Hamming polynomials and their partial derivatives

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Abstract

Hamming graphs are Cartesian products of complete graphs and partial Hamming graphs are their isometric subgraphs. The Hamming polynomial $h(G)$ of a graph $G$ is introduced as the Hamming subgraphs counting polynomial. $K_k$-derivates $\partial_k G$ ($k \geq 2$) of a partial Hamming graph are also introduced. It is proved that for a partial Hamming graph $G$, $\frac{\partial h(G)}{\partial x_k} = h(\partial_k G)$. A couple of combinatorial identities involving the coefficients of the Hamming polynomials of Hamming graphs are also proven.

Key words: Hamming graph; Hamming polynomial; Partial Hamming graph; Combinatorial identity

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1 Introduction

In this paper we introduce Hamming polynomials as counting polynomials (in several variables) that count induced subgraphs of a given graph which are isomorphic to Hamming graphs. More precisely, we define the Hamming polynomial

$$h(G) = h(G; x_2, x_3, \ldots, x_\omega)$$

of a graph $G$ as

$$\sum_{r_2, r_3, \ldots, r_\omega \geq 0} \alpha(G; r_2, r_3, \ldots, r_\omega) x_2^{r_2} x_3^{r_3} \cdots x_\omega^{r_\omega},$$

where $\alpha(G; r_2, r_3, \ldots, r_\omega)$ is the number of induced subgraphs of $G$ isomorphic to the Hamming graph $K_{r_2}^2 \square K_{r_3}^3 \square \cdots \square K_{r_\omega}^{\omega}$, and $\omega = \omega(G)$ is the clique number of $G$. For instance, $h(K_n; x_2, x_3, \ldots, x_n) = n + \sum_{r=2}^{n} \binom{n}{r} x_r$. For another example consider the graph from Fig. 1. Its Hamming polynomial is equal to $11 + 19x_2 + 5x_2^2 + 6x_3 + x_2x_3 + x_4$.

Figure 1: A partial Hamming graph

Special Hamming polynomials $\sum_{r_2 \geq 0} \alpha(G; r_2) x_2^{r_2}$ have been introduced in [5] and named cube polynomials. They have been further studied and applied in [2, 6]. In particular, using the cube polynomials one can obtain a very general form of the so-called Euler-type formulas for median graphs that were first presented in [13].

The natural environment for Hamming polynomials is formed by the isometric subgraphs of Hamming graphs, called partial Hamming graphs. These graphs have many intriguing structural properties and have been extensively studied, see [3, 4, 8, 11, 16, 17].

We proceed as follows. In the rest of this section we give definitions and concepts needed. In the next section we introduce the $K_k$-derivate $\partial_k G$ of a partial Hamming
graph $G$. For this sake we in particular extend an idea of Chung, Graham, and Saks from [7]. Our main result then asserts that the partial derivatives of the Hamming polynomials and the $K_k$-derivates of partial Hamming graphs are connected as follows. For any partial Hamming graph $G$ and for any $k$, $2 \leq k \leq \omega$,

$$\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = h(\partial^k G; x_2, \ldots, x_\omega).$$

In the last section we study the coefficients $\alpha(G; 0, \ldots, 0, d)$ where $G$ is a Hamming graph and obtain a couple of combinatorial identities. Along the way we also observe that Hamming polynomials are multiplicative on the Cartesian product of graphs.

The Cartesian product $G \Box H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where the vertex $(a, x)$ is adjacent to the vertex $(b, y)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The one vertex graph $K_1$ is the unit for Cartesian multiplication of graphs. The Cartesian product is commutative and associative, so the Cartesian product of several factors is well-defined. The product of $k$ copies of $K_2$ is called a $k$-cube while graphs of the form

$$G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_r},$$

where $r \geq 1$ and $n_i \geq 1$, $i = 1, \ldots, r$ are Hamming graphs. It is clear what we mean by $K_r^n$ for $r \geq 1$, and we define $K_0^n = K_1$.

A subgraph $H$ of a graph $G$ is called isometric if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the usual shortest path distance. In addition, $H$ is convex, if for all $u, v \in V(H)$, all shortest $u, v$-paths from $G$ belong to $H$. Isometric subgraphs of $k$-cubes are known as partial cubes while partial Hamming graphs are isometric subgraphs of Hamming graphs.

## 2 $K_k$-derivates of partial Hamming graphs

In the study of partial Hamming graphs, the following relation was used. Edges $ab, xy \in E(G)$ are in relation $\sim$ if $d(x, a) = d(y, b) = d(x, b) - 1 = d(y, b) - 1$. It was first introduced by Djoković in the context of bipartite graphs [9] where it coincides with the Winkler’s relation $\Theta$ [17]. Wilkeit [16] was the first to use the relation $\sim$ in the context of partial Hamming graphs.

The relation $\sim$ is reflexive and symmetric but it is in general not transitive, cf. $K_{2,3}$. However, in partial Hamming graphs $\sim$ is an equivalence relation on $E(G)$ and $\sim \subset \Theta$.

In this section we first extend the relation $\sim$ to a family of relations $\sim_k$ and then use them to define $K_k$-derivates of partial Hamming graphs. Then we prove our main result—Theorem 2.3.

Let $G$ be a partial Hamming graph and let $k$ be an integer, $2 \leq k \leq \omega$. Let $\mathcal{K}_k(G)$ be the set of all complete subgraphs of $G$ on $k$ vertices. Then we introduce
the relation $\sim_k$ defined on $\mathcal{K}_k(G)$ in the following way. Complete subgraphs $X, Y \in \mathcal{K}_k(G)$ on vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, respectively, are in relation $\sim_k$, if the notation of vertices can be chosen in such a way that there exists an integer $p$ such that
\[
d_G(x_i, y_j) = p + 1 \quad \text{for } i \neq j, \quad \text{and } d_G(x_i, y_i) = p.
\] (1)

Note that $\sim_2 = \sim$. We also say that $X$ and $Y$ are parallel, when they are in relation $\sim_k$. Such a parallelism relation was studied by Chung, Graham, and Saks [7], where the definition is restricted to quasi-median graphs and to maximal complete subgraphs. (Recall that quasi-median graphs form a proper subclass of partial Hamming graphs, cf. [1].)

**Lemma 2.1** Let $G$ be a partial Hamming graph and $2 \leq k \leq \omega$. Then $\sim_k$ is an equivalence relation on $\mathcal{K}_k(G)$.

**Proof.** Clearly, $\sim_k$ is reflexive and symmetric and different complete subgraphs $X, Y \in \mathcal{K}_k(G)$ that share a vertex cannot be in relation $\sim_k$. Let $X$, $Y$, and $Z$ be pairwise disjoint complete subgraphs from $\mathcal{K}_k(G)$ such that $X \sim_k Y$ and $Y \sim_k Z$. Let $x_i$, $y_i$, and $z_i$, $1 \leq i \leq k$, be the vertices of $X$, $Y$, and $Z$, respectively, where the notations are selected in the spirit of (1). Then for any $i \neq j$ the edge $x_ix_j$ of $X$ is in relation $\sim$ with the edge $y_iy_j$ of $Y$ and $y_iz_j$. Since $\sim$ is transitive, we derive $x_ix_j \sim z_iz_j$ and hence $X \sim_k Z$. We conclude that $\sim_k$ is transitive as well. □

Let $E$ be an equivalence class of the relation $\sim_k$, and let $\langle E \rangle$ be the subgraph of $G$ induced by the vertices from $E$. Note that $\langle E \rangle$ need not be connected. For instance, in $C_6$ each equivalence class with respect to $\sim_2$ consists of two disjoint $K_2$’s. We next show that $\langle E \rangle$ has a product structure.

**Lemma 2.2** Let $G$ be a partial Hamming graph, $2 \leq k \leq \omega$, and $E$ an equivalence class of $\sim_k$. Then there exists a graph $U_E$ such that $\langle E \rangle = K_k \square U_E$.

**Proof.** Let $\langle E \rangle$ be induced by complete subgraphs $X^{(1)}, X^{(2)}, \ldots, X^{(t)}$. We have already observed that these complete subgraphs are pairwise disjoint. Let $V(X^{(i)}) = \{x^{(i)}_1, x^{(i)}_2, \ldots, x^{(i)}_k\}$, where the notation is selected such that $d(x^{(i)}_1, x^{(i)}_1) < d(x^{(i)}_1, x^{(i)}_2)$ for $i = 2, \ldots, t$. Since $x^{(i)}_1x^{(2)}_1 \sim x^{(i)}_1x^{(i)}_2, x^{(i)}_1x^{(1)}_2 \sim x^{(j)}_1x^{(j)}_2$ ($i, j \neq 1, i \neq j$), and because $\sim$ is transitive we infer that $d(x^{(i)}_r, x^{(j)}_r) < d(x^{(i)}_r, x^{(j)}_s)$ for any $i \neq j$ and $r \neq s$. Set now $U_E = \langle \{x^{(1)}_1, x^{(2)}_1, \ldots, x^{(t)}_1\} \rangle$. Let $i \neq j$. Then for $r \neq s$ we have $x^{(i)}_r x^{(j)}_s \notin E(G)$ while $x^{(i)}_r x^{(j)}_s \in E(G)$ if and only if $x^{(i)}_1 x^{(j)}_1 \in E(G)$. It follows that $\langle E \rangle = K_k \square U_E$. □

Let $G$ be a partial Hamming graph and $E_1, \ldots, E_r$ the equivalence classes of relation $\sim_k$. Let $\langle E_i \rangle = K_k \square U_i$, cf. Lemma 2.2. Then we define the $K_k$-derivate
\( \partial_k G \) of \( G \) (with respect to \( k \)) as the disjoint union of the graphs \( U_i \):

\[
\partial_k G = \bigcup_{i=1}^{r} U_i.
\]

This definition is a generalization of the concept \( \partial G \) introduced for median graphs \( G \) in [4] since for median graphs \( G \), \( \partial_k G \) is defined only for \( k = 2 \) where \( \partial_2 G = \partial G \).

We are now ready for our main theorem.

**Theorem 2.3** Let \( G \) be partial Hamming graph. Then for any \( k \), \( 2 \leq k \leq \omega \),

\[
\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = h(\partial_k G; x_2, \ldots, x_\omega).
\]

**Proof.** Let \( H \) be a subgraph of \( G \) isomorphic to \( K^{r_2}_2 \Box K^{r_3}_3 \Box \cdots \Box K^{r_\omega}_\omega \), where \( r_k \geq 1 \) and \( r_i \geq 0 \) for \( i \neq k \). Clearly, \( H \) contributes 1 to the coefficient \( \alpha(G; r_2, r_3, \ldots, r_\omega) \).

For any factor graph of \( H \) isomorphic to \( K_k \), let us denote it \( K_k \), we can write

\[
H = K \Box (K^{r_2}_2 \Box \cdots \Box K^{r_k-1}_k \Box \cdots \Box K^{r_\omega}_\omega).
\]

Let \( E \) be the equivalence class of \( K_k \) with respect to \( \sim_k \). Then \( K_k \) has a vertex in common with any other factor of \( H \), hence \( K^{r_2}_2 \Box \cdots \Box K^{r_k-1}_k \Box \cdots \Box K^{r_\omega}_\omega \subseteq U_E \).

Therefore \( K_k \) contributes 1 to the coefficient \( \alpha(\partial_k G; r_2, r_3, \ldots, r_k-1, \ldots, r_\omega) \) and there are \( r_k \) copies of \( K_k \) that each belongs to its own equivalence class.

Similarly, if we have a Hamming graph \( X \) that contributes to a coefficient \( \alpha(\partial_k G; r_2, r_3, \ldots, r_k-1, \ldots, r_\omega) \), then Lemma 2.2 implies that there exists an \( U_E \) such that \( X \subseteq U_E \), where \( U_E \Box K_k \) is an induced subgraph of \( G \). Hence \( X \Box K_k \) contributes 1 to the coefficient \( \alpha(G; r_2, r_3, \ldots, r_\omega) \). Hence we conclude that

\[
\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = \sum_{r_2, \ldots, r_\omega \geq 0} r_k \alpha(G; r_2, \ldots, r_k, \ldots, r_\omega)x_2^{r_2} \cdots x_k^{r_k-1} \cdots x_\omega^{r_\omega} = h(\partial_k G; x_2, \ldots, x_\omega).
\]

\( \square \)

To illustrate Theorem 2.3 consider the partial Hamming graph \( G \) from Fig. 1. First, in Fig. 2 the equivalence classes of the relation \( \sim_2 \) are denoted.

Then \( \partial_2 G \) is shown in Fig. 3. Thus \( h(\partial_2 G) = 19 + 10x_2 + x_3 \) which is indeed equal to \( \frac{\partial h(G)}{\partial x_2} \).

The equivalence classes of the relation \( \sim_3 \) induce \( K_2 \Box K_3 \) and four triangles, that is, four copies of \( K_1 \Box K_3 \). Hence the graph \( \partial_3 G \) is the disjoint union of an edge and four isolated vertices, hence \( h(\partial_3 G) = 6 + x_2 \). Finally, \( \partial_4 G \) is just a vertex and therefore \( h(\partial_4 G) = 1 \).
Figure 2: Equivalence classes of relation \( \sim_2 \) on \( E(G) \)

Figure 3: \( K_2 \)-derivate of \( G \)

In order to be able to introduce \( K_k \)-derivates of higher order in the sense of [4], all components of every \( U_i \) should be partial Hamming graphs because then also \( \partial_k G \) would consist of partial Hamming graphs as components. If this concept can indeed be introduced in this way we pose as an open problem.

**Problem 2.4** Let \( G \) be a partial Hamming graph, and \( E \) an equivalence class of the relation \( \sim_k \). Is each connected component of \( \langle E \rangle \) a partial Hamming graph?

Note that in the case of quasi-median graphs this is true, more precisely \( \langle E \rangle \) is a convex subgraph in \( G \), hence a quasi-median graph. For instance, one can derive this fact from [1, Theorem 1].
3 Coefficients of Hamming polynomials in Hamming graphs

In this section we take a closer look to the coefficients $\alpha(G;0,\ldots,0,d)$, where $G$ is a Hamming graph. This enables us to obtain a couple of combinatorial identities. Related identities have been previously proved for hypercubes and median graphs, cf. [14, 15], as well as for quasi median graphs [4]. We first recall the following well-known fact.

**Lemma 3.1** Let $H = K_n$ be an (induced) subgraph of a Cartesian product graph $G$. Then $H$ projects isomorphically onto a factor of $G$.

With this lemma the following result follows easily.

**Proposition 3.2** For any graphs $G$ and $H$, $h(G \Box H) = h(G)h(H)$.

We next state the main result of this section. To prove it we apply a double counting of Hamming subgraphs and the binomial inversion, extending similar ideas for hypercubes from [10, 12]. Recall that the binomial inversion asserts that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad (\forall n \geq 0)$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \quad (\forall n \geq 0).$$

**Theorem 3.3** Let $G$ be the Hamming graph $K_r^n$ and let $\alpha_d = \alpha(G;0,\ldots,0,d)$. Then

$$\sum_{k=0}^{n} (-1)^k \alpha_k = (r-1)^n \quad \text{and} \quad -\sum_{k=0}^{n} (-1)^k k \alpha_k = n(r-1)^{n-1}.$$  

**Proof.** Note first that the vertices of $G = K_r^n$ can be identified with words of length $n$ over the alphabet $\{1,2,\ldots,r\}$, where two words are adjacent if and only if they differ in precisely one position. Hence by Lemma 3.1, $\alpha_d$ can be considered as the number of words of length $n$ over the alphabet $\{1,2,\ldots,r,K\}$ containing $d$ times the character $K$. Each character represents the part of a factor $K_r$ from the product we take to form a $K^d_r$. It may be either the whole clique (character $K$) or just a vertex. In the latter case, we can choose among $r$ vertices represented by $1,\ldots,r$.

We now compute $\alpha_d$ in two different ways.
First select where to put the characters $K$. We need to choose $d$ places among $n$ available. Then, in any other place, we have to choose which number $i$, $1 \leq i \leq r$, to put. Thus we get the following formula:

$$\alpha_d = \binom{n}{d} r^{n-d}.$$

On the other hand, we can first position the symbol 1. There can be any number of them from 0 to $n - d$. Let $l$ be the number of places left, $d \leq l \leq n$. Then, we place characters $K$ in $d$ places among $l$. Finally, we choose other $i$, $2 \leq i \leq r$, in the remaining $l - d$ positions. So we get

$$\alpha_d = \sum_{l=d}^{n} \binom{n}{l} \binom{l}{d} (r - 1)^{l-d}.$$

By the above double counting we have the equality

$$\binom{n}{d} r^{n-d} = \sum_{l=d}^{n} \binom{n}{l} \binom{l}{d} (r - 1)^{l-d}.$$

Applying the binomial inversion we get

$$\binom{n}{d} (r - 1)^{n-d} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \binom{l}{d} r^{l-d}.$$

When $d = 0$, this implies that

$$(r - 1)^n = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} r^l = \sum_{k=0}^{n} (-1)^k \binom{n}{k} r^{n-k} = \sum_{k=0}^{n} (-1)^k \alpha_k$$

which proves the first equality of our theorem.

For the second equality consider Equation 2 for $d = 1$:

$$n(r - 1)^{n-1} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} l r^{l-1} = \sum_{k=0}^{n} (-1)^k (n - k) \alpha_k.$$

Now using the first equality we obtain

$$- \sum_{k=0}^{n} (-1)^k k \alpha_k = nr(r - 1)^{n-1} - n(r - 1)^n = n(r - 1)^{n-1}$$

which completes the proof. \qed
References


