell(T) + \ell(T^*) - |S| - |S^*|.$$ 

and we are done.

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