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Recognizing Halved Cubes in a Constant Time Per Edge

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Graphs that can be isometrically embedded into the metric space l_1 are called l_1-graphs. Halved cubes play an important role in the characterization of l_1-graphs. We present an algorithm that recognizes halved cubes in $O(n \log^2 n)$ time.

1. INTRODUCTION

Hamming graphs are exactly those graphs which can be isometrically embedded into a Cartesian product of complete graphs. In the case in which every one of the factors is the complete graph K_2 on two vertices, one obtains an isometric embedding into a hypercube and speaks of a binary Hamming graph.

If one relaxes the condition of isometry and considers so-called scale embeddings into hypercubes a class larger than that of Hamming graphs arises. It has been characterized by Assouad and Deza [1] as the class of graphs isometrically embeddable into the metric space l_1.

By a scale embedding of a graph G into a graph H we mean a mapping

$$\psi: V(G) \to V(H)$$

for which there exists a positive integer λ such that

$$d_H(\psi(u), \psi(v)) = \lambda d_G(u, v)$$

for all $u,v \in V(G)$, where d_H and d_G denote the usual path distance in G and H, respectively. It has been proved by Deza and Grishukhin [4] and Shpectorov [13] that a graph G is an l_1-graph iff it is an isometric subgraph of a Cartesian product of complete graphs, cocktail party graphs or halved cubes, and that l_1-graphs can be recognized in polynomial time.

Aurenhammer *et al.* [2] (see also Imrich and Klavžar [11]) proved that it can be decided in $O(mn)$ time whether a given graph on n vertices and m edges is Hamming graph. Thus, the question whether this is also true for l_1-graphs arises. We cannot answer this question; in particular, we cannot efficiently decide whether a graph is an isometric subgraph of a halved cube. However, as a first step, we are able to recognize halved cubes in $O(n \log^2 n)$ steps, where n denotes the number of vertices.

2. RECOGNIZING HALVED CUBES

Let Q_d be the d-cube and let $V_1 \cup V_2$ be its bipartition as a bipartite graph. Then the *halved cube* Q'_d is the graph with $V(Q'_d) = V_1$, where u is adjacent to v in Q'_d if $d_{Q_d}(u, v) = 2$. Clearly, Q'_d has 2^{d-1} vertices and is $\binom{d}{2}$ regular (in fact, it is also distance-regular). Note that Q'_3 is isomorphic to K_4 and that Q'_4 is isomorphic to the cocktail party graph on 8 vertices. To simplify the presentation we may henceforth assume that $d \geq 5$.

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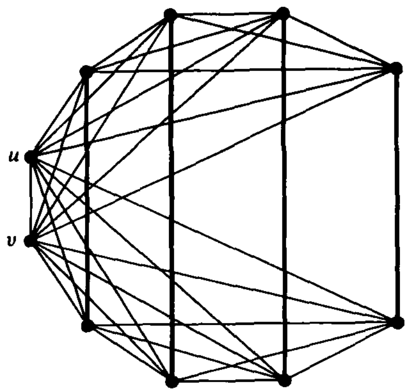

FIGURE 1. The two cliques of Q'_6 containing the edge uv.

Hemmeter [10] showed that there are only two types of cliques of Q'_d, with size 4 and d, respectively. Furthermore, every vertex of Q'_d is in precisely d cliques of size d, which implies that Q'_d has 2^{d-1} cliques of size d. Moreover, every d-clique of Q'_d is induced by the neighborhood (in Q_d) of a vertex in V_2. Based on these observations we first present a procedure TWO-CLIQUES which, for a given edge uv, computes the cliques Q and Q' to which uv belongs. The main part of the procedure is to detect a single edge $e \neq uv$ which should belong to Q if the graph in question really is a halved cube. The rest is simple: if a vertex w is adjacent to both vertices of e it should belong to Q; otherwise it should belong to Q'. We finally check if Q and Q' are indeed complete subgraphs. To follow TWO-CLIQUES more easily we have depicted the local structure of a halved cube described above in Figure 1 (for the case $d = 6$). Matching edges are indicated in bold.

TWO-CLIQUES (uv, Q, Q');

1. Let $Y = N(u) \cap N(v)$; if $|Y| \neq 2(d-2)$ then reject;
2. $Q := \{u, v\}$; $Q' := \{u, v\}$;
3. Let w_1, w_2 and w_3 be any three vertices from Y and let G_1 be the subgraph induced by w_1, w_2 and w_3;
   - 3.1. If G_1 has no edge then reject;
   - 3.2. If G_1 has three edges then $Q := Q \cup \{w_1, w_2, w_3\}$, $Y := Y - \{w_1, w_2, w_3\}$;
   - 3.3. If G_1 has one edge, say w_1w_2, then $Q := Q \cup \{w_1, w_2\}$, $Y := Y - \{w_1, w_2\}$;
   - 3.4. If G_1 is a path on three vertices, say $w_1w_2w_3$, then let w_4 be an additional vertex of Y and let G_2 be the subgraph induced by w_1, w_2, w_3 and w_4;
     - 3.4.1. If G_2 has three edges and w_4 is adjacent to either w_1 or w_3, say w_3, then $Q := Q \cup \{w_1, w_2\}$, $Q' := Q' \cup \{w_3, w_4\}$, $Y := Y - \{w_1, w_2, w_3, w_4\}$;
     - 3.4.2. If G_2 has four edges and w_4 is adjacent to either w_1 and w_2 or to w_2 and w_3, say to w_1 and w_2, then $Q := Q \cup \{w_1, w_2, w_4\}$, $Q' := Q' \cup \{w_3\}$, $Y := Y - \{w_1, w_2, w_3, w_4\}$;
     - 3.4.3. If G_2 is the 4-cycle C_4 then let w_5 be an additional vertex of Y;
       - 3.4.3.1. If w_5 is adjacent to exactly two adjacent vertices of G_2, say w_1 and w_2, then $Q := Q \cup \{w_1, w_2, w_5\}$, $Q' := Q' \cup \{w_3, w_4\}$, $Y := Y - \{w_1, w_2, w_3, w_4, w_5\}$;
       - 3.4.3.2. In all the other cases reject;
     - 3.4.4. If none of the above cases occur then reject;

4. For any vertex $w \in Y$ do: if w is adjacent to both w_1 and w_2 then $Q := Q \cup \{w\}$ else $Q' := Q' \cup \{w\}$;
5. If $|Q| \neq d$ then reject;
6. If $|E(Q)| < \binom{d}{2}$ or $|E(Q')| < \binom{d}{2}$ then reject.

We next describe a procedure which computes all d-cliques of G. The cliques of a graph will be consecutively indexed and for an edge uv let the variable $l_{u,v}$ be defined by

$$l_{u,v} = \begin{cases} 0, & uv \text{ does not belong to any clique yet;} \\ i, & uv \text{ belongs to a clique indexed } i; \\ -1, & uv \text{ belongs to two cliques.} \end{cases}$$

CLIQUES $(Q_1, Q_2, \ldots, Q_n)$;
1. If G is not a connected, $\binom{d}{2}$-regular graph on 2^{d-1} vertices for some $d \geqslant 5$, then reject;
2. $i := 0$; {clique counter}
3. For all edges $uv \in E(G)$ do $l_{u,v} := 0$;
4. While there is an edge $uv \in E(G)$ with $l_{u,v} \neq -1$ and $i < n = 2^{d-1}$ do:
   4.1. TWO-CLIQUES (uv, Q, Q');
   4.2. If $l_{u,v} = 0$ then
         $Q_{i+1} := Q$; $Q_{i+2} := Q'$;
         for all $wz \in E(Q_{i+1})$ do
            if $l_{w,z} = 0$ then $l_{w,z} := i + 1$ else
               if $l_{w,z} > 0$ then $l_{w,z} := -1$, otherwise reject;
         for all $wz \in E(Q_{i+2})$ do
            if $l_{w,z} = 0$ then $l_{w,z} := i + 2$ else
               if $l_{w,z} > 0$ then $l_{w,z} = -1$, otherwise reject;
         $i := i + 2$
      else
         if $Q = Q_{l_{u,v}}$ then $Q_{i+1} := Q'$ else
            if $Q' = Q_{l_{u,v}}$ then $Q_{i+1} := Q$ else reject;
         for all $wz \in E(Q_{i+1})$ do
            if $l_{w,z} = 0$ then $l_{w,z} := i + 1$ else
               if $l_{w,z} > 0$ then $l_{w,z} := -1$, otherwise reject;
         $i := i + 1$;
5. If $i < n$ then reject.

We have arrived to the procedure for recognizing halved cubes:

HALVED-CUBE (G);
1. CLIQUES $(Q_1, Q_2, \ldots, Q_n)$;
2. Let H be a graph which we obtain from G in the following way. To every d-clique Q of G detected by CLIQUES we add a new vertex and join it to every vertex of Q. These are the newly defined edges of H. The original edges of G are all removed.
3. If H is Q_d then G is Q'_d, otherwise reject.

THEOREM 1. *Let G be a graph on n vertices. Then it can be decided in $O(n \log^2 n)$ time whether G is a halved cube.*

PROOF. The correctness of the algorithm follows from the previous discussion.

As G is $\binom{d}{2}$-regular, Step 1 of TWO-CLIQUES can be performed in $O(d^2)$ time if the input is a preordered adjacency list. Steps 2, 3 and 5 can be computed in constant time and Step 4 needs $O(d)$ time. Finally, Step 6 can be performed in $O(d^2)$ time. Hence the total complexity of TWO-CLIQUES is $O(d^2) = O(\log^2 n)$.

Whenever we perform Step 4 of CLIQUES, at least one new d-clique is added to the list of cliques. Thus Step 4 is executed at most 2^{d-1} times. Since each call of TWO-CLIQUES requires $O(d^2)$ time and Step 4.2 can clearly be performed within the same time, the time complexity of CLIQUES is $O(n \log^2 n)$.

Finally, hypercubes can be recognized in $O(n \log n)$ time using an algorithm from [3]. □

Another approach to recognize hypercubes efficiently is the following. In [12] Jha and Slutzky gave an algorithm which recognizes median graphs in $O(n^2 \log n)$ time. As it was observed by Hagauer [8], a slight modification of this algorithm can be used to recognize hypercubes in $O(n \log n)$ time. We will not give the details here, but we wish to mention that the improvement in the complexity is due to the observation that we do not need to take care of convexity (which is the bottleneck in recognizing median graphs, cf. also [9]).

3. Concluding Remarks

In fact, without going into details, we should like to mention that by a result of Graham and Winkler [7] about so-called canonical isometric embeddings of graphs into Cartesian products together with an algorithm of Feder [6], a good algorithm for recognizing isometric subgraphs of halved cubes would suffice for a good algorithm for recognizing l_1-graphs. Such an algorithm was indeed very recently developed by Deza and Shpectorov: M. Deza and S. Shpectorov, Recognition of the l_1-graphs with complexity $O(mm)$, or football in a hypercube, Rapport de Recherche du LIENS, May (1995).

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