Orientable domination in product-like graphs

Sarah Anderson\textsuperscript{a}  Boštjan Brešar\textsuperscript{b,c}  Sandi Klavžar\textsuperscript{d,b,c}
Kirsti Kuenzel\textsuperscript{e}  Douglas F. Rall\textsuperscript{f}

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\textsuperscript{a} Department of Mathematics, University of St. Thomas, St. Paul, MN, USA
\textsuperscript{b} Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
\textsuperscript{c} Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
\textsuperscript{d} Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
\textsuperscript{e} Department of Mathematics, Trinity College, Hartford, CT, USA
\textsuperscript{f} Department of Mathematics, Furman University, Greenville, SC, USA

Abstract

The orientable domination number, $\text{DOM}(G)$, of a graph $G$ is the largest domination number over all orientations of $G$. In this paper, $\text{DOM}$ is studied on different product graphs and related graph operations. The orientable domination number of arbitrary corona products is determined, while sharp lower and upper bounds are proved for Cartesian, direct and lexicographic products. A result of Chartrand et al. from 1996 is extended by establishing the values of $\text{DOM}(K_{n_1,n_2,n_3})$ for arbitrary positive integers $n_1, n_2$ and $n_3$. While considering the orientable domination number of the lexicographic product of cycles and edgeless graphs, we answer in the negative a question concerning domination and packing numbers in acyclic digraphs posed in [Domination in digraphs and their direct and Cartesian products, J. Graph Theory 99 (2022) 359–377].

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1 Introduction

Domination is one of the most explored topics in graph theory. On the other hand, this concept has not received as much attention in directed graphs. There are several ways in which domination in graphs transfers to directed graphs, notably indomination, out-domination, twin domination, and reverse domination. The most standard one, however, is out-domination, which is then referred to simply as domination. For some recent papers on domination in digraphs see papers [6, 9, 10] and a recent survey [11].

Chartrand, VanderJagt and Yue defined and studied two invariants on undirected graphs that are based on the domination number of the orientations of the graph [5]. The concept presented in [5], which is the main topic of this paper, is defined as follows. Given an undirected graph $G$, the orientable domination number of $G$ is

$$\text{DOM}(G) = \max \{\gamma(D) : D \text{ is an orientation of } G\}.$$ 

Replacing max with min in the above definition gives the (ordinary) domination number $\gamma(G)$ of $G$. The first result on orientable domination number, although stated in a different language, was proved by Erdős [7] who found the following bounds for $\text{DOM}(K_n)$, where $n \geq 2$: $\log_2 n - 2 \log_2 (\log_2 n) \leq \text{DOM}(K_n) \leq \log_2 (n + 1)$. Interestingly, the exact values of the orientable domination number of complete graphs are still not known. Szekeres and Szekeres [17] improved the upper bound of Erdős to arrive at the following result:

$$\log_2 n - 2 \log_2 (\log_2 n) \leq \text{DOM}(K_n) \leq \log_2 n - \log_2 (\log_2 n) + 2. \quad (1)$$

Lu, Wang and Wong [12] gave a short proof of the above upper bound. Dominating sets in tournaments were studied in [15] and used in [1].

One of the main open problems in domination in graphs is Vizing’s conjecture from [18], which considers domination in the Cartesian product of graphs (see the survey paper [2] and the references therein). A number of papers are devoted not only to the conjecture itself, but also to several other domination invariants in various graph products. With this paper we initiate the study of orientable domination in graph products and with respect to related graph operations.

In the next section, we establish notation and mention some preliminary results needed throughout the paper. In Section 3, we determine the orientable domination number of the corona $G \odot H$ of arbitrary graphs $G$ and $H$, which is expressed as a function of the orientable domination numbers of $G$ and $H$. In Section 4, we prove a sharp lower and upper bound on the orientable domination number of the Cartesian product of two graphs $G$ and $H$. We also pose the Vizing-like problem,
whether \(\text{DOM}(G \Box H) \geq \text{DOM}(G)\text{DOM}(H)\) holds for all graphs \(G\) and \(H\), and observe that it holds if at least one of \(G\) or \(H\) is bipartite. Then, in Section 5, we consider the direct product of graphs and present several lower and upper bounds on its orientable domination number. In Section 6, we prove a sharp lower and upper bound on \(\text{DOM}(G \circ H)\), where \(G \circ H\) is the lexicographic product of graphs \(G\) and \(H\).

We continue with generalized lexicographic products, where the main focus is given to complete multipartite graphs. We establish the values of \(\text{DOM}(K_{n_1,n_2,n_3})\) for arbitrary positive integers \(n_1, n_2,\) and \(n_3\), by which we extend a result of Chartrand et al. [5]. We also study the orientable domination number in specific classes of lexicographic product graphs. In particular, a specific orientation of the graph \(C_{2k+1} \circ \overline{K}_s\) allows us to provide a negative answer to the problem from [3] asking whether the domination number of an acyclic digraph is equal to its packing number.

\section{Notation and preliminaries}

A digraph \(D\) is an ordered pair \((V(D), A(D))\), where \(V(D)\) is a set of vertices of \(D\) and \(A(D)\) a set of arcs, where \(A(D) \subseteq V(D) \times V(D)\). We will assume that the binary relation \(A(D)\) is irreflexive. If \((u, v) \in A(D)\), then we say that \(u\) dominates \(v\) or that \(v\) is dominated by \(u\). A set \(S \subseteq V(D)\) is a dominating set of \(D\) if each vertex in \(V(D) - S\) is dominated by a vertex in \(S\). The domination number, \(\gamma(D)\), of \(D\) is the smallest cardinality of a dominating set of \(D\). A dominating set of \(D\) of cardinality \(\gamma(D)\) is a \(\gamma\)-set of \(D\) or simply a \(\gamma(D)\)-set. An undirected graph \(G\) can be considered as a digraph in which \(A(G)\) is a symmetric binary relation on \(V(G)\). The order of a (di)graph \(G\) will be denoted by \(n(G)\).

A vertex \(u\) is an in-neighbor of \(v\) if \((u, v) \in A(D)\) and an out-neighbor of \(v\) if \((v, u) \in A(D)\). The open out-neighborhood of \(v\) is the set of out-neighbors of \(v\) and is denoted by \(N_D^+(v)\). The closed out-neighborhood of \(v\) is the set \(N_D[v]\) defined by \(N_D[v] = N_D^+(v) \cup \{v\}\). In a similar manner one defines the open in-neighborhood \(N_D^-(v)\) of \(v\) and the closed in-neighborhood \(N_D[v]\) of \(v\). The in-degree of \(v\) is the number \(|N_D^-(v)|\) and the out-degree of \(v\) is \(|N_D^+(v)|\). If the digraph \(D\) is clear from the context, then we may omit the subscript \(D\) of the above notations.

Let \(G\) be an undirected graph. An orientation of \(G\) is a digraph in which every edge from \(G\) is directed in one of the two possible directions. Formally, an orientation of \(G\) is defined by a mapping \(f : E(G) \to V(G) \times V(G)\), such that if \(uv \in E(G)\), then \(f(uv) \in \{(u, v), (v, u)\}\). We denote this orientation of \(G\) by \(G_f\), while we refer to \(f\) as the orienting mapping. Note that by this definition, we can formulate \(\text{DOM}(G)\) as \(\max\{\gamma(G_f) : f\text{ is an orientating mapping of } G\}\).

The following observations are from [4, Observations 3 and 4].
Lemma 2.1. Let $G$ and $H$ be two graphs.

(i) If $H$ is an induced subgraph of $G$, then $\text{DOM}(G) \geq \text{DOM}(H)$.

(ii) If $H$ is an spanning subgraph of $G$, then $\text{DOM}(G) \leq \text{DOM}(H)$.

We also recall the following result from [4, Lemma 3].

Lemma 2.2. Let $G$ be a graph, and let $V_1, \ldots, V_k$ be subsets of $V(G)$ such that $V(G) = V_1 \cup \cdots \cup V_k$. Letting $G_i$ be the subgraph of $G$ induced by $V_i$ for all $i \in [k]$, we get

$$\text{DOM}(G) \leq \sum_{j=1}^{k} \text{DOM}(G_i).$$

Let $G$ be a graph. The cardinality of a largest independent set of vertices in $G$ is the independence number, $\alpha(G)$, of $G$. An independent set of cardinality $\alpha(G)$ is an $\alpha$-set of $G$. A set $M \subseteq E(G)$ is a matching in $G$ if no two edges in $M$ have a common end-vertex. The cardinality of a largest matching in $G$, $\alpha'(G)$, is the matching number of $G$. A set $C \subseteq V(G)$ is a vertex cover in $G$ if $G - C$ is an edgeless graph. The vertex cover number, $\beta(G)$, is the cardinality of a small vertex cover in $G$. A set $A \subseteq E(G)$ is an edge cover in $G$ if every vertex in $G$ is an end-vertex of an edge in $A$. The edge cover number, $\beta'(G)$, is the cardinality of a smallest edge cover in $G$ (note that $G$ has to be without isolated vertices for this definition to make sense). It is well-known that $\alpha(G) + \beta(G) = n(G)$ and $\alpha'(G) + \beta'(G) = n(G)$ in any graph $G$. If $G$ is bipartite, then $\alpha'(G) = \beta(G)$ by the famous König-Egervary theorem.

We will use yet another result of Caro and Henning [4, Theorem 2(a) and Theorem 4(a)].

Theorem 2.3. If $G$ is a graph, then

(i) $\text{DOM}(G) \geq \alpha(G)$, and equality holds if $G$ is bipartite.

(ii) $\text{DOM}(G) \leq n(G) - \alpha'(G)$.

3 Corona graphs

Let $G$ and $H$ be two graphs. The corona, $G \odot H$, of $G$ and $H$ is the graph obtained from the disjoint union of $G$ and $n(G)$ copies of $H$, which we denote by $H_u$ for every $u \in V(G)$, and then joining each $u \in V(G)$ to all vertices of $H_u$. The join of $G$ and $H$ is the graph $G + H$ obtained from the disjoint union of $G$ and $H$ by connecting each vertex of $G$ with each vertex of $H$.

We start with a simple observation about the join with $K_1$. 

4
Lemma 3.1. If $G$ is a graph, then $\text{DOM}(G + K_1) \in \{\text{DOM}(G), \text{DOM}(G) + 1\}$.

Proof. By Lemma 2.1(i), $\text{DOM}(G + K_1) \geq \text{DOM}(G)$, since $G$ is an induced subgraph of $G + K_1$. Let $V(K_1) = \{u\}$, and consider an arbitrary orientation $(G + K_1)_f$. The restriction of this orientation to $G$ can be dominated by a set $S$ of at most $\text{DOM}(G)$ vertices. Then $S \cup \{u\}$ dominates $(G + K_1)_f$. We conclude that $\text{DOM}(G + K_1) \leq \text{DOM}(G) + 1$.

Clearly, $\text{DOM}(K_2) = 1$ and $\text{DOM}(K_3) = 2$. By (1), for every $k \geq 2$ there exists $n_k \in \mathbb{N}$ such that $\text{DOM}(K_{n_k}) = k$ and $\text{DOM}(K_{n_k+1}) = k + 1$. Hence, the sequence of complete graphs contains an infinite subsequence for which the larger value in the conclusion of Lemma 3.1 is attained. (Since the bounds in (1) increase slowly, indices $n_k$ with $\text{DOM}(K_{n_k}) = \text{DOM}(K_{n_k+1})$ appear more often.) More generally, let $G$ be an arbitrary graph. Combining Lemma 2.1(i) and Eq. (1), we infer that the sequence of graphs $(G + K_n)_{n \geq 1}$ contains a subsequence of graphs which attain the larger value in the conclusion of Lemma 3.1.

The condition in the conclusion of Lemma 3.1 is crucial for determining the orientable domination number of corona graphs, as shown in Theorem 3.2 below. Another, more explicit example of a sequence of graphs that enjoy the studied condition, is obtained by the class of even paths.

Let $n$ be any even positive integer, and let $G_n = P_n + K_1$. We also let $V(G_n) = \{x, x_1, \ldots, x_n\}$, where $x$ is the universal vertex of $G_n$, and vertices $x_1, \ldots, x_n$ of the path of order $n$ are indexed in the natural order. Consider the orienting mapping $f$ of $G_n$ defined as follows: $f(x; i + 1) = (x_i, x_{i+1})$ for $i \in [n - 1]$ and

$$f(x; j) = \begin{cases} (x, x_j); & j \text{ odd}, \\ (x_j, x); & j \text{ even}. \end{cases}$$

Note that $\text{DOM}(P_n) = n/2$, and we claim that $\text{DOM}(G_n) = n/2 + 1$. By Lemma 2.2, $\text{DOM}(G_n) \leq \text{DOM}(P_n) + \text{DOM}(K_1) = n/2 + 1$. Let $D$ be a minimum dominating set of $(G_n)_f$. Since $V((G_n)_f) - N^+[x] = \{x_i : i \text{ is even}\}$, and no two vertices of this set are dominated in $(G_n)_f$ by a single vertex, we infer that $x \in D$ implies $|D| = n/2 + 1$. On the other hand, if $x \notin D$, then there exists an even integer $i \in [n]$ such that $x_i \in D$. The vertices that are not dominated by $x_i$ induce either two directed paths, each of odd order (namely, of order $i - 1$ and $n - (i + 1)$), or, if $i = n$, a directed path of order $n - 1$. Since $x$ is not in $D$, we can assume that the vertices from $D$ that dominate the two odd paths (respectively, the odd path of order $n - 1$) lie in those paths (respectively, in the path of order $n - 1$). Hence,

$$|D| \geq 1 + \frac{(i - 1) + 1}{2} + \frac{n - (i + 1) + 1}{2} = \frac{n}{2} + 1.$$ 

We conclude that $\text{DOM}(P_n) = n/2$ and $\text{DOM}(P_n + K_1) = n/2 + 1$. 


Theorem 3.2. If $G$ and $H$ are two graphs, then

$$\text{DOM}(G \odot H) = \begin{cases} 
\text{DOM}(H)n(G); & \text{DOM}(H + K_1) = \text{DOM}(H), \\
\text{DOM}(H)n(G) + \text{DOM}(G); & \text{DOM}(H + K_1) = \text{DOM}(H) + 1.
\end{cases}$$

Proof. By Lemma 3.1, $\text{DOM}(H + K_1) \in \{\text{DOM}(H), \text{DOM}(H) + 1\}$.

First, consider the case when $\text{DOM}(H + K_1) = \text{DOM}(H)$. Consider the subgraph $X$ of $G \odot H$ induced by $\bigcup_{u \in V(G)} V(H_u)$. Clearly, $X$ is the disjoint union of $n(G)$ copies of $H$, hence $\text{DOM}(X) = n(G)\text{DOM}(H)$. By Lemma 2.1(i), $\text{DOM}(G \odot H) \geq n(G)\text{DOM}(H)$. Consider next the spanning subgraph $Y$ of $G \odot H$ obtained from $G \odot H$ by removing all the edges of $G$. Then, $Y$ is the disjoint union of $n(G)$ copies of $H \odot K_1$, and so $\text{DOM}(Y) = n(G)\text{DOM}(H \odot K_1) = n(G)\text{DOM}(H)$. By Lemma 2.1(ii), $\text{DOM}(G \odot H) \leq \text{DOM}(Y) = n(G)\text{DOM}(H)$.

Second, let $\text{DOM}(H + K_1) = k + 1$, where $k = \text{DOM}(H)$. Considering the induced subgraphs $G$ and $H_u$ for all $u \in V(G)$, Lemma 2.2 implies that $\text{DOM}(G \odot H) \leq \text{DOM}(H)n(G) + \text{DOM}(G)$. For the reversed inequality, we construct an orientation $f$ of $G \odot H$ as follows. Let $h$ be an orientation of $H + K_1$ such that $\gamma((H + K_1)_h) = \text{DOM}(H + K_1)$, and let $g$ be an orientation of $G$ such that $\gamma(G_g) = \text{DOM}(G)$. Now, the orientation $f$ of $G \odot H$ is defined by using $h$ on the edges of the subgraph $H'_u$ induced by $V(H_u) \cup \{u\}$ for each $u \in V(G)$, and by using $g$ on the edges of $G$ in $G \odot H$. Let $D$ be a minimum dominating set of $(G \odot H)_f$. We claim that $|D \cap V(H'_u)| \geq k$. The inequality follows from the fact that all vertices in $V(H_u)$ are dominated only by vertices in $H'_u$, and $\gamma((H + K_1)_h) = k + 1$. Indeed, if $|D \cap V(H'_u)| \leq k - 1$, then we get a contradiction because $(D \cap V(H'_u)) \cup \{u\}$ would be a dominating set of $(H'_u)_h$ of size at most $k$. Let $V_1 = \{u \in V(G) : |D \cap V(H'_u)| = k\}$, and $V_2 = V(G) - V_1$. We note that $u \in V_1$ implies $u \notin D$ for otherwise $(H'_u)_h$ would be dominated by $k$ vertices. For the same reason, every $u \in V_1$ has to be dominated by a vertex in $D \cap V_2$. This implies that $V_2$ is a dominating set of $G_g$, which yields $|V_2| \geq \text{DOM}(G)$. Therefore,

$$|D| \geq |V_1|k + |V_2|(k + 1)$$

$$= k(|V_1| + |V_2|) + |V_2|$$

$$= \text{DOM}(H)n(G) + |V_2|$$

$$\geq \text{DOM}(H)n(G) + \text{DOM}(G),$$

and this completes the proof. □
4 Cartesian products

Recall that the Cartesian product of two graphs $G$ and $H$, denoted $G \square H$, is the graph with vertex set $V(G \square H) = V(G) \times V(H)$, where two vertices $(u, v)$ and $(x, y)$ are adjacent in $G \square H$ if either $u = x$ and $vy \in E(H)$, or $v = y$ and $ux \in E(G)$. The Cartesian product of digraphs is defined analogously; see [8].

We first give general bounds for $\text{DOM}(G \square H)$.

**Theorem 4.1.** For any graphs $G$ and $H$,

$$\text{DOM}(G \square H) \geq \max\{\text{DOM}(G)\alpha(H), \alpha(G)\text{DOM}(H)\}$$

$$\text{DOM}(G \square H) \leq \min\{\text{DOM}(G)n(H), n(G)\text{DOM}(H)\},$$

and the bounds are sharp.

**Proof.** Let $A$ be an $\alpha(H)$-set and let $G_f$ be an orientation of $G$ so that $\gamma(G_f) = \text{DOM}(G)$. Let $H_g$ be an arbitrary orientation of $H$. Define the following mapping $h : E(G \square H) \rightarrow V(G \square H) \times V(G \square H)$ by

$$h((u_i, v_j)(u_k, v_\ell)) = \begin{cases} ((u_i, v_j), (u_k, v_j)); & j = \ell \text{ and } f(u_i u_k) = (u_i, u_k), \\ ((u_i, v_j), (u_i, v_\ell)); & i = k \text{ and } v_j \in A, \\ ((u_i, v_j), (u_i, v_\ell)); & i = k, \{v_j, v_\ell\} \cap A = \emptyset \text{ and } g(v_j v_\ell) = (v_j, v_\ell). \end{cases}$$

Considering the orientation $(G \square H)_h$, for each $v \in A$, the only way to dominate vertices in $V(G) \times \{v\}$ is by vertices within $V(G) \times \{v\}$. Thus, $\gamma((G \square H)_h) \geq \alpha(H)\text{DOM}(G)$. Reversing the roles of $G$ and $H$, we have

$$\text{DOM}(G \square H) \geq \max\{\text{DOM}(G)\alpha(H), \alpha(G)\text{DOM}(H)\}.$$ 

Partition the vertex set of $G \square H$ into subsets $V(G) \times \{v\}$, for all $v \in V(H)$, and denote the subgraphs induced by these subsets by $G_v$. Applying Lemma 2.2, we get

$$\text{DOM}(G \square H) \leq \sum_{v \in V(H)} \text{DOM}(G_v) = n(H)\text{DOM}(G).$$

To see that the upper bound is sharp, consider bipartite graphs $G$ and $H$ such that $\alpha(H) = n(H)/2$ and $\alpha(G) = n(G)/2$. Thus, $V(H)$ can be partitioned into $\alpha$-sets $B$ and $B'$, and $V(G)$ can be partitioned into $\alpha$-sets $A$ and $A'$. Since $G \square H$ is bipartite, we get $\text{DOM}(G \square H) = \alpha(G \square H)$, by Theorem 2.3(i). Note that $(A \times B) \cup (A' \times B')$ is an $\alpha$-set of $G \square H$, thus

$$\text{DOM}(G \square H) = |A| \cdot |B| + |A'| \cdot |B'| = (|A| + |A'|)n(H)/2 = n(G)n(H)/2.$$
On the other hand,
\[
\min\{\DOM(G)n(H), n(G)\DOM(H)\} = \min\{\alpha(G)n(H), \alpha(H)n(G)\} = n(G)n(H)/2,
\]
which shows that the upper bound is indeed attained for such graphs \(G\) and \(H\).

The lower bound is also sharp and can be attained by taking the graphs \(K_3\) and \(P_3\). Let \(X = P_3 \square K_3\). Set \(V(P_3) = \{a, x, y\}\) with \(E(P_3) = \{ax, ay\}\), and \(V(K_3) = [3]\). Consider an arbitrary orientation \(X_f\) of \(X\). Assume without loss of generality that \((a,1), (a,2)\) \(\in A(X_f)\). If \((a,1), (x,1)\) \(\in A(X_f)\), then the vertex \((a,1)\) together with one vertex from each of the edges of the matching \(\{(x,2)(x,3), (a,3)(y,3), (y,2)(y,1)\}\) gives us \(\gamma(X_f) \leq 4\). More generally, if a vertex of \(X\) dominates a vertex in its copy of \(K_3\) and a vertex in its copy of \(P_3\), then we can find a dominating set of cardinality at most 4. Using this argument, we can assume that \((a,1), (a,1)\) \(\in A(X_f)\), \((x,2), (x,1)\) \(\in A(X_f)\), \((a,2), (x,2)\) \(\in A(X_f)\), \((a,3), (a,2)\) \(\in A(X_f)\), \((x,3), (a,3)\) \(\in A(X_f)\) and \((x,2), (x,3)\) \(\in A(X_f)\). Now, the vertex \((x,2)\) together with one vertex from each of the edges of the matching \(\{(a,3)(y,3), (a,2)(y,2), (a,1)(y,1)\}\) yields \(\gamma(X_f) \leq 4\). We have thus seen that \(\DOM(P_3 \square K_3) \leq 4\). To see that \(\DOM(P_3 \square K_3) \geq 4\), consider the orientation \(X_f\) of \(X\), which restricted to \(\{x\} \times [3]\) and \(\{y\} \times [3]\) yields \(\overrightarrow{C_3}\), and all arcs between \(\{x,y\} \times [3]\) and \(\{a\} \times [3]\) are oriented towards a vertex in \(\{a\} \times [3]\). We derive

\[
\DOM(P_3 \square K_3) = 4
\]

and we are done.

\[\square\]

We think that the lower bound in Theorem 4.1 might not be attainable for graphs \(G\) and \(H\) with large enough order. We can verify that the lower bound is not attained in the case when \(G\) and \(H\) are nontrivial, connected bipartite graphs. The lower bound then reads \(\alpha(G \square H) \geq \alpha(G)\alpha(H)\), by Theorem 2.3(i). It was proved by Vizing that \(\alpha(G \square H) \geq \alpha(G)\alpha(H) + \min\{n(G) - \alpha(G), n(H) - \alpha(H)\}\) for any non-trivial graphs \(G\) and \(H\); see [8]. Thus, the lower bound in Theorem 4.1 is not attained if both \(G\) and \(H\) are nontrivial, connected bipartite graphs.

Vizing’s conjecture [18] from 1968 concerning the ordinary domination number of a graph in the Cartesian product of graphs is one of the main open problems in graph domination. The conjecture states that for any two graphs \(G\) and \(H\) the domination number \(\gamma(G \square H)\) of the Cartesian product of \(G\) and \(H\) is at least as big as the product \(\gamma(G)\gamma(H)\) of their domination numbers. The inequality \(\psi(G \ast H) \geq \psi(G)\psi(H)\), where \(\psi\) is a graph invariant and \(\ast\) is a product operation in graphs is often referred to as a Vizing-like inequality.

We point out that if \(G\) or \(H\) is bipartite, then Theorem 4.1 says

\[
\DOM(G)\DOM(H) = \max\{\DOM(G)\alpha(H), \alpha(G)\DOM(H)\} \leq \DOM(G \square H).
\]
We suspect a Vizing-like bound holds for $\text{DOM}(G \Box H)$ which we formally pose as a problem.

**Problem 1.** Is it true that $\text{DOM}(G \Box H) \geq \text{DOM}(G)\text{DOM}(H)$ holds for any two graphs $G$ and $H$?

In studying the above problem, it is important to note that given two directed graphs $D_1$ and $D_2$ it may be the case that $\gamma(D_1 \Box D_2) < \gamma(D_1)\gamma(D_2)$. For example, we know that $\gamma(\overrightarrow{C_3}) = 2$ for the directed cycle $\overrightarrow{C_3}$. However, one can easily verify that $\gamma(\overrightarrow{C_3 \Box C_3}) = 3$.

On the other hand, we can find an orientation showing that $\text{DOM}(K_3 \Box K_3) \geq 4$; see Fig. 1. Let $G = K_3 \Box K_3$. Note that the black vertices are a dominating set of $G_f$. To see that $\gamma(G_f) \geq 4$, suppose there exists a dominating set $D$ of $G_f$ of cardinality 3. Since each vertex of $G_f$ has out-degree 2, then $D$ is an independent set in $K_3 \Box K_3$. One can easily verify that each of the six independent sets of $K_3 \Box K_3$ is not a dominating set of $G_f$. Thus, $\gamma(G_f) = 4 \leq \text{DOM}(G)$. Noting that $P_3 \Box K_3$ is a spanning subgraph of $K_3 \Box K_3$, combined with (2), and using Lemma 2.1(ii), we infer $\text{DOM}(K_3 \Box K_3) \leq 4$. We conclude that $\text{DOM}(K_3 \Box K_3) = 4$.

![Figure 1: The orientation $G_f$ of $G = K_3 \Box K_3$](image)

Next, we focus on the prism of a cycle. Theorem 4.1 implies that $\text{DOM}(C_n \Box K_2) \leq n$.

**Proposition 4.2.** If $n \geq 2$, then $\text{DOM}(C_n \Box K_2) = n$.

**Proof.** Let $V(C_n) = \{v_1, \ldots, v_n\}$ with vertices ordered in the natural order and let $V(K_2) = [2]$. Let $(C_n \Box K_2)_h$ be the orientation of $C_n \Box K_2$ with arcs

$$\{(v_i, j), (v_{i+1}, j) : i \in [n], j \in [2], i \text{ taken modulo } n\} \bigcup \{(v_i, 1), (v_i, 2) : i \in [n]\}.$$
Let $D$ be a dominating set of $(C_n \Box K_2)_h$, and $D_i = D \cap \{(v, i) : v \in V(C_n)\}$, for $i \in [2]$. Note that each vertex of the form $(v_i, 1)$ is dominated by a vertex in $D_1$, which implies that there exists no $i \in [n]$ such that $\{(v_i, 1), (v_{i+1}, 1)\} \cap D = \emptyset$. Hence, the set $S$ of vertices in $\{(v, 2) : v \in V(C_n)\}$ that are not dominated by $D_1$ is independent. We derive that in order to dominate the vertices of $S$ (from $D_2$) one needs at least $|S|$ vertices, which implies $|D| = |D_1| + |D_2| \geq |D_1| + |S| = n(G)$. \hfill \Box

Given a graph $G$ let $\alpha_2(G)$ denote the 2-independence number of $G$, which is the maximum order of a bipartite induced subgraph of $G$.

**Proposition 4.3.** If $G$ is a graph, then

$$\alpha_2(G) \leq \text{DOM}(G \Box K_2) \leq n(G).$$

**Proof.** The upper bound follows from Theorem 4.1. For the lower bound consider the largest bipartite induced subgraph of $G$ and let $S$ and $T$ be the sets of its bipartition. Since $(S \times \{1\}) \cup (T \times \{2\})$ is an independent set, it follows from Theorem 2.3(i) that $\text{DOM}(G \Box K_2) \geq |S| + |T| = \alpha_2(G)$. \hfill \Box

If $G$ is bipartite, then $\alpha_2(G) = n(G)$, thus Proposition 4.3 implies $\text{DOM}(G \Box K_2) = n(G)$.

## 5 Direct products

Let $G$ and $H$ be two graphs. The direct product, $G \times H$, of $G$ and $H$ is the graph with $V(G \times H) = V(G) \times V(H)$. Two vertices $(x, y)$ and $(u, v)$ are adjacent in $G \times H$ if both $xu \in E(G)$ and $yv \in E(H)$. Recall the following known facts regarding the direct product of two graphs; see [8]. If $G$ and $H$ are graphs, then

$$\alpha(G \times H) \geq \max\{\alpha(G)n(H), \alpha(H)n(G)\} \tag{3}$$  
$$\alpha'(G \times H) \geq 2\alpha'(G)\alpha'(H). \tag{4}$$

Now we can bound $\text{DOM}(G \times H)$ as follows.

**Proposition 5.1.** If $G$ and $H$ are graphs, then

$$\max\{\alpha(G)n(H), \alpha(H)n(G)\} \leq \text{DOM}(G \times H) \leq n(G)n(H) - 2\alpha'(G)\alpha'(H).$$

**Proof.** The lower bound follows from Theorem 2.3(i) and (3). For the upper bound, $\text{DOM}(G \times H) \leq n(G)n(H) - \alpha'(G \times H)$ by Theorem 2.3(ii). Now the result follows by (4). \hfill \Box
The following result presents a large class of graphs for which the lower and upper bounds in Proposition 5.1 coincide.

**Corollary 5.2.** If $G$ and $H$ both have a perfect matching and at least one of them has independence number one-half its order, then $\text{DOM}(G \times H) = n(G)n(H)/2$.

**Proof.** Without loss of generality, assume that $\alpha(G) = n(G)/2$. Then, by using Proposition 5.1 we can estimate as follows.

$$\frac{n(G)}{2}n(H) = \alpha(G)n(H) \leq \max\{\alpha(G)n(H), \alpha(H)n(G)\} \leq \text{DOM}(G \times H) \leq n(G)n(H) - 2\alpha'(G)\alpha'(H) \leq n(G)n(H) - (n(G)n(H))/2 = \frac{n(G)}{2}n(H).$$

Hence, equality holds throughout, and the result follows. \qed

If only one of the factors of a direct product contains a perfect matching, then Proposition 5.1 yields the following consequence.

**Corollary 5.3.** If $G$ and $H$ are graphs and $G$ contains a perfect matching, then $\text{DOM}(G \times H) \leq n(G)\beta'(H)$. Moreover, if $\alpha(H) = \beta'(H)$, then $\text{DOM}(G \times H) = n(G)\alpha(H)$.

**Proof.** By Proposition 5.1,

$$\text{DOM}(G \times H) \leq n(G)n(H) - 2\alpha'(G)\alpha'(H) = n(G)n(H) - n(G)\alpha'(H) = n(G)(n(H) - \alpha'(H)) = n(G)\beta'(H).$$

Now, if $\alpha(H) = \beta'(H)$, then since

$$n(G)\alpha(H) \leq \text{DOM}(G \times H) \leq n(G)\beta'(H) = n(G)\alpha(H),$$

we conclude that $\text{DOM}(G \times H) = n(G)\alpha(H)$. \qed
6  Lexicographic products

Let $G$ and $H$ be two graphs. The lexicographic product $G \circ H$ of $G$ and $H$ is the graph with $V(G \circ H) = V(G) \times V(H)$ and two vertices $(x, y)$ and $(u, v)$ are adjacent in $G \circ H$ if either $xu \in E(G)$, or $x = u$ and $yv \in E(H)$.

**Proposition 6.1.** If $G$ and $H$ are arbitrary graphs, then

$$\alpha(G)\text{DOM}(H) \leq \text{DOM}(G \circ H) \leq \min\{\text{DOM}(G)n(H), \text{DOM}(H)n(G)\}.$$ 

**Proof.** To prove the lower bound consider the following orientation of $G \circ H$. Let $H_f$ be an orientation of $H$ such that $\gamma(H_f) = \text{DOM}(H)$. For each $x \in V(G)$ consider the subgraph of $G \circ H$ induced by the set \{(x, y) : y \in V(H)\}, and denote it by $H^x$. Clearly, $H^x$ is isomorphic to $H$, and for every $x$ orient the edges of $H^x$ by following the orienting mapping $f$. (That is, if $f$ maps $ab \in E(H)$ to the arc $(a, b) \in V(H) \times V(H)$, then let $(x, a)(x, b) \in E(G \circ H)$ be mapped to $((x, a), (x, b)) \in V(G \circ H) \times V(G \circ H)$.) The edges among vertices with distinct first coordinates are oriented as follows. Let $A$ be an $\alpha$-set of $G$. For each $u \in A$ and $v \in N_G(u)$ and any $h, h' \in V(H)$, let the edges $(u, h)(v, h') \in E(G \circ H)$ be oriented from $(u, h)$ to $(v, h')$. The latter orientation yields that vertices in $H^x$, where $x \in A$, can only be dominated by vertices in $H^x$. This establishes the lower bound.

To see that $\text{DOM}(G \circ H) \leq \text{DOM}(G)n(H)$, partition the vertex set of $G \circ H$ into the sets $\{(x, y) : x \in V(G)\}$ for all $y \in V(H)$. These sets induce subgraphs isomorphic to $G$, and hence, by Lemma 2.2, $\text{DOM}(G \circ H) \leq \text{DOM}(G)n(H)$. Similarly, the partition of $V(G \circ H)$ into the sets $\{(x, y) : y \in V(H)\}$ for all $x \in V(G)$ yields $\text{DOM}(G \circ H) \leq \text{DOM}(H)n(G)$. \hfill $\Box$

If $G$ is bipartite, then Theorem 2.3(i) and Proposition 6.1 imply

$$\text{DOM}(G)\text{DOM}(H) \leq \text{DOM}(G \circ H) \leq \text{DOM}(G)n(H),$$

which in turn implies that

$$\text{DOM}(G \circ \overline{K_s}) = \text{DOM}(G)\text{DOM}(\overline{K_s}) = s \cdot \text{DOM}(G).$$

This shows that both bounds in Proposition 6.1 are sharp. In particular, the “Vizing-like” bound $\text{DOM}(G)\text{DOM}(H) \leq \text{DOM}(G \circ H)$ does not hold in general, as can be seen by taking $G = K_3 = H$. Note that $\text{DOM}(K_3 \circ K_3) = \text{DOM}(K_9) = 3$, while $\text{DOM}(K_3)^2 = 4$.

The situation when $G$ is not bipartite is much more complex. Proposition 6.1 gives the upper bound $ks + s$ for $\text{DOM}(C_{2k+1} \circ \overline{K_s})$, which we are able to improve as follows.
Proposition 6.2. If \( k \geq 2 \) and \( s \geq 2 \), then

\[
ks \leq \text{DOM}(C_{2k+1} \circ \overline{K_s}) \leq ks + \left\lfloor \frac{s+1}{2} \right\rfloor.
\]

Proof. Throughout the proof we write \( G = C_{2k+1} \circ \overline{K_s} \). The lower bound follows directly from the lower bound in Proposition 6.1. For the upper bound note that \( G \) is a Hamiltonian graph. Hence, if \( s \) is even, \( n(G) \) is even, and thus has a perfect matching. By Theorem 2.3(ii), \( \text{DOM}(G) \leq n(G) - n(G)/2 = ks + s/2 = ks + \left\lfloor \frac{s+1}{2} \right\rfloor \). On the other hand, \( s \) odd implies that \( \alpha'(G) = n(G) - 1 \), and so \( \text{DOM}(G) \leq \frac{(2k+1)s+1}{2} = ks + \left\lfloor \frac{s+1}{2} \right\rfloor \). \( \square \)

6.1 Generalized lexicographic products

The lexicographic product \( G \circ H \) of graphs \( G \) and \( H \) can be described as the graph obtained from \( G \) by replacing each vertex \( u \) of \( G \) with an isomorphic copy of \( H \), say \( H_u \), and adding all the edges between \( H_u \) and \( H_v \) whenever \( uv \in E(G) \). This can be generalized by replacing each vertex \( u \) of \( G \) by an arbitrary graph \( H_u \). If \( \mathcal{H} = \{H_u : u \in V(G)\} \) is a collection of graphs associated with the vertices of \( G \), then the graph constructed in this way is called the generalized lexicographic product and is denoted by \( G \circ \mathcal{H} \); see [16].

Let \( G \) be a graph and \( \mathcal{H} = \{H_u : u \in V(G)\} \) be a collection of graphs associated with the vertices of \( G \). Then arguments of the proof of Proposition 6.1 yield the following bounds. If \( X \) is an independent set of \( G \), then

\[
\sum_{u \in X} \text{DOM}(H_u) \leq \text{DOM}(G \circ \mathcal{H}) \leq \sum_{u \in V(G)} \text{DOM}(H_u).
\]

Chartrand et al. [5] considered the orientable domination number in several families of graphs. In particular, they determined the orientable domination number of the graphs \( K_{n,n,n} = K_k \circ \overline{K_n} \) for all positive integers \( n \). In this subsection, we extend their result by considering arbitrary complete multipartite graphs. Note that \( K_{n_1,\ldots,n_k} \) is the generalized lexicographic product with the first factor \( K_k \) and the collection of \( k \) edgeless graphs of order \( n_1,\ldots,n_k \), respectively.

Given a non-decreasing sequence \( n_1 \leq n_2 \leq \cdots \leq n_k \) of positive integers, where \( k \geq 2 \), the vertices of the graph \( K_{n_1,\ldots,n_k} \) can be partitioned into independent sets \( A_1,\ldots,A_k \) such that \( |A_i| = n_i \) for every \( i \in [k] \) and for \( 1 \leq i < j \leq k \), every vertex of \( A_i \) is adjacent to every vertex of \( A_j \). We use this notation throughout the remainder of this subsection. If \( k = 2 \), then \( K_{n_1,n_2} \) is a complete bipartite graph, and \( \text{DOM}(K_{n_1,n_2}) = n_2 \) by Theorem 2.3(i).
We start by proving a lower and an upper bound that hold for an arbitrary complete multipartite graph.

**Proposition 6.3.** Given a non-decreasing sequence \( n_1 \leq n_2 \leq \cdots \leq n_k \) of positive integers, where \( k \geq 2 \), we have

\[
n_k \leq \text{DOM}(K_{n_1,\ldots,n_k}) \leq \max\{n_k, k\}.
\]

**Proof.** The lower bound follows from Theorem 2.3(i), since \( n_k = \alpha(K_{n_1,\ldots,n_k}) \). Set \( X = K_{n_1,\ldots,n_k} \). For the proof of the upper bound, consider an orientation \( X_f \) of \( X \) such that \( \gamma(X_f) = \text{DOM}(X) \). If one of the sets \( A_i \), where \( i \in [k] \), is a dominating set of \( X_f \), then \( \text{DOM}(X) \leq n_i \leq n_k \), as claimed. Otherwise, for each \( i \in [k] \), there exists a vertex \( x_j \in V(X) - A_i \) such that \( x_j \) dominates all vertices of \( A_i \). Hence, the set \( \{x_1, \ldots, x_k\} \) is a dominating set \( X_f \), giving \( \text{DOM}(X) \leq k \), as claimed. \( \square \)

The following result immediately follows from Proposition 6.3, and resolves the orientable domination of a large class of complete multipartite graphs.

**Corollary 6.4.** If \( n_1 \leq n_2 \leq \cdots \leq n_k \) is a non-decreasing sequence of positive integers, where \( n_k \geq k \geq 2 \), then \( \text{DOM}(K_{n_1,\ldots,n_k}) = n_k = \alpha(K_{n_1,\ldots,n_k}) \).

Finally, we concentrate on complete tripartite graphs and extend the result on \( \text{DOM}(K_{n,n,n}) \) from [5].

**Theorem 6.5.** If \( 1 \leq n_1 \leq n_2 \leq n_3 \), then

\[
\text{DOM}(K_{n_1,n_2,n_3}) = \begin{cases} 
n_3; & n_3 \geq 3, \\
3; & n_1 = n_2 = n_3 = 2, \\
2; & \text{otherwise.}
\end{cases}
\]

**Proof.** If \( n_3 \geq 3 \), then \( \text{DOM}(K_{n_1,n_2,n_3}) = n_3 \), by Corollary 6.4. Hence, let \( n_3 \leq 2 \), and consider the following cases. To see that \( \text{DOM}(K_{2,2,2}) = 3 \), first note that \( \text{DOM}(K_{2,2,2}) \leq 3 \), by Proposition 6.3. The orientation \( (K_{2,2,2})_f \) of \( K_{2,2,2} \) depicted in the following table, where we list the closed out-neighborhoods of each of the vertices of \( K_{2,2,2} \):

<table>
<thead>
<tr>
<th>( u )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N^+[u] )</td>
<td>( x_1, x_5, x_6 )</td>
<td>( x_2, x_4, x_6 )</td>
<td>( x_1, x_2, x_3 )</td>
<td>( x_1, x_4, x_5 )</td>
<td>( x_2, x_3, x_5 )</td>
<td>( x_3, x_4, x_6 )</td>
</tr>
</tbody>
</table>

shows that \( \gamma((K_{2,2,2})_f) = 3 \), since no two vertices dominate the oriented graph.

Consider any orientation \((K_{1,2,2})_f\) of \( K_{1,2,2} \). If either of \( A_2 \) or \( A_3 \) is a dominating set of \((K_{1,2,2})_f\), then \( \gamma((K_{1,2,2})_f) \leq 2 \). Therefore assume that neither of \( A_2 \) or \( A_3 \) is a dominating set of \((K_{1,2,2})_f\). Thus \( A_2 \), resp. \( A_3 \), is dominated by a vertex \( x_2 \), resp. \( x_3 \).
If \( x_2 = x_3 \), then \( x_2 \) dominates the given orientation, and so \( \gamma((K_{1,2,2})_f \leq 2. \) On the other hand, if \( x_2 \neq x_3 \), then \( x_2 \notin A_3 \) or \( x_3 \notin A_2 \), and in either case \( \gamma((K_{1,2,2})_f) = 2. \) Therefore, \( \text{DOM}(K_{1,2,2}) = 2. \) Note that \( K_{1,1,2} \) is isomorphic to \( K_4 - e \) while \( K_{1,1,1} \) is isomorphic to \( K_3 \), and clearly \( \text{DOM}(K_4 - e) = 2 = \text{DOM}(K_3) \), which concludes the proof.

6.2 Domination and packing in an acyclic orientation of \( C_{2k+1} \circ \overline{K_s} \)

The classical result of Meir and Moon [13] states that the domination number of a tree \( T \) is equal to the 2-packing number of \( T \), which is defined as the maximum number of pairwise disjoint closed neighborhoods in \( T \). In [3] this result was extended to the context of digraphs. (See also Mojdeh, Samadi and G. Yero [14, Theorem 5] where the special case of this result was proved for orientations of trees.) The extension uses the following notion – the digraph version of a 2-packing. A subset \( P \) of \( V(D) \) is a packing of a digraph \( D \) if there are no arcs joining vertices of \( P \) and for every two vertices \( x, y \in P \) there does not exist \( v \in V(D) \) such that \( \{(v, x), (v, y)\} \subseteq A(D) \). The packing number, \( \rho(D) \), of \( D \) is the cardinality of a largest packing in \( D \).

The mentioned extension to digraphs [3, Theorem 1.2] asserts that if \( T \) is a digraph whose underlying graph is a tree, then \( \rho(T) = \gamma(T) \). The authors then asked whether the result can be extended to all acyclic digraphs; see [3, Problem 1]. Recall that a digraph is acyclic if it contains no directed cycles. In particular, an acyclic digraph contains no opposite arcs, hence it is an orientation of an undirected graph. The mentioned problem reads as follows: Is \( \rho(D) = \gamma(D) \) if \( D \) is an acyclic digraph?

We now use the lexicographic product to answer this question in the negative. The following construction will be used. Consider the lexicographic product \( C_{2k+1} \circ \overline{K_s} \), where \( k \geq 2 \) and \( s \geq 2 \). Let \( V(C_{2k+1}) = \{v_1, \ldots, v_{2k+1}\} \), and let \( V(\overline{K_s}) = \{w_1, \ldots, w_s\} \). Let \( X = C_{2k+1} \circ \overline{K_s} \), and define the orientation using the following mapping \( f : E(X) \rightarrow V(X) \times V(X) \). For every \( i \in [2k] \) and \( j, k \in [s] \), let
\[
f((v_i, w_j)(v_{i+1}, w_k)) = ((v_i, w_j), (v_{i+1}, w_k)),
\]
and for every \( j, k \in [s] \), let
\[
f((v_1, w_j)(v_{2k+1}, w_k)) = ((v_1, w_j), (v_{2k+1}, w_k)).
\]

Theorem 6.6. If \( k \geq 2, s \geq 2, \) and \( f \) is the orienting mapping as defined above, then
\[
\gamma((C_{2k+1} \circ \overline{K_s})_f) = s + 2k - 2 \quad \text{and} \quad \rho((C_{2k+1} \circ \overline{K_s})_f) = s + k - 1.
\]
Proof. Set $X = C_{2k+1} \circ \overline{K}_s$ for this proof, and for each $i \in [2k+1]$ let $V_i = \{(v_i, w_j) : j \in [s]\}$.

Let $D$ be a minimum dominating set of $X_f$. Note that the in-degree of each vertex from $V_1$ is 0, which implies that these vertices belong to every dominating set of $X_f$, thus also to $D$. Hence the vertices from $V_1 \cup V_2 \cup V_{2k+1}$ are dominated by the vertices from $V_1$, where $V_1 \subseteq D$. Consider an arbitrary set $V_i$, where $i \in \{3, \ldots, 2k\}$. In order to dominate vertices of $V_i$ either all of them lie in $D$ or $V_{i+1} \cap D \neq \emptyset$. If $V_i$ is dominated by a vertex $(v_{i-1}, w_j)$ (from $V_{i-1} \cap D$), then we set $x_i = (v_{i-1}, w_j)$. Otherwise, if $D \cap V_i = V_i$, then set $x_i = (v_i, w_1)$. In this case, further set $x_{i+1} = (v_i, w_2)$ (note that $x_{i+1}$ dominates $V_{i+1}$). The resulting mapping, which assigns $x_i$ to $V_i$ is one-to-one, which implies that $V_1 \cup \{x_3, \ldots, x_{2k}\} \subseteq D$, hence $\gamma(X_f) \geq s + (2k+1) - 3 = s + 2k - 2$.

On the other hand, $V_1 \cup \{(v_2, w_1), (v_3, w_1), \ldots, (v_{2k-1}, w_1)\}$ is a dominating set of cardinality $s + 2k - 2$, which gives the first formula.

To establish the packing number of $X_f$, first note that in every packing $P$ of $X_f$, we have $|P \cap V_i| \leq 1$ holds for all $i \in \{2, 3, \ldots, 2k+1\}$. Indeed, if $(v_i, w_j), (v_i, w_k) \subseteq P$, then the arcs going from $(v_{i-1}, w_1)$ to each of the two vertices imply that $P$ is not a packing. Now, if $|V_i \cap P| = 1$, then $|V_{i-1} \cap P| = 0 = |V_{i+1} \cap P|$. Hence, $|P \cap (V_2 \cup \cdots \cup V_{2k})| \leq k$. Moreover, if $|P \cap (V_2 \cup \cdots \cup V_{2k})| = k$, then either $V_2 \cap P \neq \emptyset$ or $V_{2k} \cap P \neq \emptyset$, which implies that $V_1 \cap P = \emptyset$. This yields $|P| = k$. However, the set $V_1 \cup \{(v_3, w_1), (v_5, w_1), \ldots, (v_{2k-1}, w_1)\}$ is a packing of cardinality $s + k - 1$, and is thus a maximum packing of $X_f$. Thus, $\rho(X_f) = s + k - 1$. \hfill \Box

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