On local colorings of Cartesian product graphs

Sandi Klavžar \textsuperscript{a,b,c} Zehui Shao \textsuperscript{d,e}

\textsuperscript{a} Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
\textsuperscript{b} Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
\textsuperscript{c} Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
\texttt{sandi.klavzar@fmf.uni-lj.si}
\textsuperscript{d} Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, China
\textsuperscript{e} School of Information Science and Technology, Chengdu University, Chengdu, 610106, China
\texttt{zshao@cdu.edu.cn}

Abstract

A local coloring of a graph $G$ is a function $c : V(G) \to \mathbb{N}$ such that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist $u, v \in S$ such that the colors of $u$ and $v$ differ by at least the size of the subgraph induced by $S$. Note that a local coloring is in particular a proper usual vertex coloring. The maximum color assigned by a local coloring $c$ to a vertex of $G$ is called the \textit{value} of $c$ and denoted by $\chi_\ell(c)$. The \textit{local chromatic number} $\chi_\ell(G)$ of $G$ is the minimum value over all local colorings $c$ of $G$.

A local coloring is thus a usual coloring with two additional conditions: any induced path of length 2 must contain two vertices with colors differing by at least

1 Introduction

A \textit{local coloring} of a graph $G$ is a function $c : V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \leq |S| \leq 3$, there exist vertices $u, v \in S$ such that the colors of $u$ and $v$ differ by at least the size of the subgraph induced by $S$. Note that a local coloring is in particular a proper usual vertex coloring. The maximum color assigned by a local coloring $c$ to a vertex of $G$ is called the \textit{value} of $c$ and denoted by $\chi_\ell(c)$. The \textit{local chromatic number} $\chi_\ell(G)$ of $G$ is the minimum value over all local colorings $c$ of $G$.

A local coloring is thus a usual coloring with two additional conditions: any induced path of length 2 must contain two vertices with colors differing by at least
2 and any triangle contains two vertices with colors that differ by at least 3. If the latter condition is dropped, one speaks of the so-called semi-matching colorings which were studied in [4]. In the class of triangle-free graphs, local colorings and semi-matching colorings thus form the same concept. We add that local colorings are of similar nature as $L(p, q)$-labelings in which labels of adjacent vertices differ by at least $p$ and labels of vertices at distance 2 differ by at least $q$. More precisely, local colorings are similar to $L(1, 2)$-labelings. For more information on $L(p, q)$-labelings we refer to recent papers [3, 10] in which graph products were studied.

Local colorings were introduced by Chartrand et al. in [1] (although the next paper written on the topic [2] was eventually published two years earlier). In the seminar paper it was shown that the study of the local chromatic number cannot be reduced to 2-connected graphs. Several exact values of the invariant were obtained, as for complete multipartite graphs, and so in particular for complete graphs. In the subsequent paper [2] the emphasize was on regular graphs, where Cartesian products with one factor being a hypercube played the central role. In [7] it was proved that $\chi_\ell(G) \leq \Delta(G) - 2$ holds for any graph with $\Delta(G) \geq 3$ and different from $K_4$ and $K_5$. This result in particular confirms Conjecture 4.2 from [1] asserting that $\chi_\ell(G) = 4$ holds for cubic, non-bipartite, and non-complete graphs. In the subsequent paper [2] the emphasize was on regular graphs, where Cartesian products with one factor being a hypercube played the central role. In [7] it was proved that $\chi_\ell(G) \leq \Delta(G) - 2$ holds for any graph with $\Delta(G) \geq 3$ and different from $K_4$ and $K_5$. This result in particular confirms Conjecture 4.2 from [1] asserting that $\chi_\ell(G) = 4$ holds for cubic, non-bipartite, and non-complete graphs. In [8] local colorings of Kneser graphs were studied, this line of research was continued in [4] through the perspective of semi-matching colorings.

In this note we are interested in the local coloring of Cartesian products of graphs, primarily motivated with investigations in [7] where exact local chromatic numbers were determined for some specific products. We proceed as follows. In the next section concepts, definitions, and results needed are recalled. In Section 3 the local chromatic number is determined for Cartesian products of 3-chromatic graphs. In particular, the local chromatic number of products of cycles is extracted, a result that in one part corrects an assertion from [7]. In the final section we then prove that if $G$ and $H$ are graphs such that $\chi(G) \leq \lceil \chi_\ell(H)/2 \rceil$ and $H$ is triangle-free, then $\chi_\ell(G \Box H) = \chi_\ell(H)$.

## 2 Preliminaries

We will use the notation $[n]$ for the set $\{1, \ldots, n\}$. Graphs considered here are simple. If $G = (V(G), E(G))$ is a graph, then the order of $G$ is $|V(G)|$ and the size of $G$ is $|E(G)|$. As usual, $\chi(G)$ is the chromatic number of $G$. If a coloring uses $k$ colors then it is a $k$-coloring and if $k = \chi(G)$ then it is a $\chi$-coloring. $G$ is called $k$-chromatic if $\chi(G) \leq k$. Similarly, a local coloring that uses $k$ colors is a $k$-local coloring and if $k = \chi_\ell(G)$ then it is a $\chi_\ell$-coloring.

The Cartesian product of graphs $G$ and $H$ is the graph $G \Box H$ with the vertex set $G \times H$, and $(g, h)(g', h') \in E(G \Box H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product is commutative and associative, having the one vertex graph as a unit. The subgraph of $G \Box H$ induced by $g \times V(H)$,
where $g \in V(G)$, is isomorphic to $H$. It is called an $H$-layer (over $g$) and denoted $^gH$. Similarly, the subgraph of $G \square H$ induced by $V(G) \times h$, where $h \in V(H)$, is isomorphic to $G$, called a $G$-layer (over $h$) and denoted $G^h$. For more information on the Cartesian product of graphs see [5].

We now recall some results on the local chromatic number. Note first that if $G$ is a subgraph of $H$, then $\chi_\ell(G) \leq \chi_\ell(H)$. In the rest we will use the fact that a labeling $c$ of $V(G)$ is a local coloring if and only if (i) $c$ is a (usual) vertex coloring, (ii) every induced $P_3$ contains two vertices with colors at least two apart, and (iii) every induced triangle contains two vertices with colors at least three apart. We will also use the following:

**Proposition 1** ([1]) If $G$ is a connected bipartite graph of order at least 3, then $\chi_\ell(G) = 3$.

It is easy to prove Proposition 1: the lower bound follows because $G$ contains at least one induced $P_3$, the upper bound is obtained by coloring each vertex of one bipartition set of $G$ with 1 and each vertex of the other bipartition set with 3. We will also need the following result which follows by replacing color $i$ with $2i - 1$ for any $1 \leq i \leq \chi(G)$ in a $\chi$-coloring of $G$.

**Proposition 2** ([1]) For any graph $G$, $\chi(G) \leq \chi_\ell(G) \leq 2\chi(G) - 1$.

As already mentioned, the local chromatic number of complete multipartite graphs was determined in [2], for further use we state the following special case:

**Theorem 3** If $n \geq 1$, then $\chi_\ell(K_n) = \lceil (3n - 1)/2 \rceil$.

We will also often use (without explicitly mentioning it) the 1957 Sabidussi’s result [9] asserting that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$, cf. [5, Theorem 26.1].

### 3 Products of 3-chromatic graphs

**Theorem 4** Let $G$ and $H$ be 3-chromatic graphs with at least one edge.

(i) If $\chi(G) = \chi(H) = 2$, then $\chi_\ell(G \square H) = 3$.

(ii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 3$, then $\chi_\ell(G \square H) = 4$.

(iii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 4$, then $4 \leq \chi_\ell(G \square H) \leq 5$.

(iv) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 5$, then $\chi_\ell(G \square H) = 5$.

(v) If $\chi(G) = \chi(H) = 3$, then $\chi_\ell(G \square H) = 5$. 


Proof. (i) Since $G$ and $H$ each have at least one edge, $G \Box H$ contains at least one $C_4$ and since $G$ and $H$ are bipartite, $G \Box H$ is bipartite as well. Hence $\chi_\ell(G \Box H) = 3$ follows by Proposition 1.

(ii) Let $c_G$ be a $\chi$-coloring of $G$ (so $c_G : V(G) \to [2]$) and let $c_H$ be a $\chi_\ell$-coloring of $H$ (so $c_H : V(H) \to [3]$). Then define $c : V(G \Box H) \to [4]$ as follows:

$$c(g, h) = \begin{cases} 
1; & c_G(g) = 1, c_H(h) = 1 \text{ or } c_G(g) = 2, c_H(h) = 3, \\
2; & c_G(g) = 1, c_H(h) = 2, \\
3; & c_G(g) = 2, c_H(h) = 1 \text{ or } c_G(g) = 1, c_H(h) = 3, \\
4; & c_G(g) = 2, c_H(h) = 2.
\end{cases}$$

Note that the endvertices of any edge from a $G$-layer receive colors that differ by at least 2. Moreover, a possible triangle of $G \Box H$ can only lie in an $H$-layer, but $H$ is triangle-free because $\chi(H) = 3$. It now readily follows that $c$ is a $4$-$\chi_\ell$-coloring of $G \Box H$ and thus $\chi_\ell(G \Box H) \leq 4$. To see that $\chi_\ell(G \Box H) \geq 4$, consider a subgraph $X = K_2 \Box C_{2k+1}$, $k \geq 1$. (Such a subgraph exists because $\chi(H) = 3$.) Assume for a moment that $\chi_\ell(X) = 3$. Then each of the 4-cycles of $X$ must be colored with consecutive colors 1, 3, 1, 3. But this clearly leads to a contradiction after considering all the 4-cycles above the edges of $C_{2k+1}$. Therefore $\chi_\ell(X) \geq 4$ and hence also $\chi_\ell(G \Box H) \geq 4$. We conclude that $c$ is a $\chi_\ell$-coloring of $G \Box H$.

(iii) and (iv) If $\chi_\ell(H) = 4$, then $\chi_\ell(G \Box H) \geq 4$ and if $\chi_\ell(H) = 5$, then $\chi_\ell(G \Box H) \geq 5$. On the other hand, since $\chi(G \Box H) = 3$, Proposition 1 implies that $\chi_\ell(G \Box H) \leq 5$.

(v) Since $\chi(G) = \chi(H) = 3$, we have $\chi(G \Box H) = 3$ and hence Proposition 2 implies that $\chi_\ell(G \Box H) \leq 5$. It thus remains to prove that $\chi_\ell(G \Box H) \geq 5$. For this sake it suffices to prove that $\chi_\ell(C_{2m+1} \Box C_{2n+1}) \geq 5$ holds for any $m, n \geq 1$. Indeed, since $\chi(G) = \chi(H) = 3$, $G$ contains an (induced) odd cycle $C_{2m+1}$ and $H$ contains an (induced) odd cycle $C_{2n+1}$, hence $G \Box H$ contains an induced $C_{2m+1} \Box C_{2n+1}$.

Set $X = C_{2m+1} \Box C_{2n+1}$, and let $V(C_k) = [k]$ with natural adjacencies, so that $V(X) = [2m+1] \times [2n+1]$. Suppose on the contrary that $c$ is a 4-local-coloring of $X$.

We first claim that if $e = xy \in E(X)$, then $c(\{u, v\}) \neq \{1, 2\}$ and $c(\{u, v\}) \neq \{3, 4\}$. We may assume without loss of generality that $e$ lies in a $C_{2m+1}$-layer, that is, $x = (i, j)$ and $y = (i+1, j)$ for some $i \in [2m+1]$ and some $j \in [2n+1]$ (indices modulo $2m+1$ and $2n+1$, respectively). Suppose that $c(i, j) = 1$ and $c(i+1, j) = 2$. Then by the definition of the local coloring, $c(\{(i, j+1), (i+1, j+1)\}) = \{3, 4\}$. In the same way we get $c(\{(i+1, j+1), (i+2, j+2)\}) = \{1, 2\}$. Continuing the argument and having in mind that $C_{2n+1}$ is an odd cycle, we arrive at $c(i, j-1), c(i+1, j-1) = \{1, 2\}$, which is a clear contradiction. We analogously arrive at a contradiction if $c(i, j) = 3$ and $c(i+1, j) = 4$. This proves the claim.

Let $A_i = (c^{-1}(1) \cup c^{-1}(2)) \cap C_{2m+1}$ and $B_i = (f^{-1}(3) \cup f^{-1}(4)) \cap C_{2m+1}$. Then $|A_i| = |B_{i+1}|$ and $|B_i| = |A_{i+1}|$ hold for any $1 \leq i \leq 2n + 1$. Indeed, this follows from the above claim because $c(i, j) \in \{1, 2\}$ if and only if $c(i+1, j) \in \{3, 4\}$.
Clearly, \(|A_i| + |B_i| = 2m + 1\) and hence \(|A_i| \neq |B_i|\). Assume without loss of generality that \(|A_1| > |B_1|\). Then \(|A_i| \geq m + 1\) and \(|B_i| \leq m\) hold for \(i \in \{1, 3, \ldots, 2m + 1\}\). It follows that \(2m + 1 = |A_1| + |B_1| = |A_1| + |A_m| \geq 2m + 2\), the final contradiction. \(\square\)

Note that by the commutativity of the Cartesian product, Theorem 4 covers all possible products with 3-chromatic factors.

If \(\chi(G) = 2\), \(\chi(H) = 3\), and \(\chi_k(H) = 4\), Theorem 4 offers two possibilities: \(\chi_k(G \Box H) = 4\) or \(\chi_k(G \Box H) = 5\). Both can indeed happen.

To see that the first possibility can happen, recall that \(\chi_k(K_3) = 4\) and observe that \(\chi_k(K_2 \Box K_3) = 4\).

Consider now the product \(P_3 \Box K_4 - e\), where \(K_4 - e\) is the graph obtained from \(K_4\) by deleting one of its edges. Note first that \(\chi_k(K_4 - e) = 4\). Moreover, since \(K_3\) has exactly two \(\chi_k\)-colorings (with colors 1, 2, 4 and 1, 3, 4, respectively), it follows that \(K_4 - e\) has an essentially unique \(\chi_k\)-coloring: the vertices of degree 3 must be colored with 1 and 4, while the remaining two vertices are colored with 2 and 3. Therefore, the two non-adjacent \((K_4 - e)\)-layers in a 4-local coloring of \(P_3 \Box K_4 - e\) must have identical coloring and consequently one \(P_3\)-layer is colored with colors 2, 3, 2, which is not possible. We conclude that \(\chi_k(P_3 \Box K_4 - e) > 4\) and so \(\chi_k(P_3 \Box K_4 - e) = 5\).

**Corollary 5** If \(m, n \geq 3\), then

\[
\chi_k(C_m \Box C_n) = \begin{cases} 
3; & m, n \text{ are both even,} \\
4; & \text{exactly one of } m \text{ and } n \text{ is even,} \\
5; & m, n \text{ are both odd.}
\end{cases}
\]

**Proof.** Recall that \(\chi_k(C_n) = 3\) holds for any \(n \geq 4\). Then Theorem 4 (i), (ii), and (v) cover all the cases, except the case \(C_3 \Box C_{2n}, n \geq 2\), because \(\chi_k(C_3) = 4\) and we thus have two possibilities due to Theorem 4 (iii). Now color the first \(C_3\)-layer of \(C_3 \Box C_{2n}\) with 1, 2, 4, the second layer with 3, 4, 1, and alternately continue with this coloring. This is a local coloring and hence \(\chi_k(C_3 \Box C_{2n}) = 4\). \(\square\)

Corollary 5 corrects [8, Theorem 5] where it is claimed that \(\chi_k(C_{2m+1} \Box C_{2n+1}) = 4\). A coloring is proposed there for which it is claimed that it is easy to see to be a local coloring of value 4. An example of the proposed coloring is shown in Fig. 1, where it can be seen that the problem is the most outer 4-cycle which is colored 1, 2, 1, 2.

**4 More exact values**

For a long time (cf. [6]) it is known that there exist triangle-free graphs with an arbitrary high chromatic number. Consequently, there exist triangle-free graphs with an arbitrary high local chromatic number. In this section we prove:
Figure 1: A coloring of $C_7 \Box C_5$ which is not local

**Theorem 6** If $G$ and $H$ are graphs such that $\chi(G) \leq \left\lfloor \frac{\chi(H)}{2} \right\rfloor$ and $H$ is triangle-free, then $\chi_\ell(G \Box H) = \chi_\ell(H)$.

**Proof.** $\chi_\ell(G \Box H) \geq \chi_\ell(H)$ holds because $H$ is an induced subgraph of $G \Box H$. To prove the reverse inequality, let $c_G$ be a $\chi$-coloring of $G$, let $c_H$ be a $\chi_\ell$-coloring of $H$, and define a coloring $c$ of $G \Box H$ as

$$c(g, h) = 2c_G(g) + c_H(h) - 2,$$

where computations are done modulo $\ell = \chi_\ell(H)$. As $c : V(G \Box H) \to [\ell]$ it suffices to prove that $c$ is a local coloring.

Consider vertices $(g, h)$ and $(g', h)$, $g \neq g'$, such that $c_G(g) \neq c_G(g')$. Suppose first that $c_H(h) = 1$. Since $\chi(G) \leq \left\lfloor \frac{\chi(H)}{2} \right\rfloor$, we have

$$c(g, h) = 2c_G(g) - 1 \leq 2 \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \leq \ell - 1.$$

By the same argument, $c(g', h) \leq \ell - 1$. Since $c_G(g) \neq c_G(g')$ it follows that $|c(g, h) - c(g', h)| \geq 2$. Moreover, because $c(g, h), c(g', h) \leq \ell - 1$ we also have $|c(g, h) - c(g', h)| \leq \ell - 2$. By the definition of the coloring $c$ it follows that

$$|c(g, h) - c(g', h)| \geq 2 \quad (1)$$

holds for any vertex $h \in V(H)$.

Consider now an arbitrary triangle $T$ of $G \Box H$. By the definition of the Cartesian product, $T$ is an induced subgraph in either a $G$-layer or in an $H$-layer. Moreover, since $H$ is triangle-free, we infer that $T \subseteq G^h$ for some $h \in V(H)$. Then by (1), the vertices of $T$ are colored with colors that are pairwise at least 2 apart. So $T$ is properly locally colored.
Let \( P \) be an induced subgraph of \( G \) isomorphic to \( P_3 \). If \( P \subset G \) or \( P \subseteq G \), then we can argue as above for \( T \) that \( P \) is properly locally colored. Suppose now that \( P \) is induced on vertices \((g, h), (g', h), \) and \((g', h')\), where \( g \neq g' \) and \( h \neq h' \). Considering the edge \((g, h)(g', h)\) and applying (1) yield the required conclusion. \( \Box \)

Consider the product \( K_4 \square K_6 \). Since \( \chi_\ell(K_6) = \lfloor (3 \cdot 6 - 1)/2 \rfloor = 8 \), the condition \( \chi(K_4) \leq \lfloor \chi_\ell(K_6)/2 \rfloor \) is satisfied. On the other hand, by [8, Theorem 7], \( \chi_\ell(K_4 \square K_6) > \chi_\ell(K_6) \). This demonstrates that the condition for \( H \) to be triangle-free cannot be dropped in general.

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