BIANALYTIC MAPS BETWEEN FREE SPECTRAHEDRA

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Abstract. Linear matrix inequalities (LMIs) $I_d + \sum_{j=1}^g A_j x_j + \sum_{j=1}^g A_j^* x_j^* \succeq 0$ play a role in many areas of applications. The set of solutions of an LMI is a spectrahedron. LMIs in (dimension–free) matrix variables model most problems in linear systems engineering, and their solution sets are called free spectrahedra. Free spectrahedra are exactly the free semialgebraic convex sets.

This paper studies free analytic maps between free spectrahedra and, under certain (generically valid) irreducibility assumptions, classifies all those that are bianalytic. The foundation of such maps turns out to be a very small class of birational maps we call convexotonic. The convexotonic maps in g variables sit in correspondence with g-dimensional algebras. If two bounded free spectrahedra $\mathcal{D}_A$ and $\mathcal{D}_B$ meeting our irreducibility assumptions are free bianalytic with map denoted p, then p must (after possibly an affine linear transform) extend to a convexotonic map corresponding to a g-dimensional algebra spanned by $(U-I)A_1, \dots, (U-I)A_g$ for some unitary U. Furthermore, B and UA are unitarily equivalent.

The article also establishes a Positivstellensatz for free analytic functions whose real part is positive semidefinite on a free spectrahedron and proves a representation for a free analytic map from $\mathcal{D}_A$ to $\mathcal{D}_B$ (not necessarily bianalytic). Another result shows that a function analytic on any radial expansion of a free spectrahedron is approximable by polynomials uniformly on the spectrahedron. These theorems are needed for classifying free bianalytic maps.

1. Introduction

Given a tuple $A = (A_1, \dots, A_g)$ of complex $d \times d$ matrices and indeterminates $x = (x_1, \dots, x_g)$, the expression

$$L_A(x) = I_d + \sum_{j=1}^{g} A_j x_j + \sum_{j=1}^{g} A_j^* x_j^*$$

is a monic linear pencil. The set

$$\mathcal{D}_A(1) = \{z \in \mathbb{C}^g : L_A(z) \text{ is positive semidefinite}\}$$

is known as a **spectrahedron** (synonymously **LMI domain**). Spectrahedra play a central role in semidefinite programming, convex optimization and in real algebraic geometry [BPR13]. They also figure prominently in the study of determinantal representations [Bra11, GK-VVW16, NT12, Vin93], the solution of the Lax conjecture [HV07], in the solution of the Kadison-Singer paving conjecture [MSS15], and in systems engineering [BGFB94, SIG96]. The monic linear pencil L_A is naturally

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evaluated at a tuple $X = (X_1, \dots, X_g)$ of $n \times n$ matrices using the Kronecker product as

$$L_A(X) = I_d \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j + \sum_{j=1}^{g} A_j^* \otimes X_j^*$$

with output a $dn \times dn$ self-adjoint matrix. Let $M_n(\mathbb{C})$ denote the $n \times n$ matrices with entries from $\mathbb{C}$ and $M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices. We call the sequence $(\mathcal{D}_A(n))_n$, where

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A(X) \text{ is positive semidefinite}\}$$

a **free spectrahedron** (or a **free LMI domain**). Free spectrahedra arise naturally in many systems engineering problems described by a signal flow diagram [dOHMP09]. They are also canonical examples of matrix convex sets [EW97, HKM17] and thus are intimately connected to the theory of completely positive maps and operator systems and spaces [Pau02].

In this article we study bianalytic maps p between free spectrahedra. Our belief, supported by the results in this paper and our experience with free spectrahedra (see for instance [HM12], [HKMS] and [HKM12b]), is that the existence of bianalytic maps imposes rigid, but elegant, structure on both the free spectrahedra as well as the map p. Motivation for this study comes from several sources. Free analysis, including free analytic functions, is a recent development [KVV14, Tay72, Pas14, AM15, BGM06, Pop06, Pop10, KŠ17, HKM12b, BKP16] with close ties to free probability [Voi04, Voi10] and quantum information theory [NC11, HKM17]. In engineering systems theory certain model problems can be described by a system of matrix inequalities. For optimization and design purposes, it is hoped that these inequalities have a convex solution set. In this case, under a boundedness hypothesis, the solution set is a free spectrahedron [HM12]. If the domain is not convex one might replace it by its matrix convex hull [HKM16] or map it bianalytically to a free spectrahedron. Two such maps then lead to a bianalytic map between free spectrahedra.

Studying bianalytic maps between free spectrahedra is a free analog of rigidity problems in several complex variables [DAn93, For89, For93, HJ01, HJY14, Kra01]. Indeed, there is a large literature on bianalytic maps on convex sets. For instance, Faran [Far86] showed that any proper analytic map from the unit ball in $\mathbb{C}^n$ to the unit ball in $\mathbb{C}^N$ with $N \le 2n-2$ that is real analytic up to the boundary, is (up to automorphisms of the domain and codomain) the standard linear embedding $z \mapsto (z, 0)$. When $N = 2n-1$, Huang and Ji [HJ01] proved this map and the Whitney map $z = (z', z_n) \mapsto (z', z_n z)$ are the only such maps. Forstnerič [For93] showed that any proper analytic map between balls with sufficient regularity at the boundary must be rational. We refer to [HJY14] for further recent developments.

The remainder of this introduction is organized as follows. Basic terminology and background appear in Subsection 1.1. A novel family of maps we call convexotonic maps and that we believe comprise, up to affine linear equivalence, exactly the bianalytic maps between free spectrahedra, is described in Subsection 1.2. Subsection 1.2 also contains the main result of the article, Theorem 1.5 on bianalytic mappings between free spectrahedra. Subsections 1.3 and 1.4 describe Positivstellensätze and results related to recent free Oka-Weil theorems [AM14, BMV] on (uniform) polynomial approximation of free spectrahedra and functions analytic in a suitable neighborhood of a spectrahedron. Both are ingredients in the proof of Theorem 1.5.

1.1. **Basic definitions.** Notations, definitions and background needed, but not already introduced, to describe the results in this paper are collected in this section.

1.1.1. *Free polynomials.* Let $x = (x_1, \dots, x_g)$ denote g freely noncommuting letters and $\langle x \rangle$ the set of **words** in x, including the empty word denoted by either 1 or $\varnothing$. The **length of a word** $w \in \langle x \rangle$ is denoted by $|w|$. Let $\mathbb{C}\langle x \rangle = \mathbb{C}\langle x_1, \dots, x_g \rangle$ denote the $\mathbb{C}$-algebra freely generated by x. Its elements are linear combinations of words in x and are called **analytic free polynomials**. We shall also consider the **free polynomials** $\mathbb{C}\langle x, x^*\rangle$ in both the variables $x = (x_1, \dots, x_g)$ and their formal adjoints, $x^* = (x_1^*, \dots, x_g^*)$. For instance, $x_1x_2 + x_2x_1 + 5x_1^3$ is analytic, but $x_1^*x_2 + 3x_2x_1^5$ is not. A polynomial is **hereditary** provided all the x^* variables, if any, always appear on the left of all x variables. Thus an hereditary polynomial is a finite linear combination of terms v^*w where v and w are words in x. A special case are polynomials of the form analytic plus anti-analytic; that is $f + g^*$ for some $f, g \in \mathbb{C}\langle x\rangle$. These definitions naturally extend to matrices over polynomials.

Given a word $w = x_{i_1}x_{i_2}\cdots x_{i_m}$ and a tuple $X = (X_1, \dots, X_g) \in M_n(\mathbb{C})^g$, let $w(X) = X^w = X_{i_1}X_{i_2}\cdots X_{i_m}$. A matrix-valued free polynomial $p = \sum p_w w$ is evaluated at X using the Kronecker product as

$$p(X) = \sum p_w \otimes w(X).$$

ree domains

1.1.2. *Free domains, matrix convex sets and spectrahedra.* Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A **subset** Γ of $M(\mathbb{C})^g$ is a sequence $(\Gamma(n))_n$ where $\Gamma(n) \subseteq M_n(\mathbb{C})^g$. The subset Γ is a **free set** if it is closed under direct sums and unitary similarity; that is, if $X \in \Gamma(n)$ and $Y \in \Gamma(m)$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in \Gamma(n+m)$$

and if U is an $n \times n$ unitary matrix, then

$$U^*XU = \big(U^*X_1U, \dots, U^*X_gU\big) \in \Gamma(n).$$

The free set Γ is a **matrix convex set** (alternately **free convex set**) if it is also closed under simultaneous conjugation by isometries; i.e., if $X \in \Gamma(n)$ and V is an $n \times m$ isometry, then $V^*XV \in \Gamma(m)$. In the case that $0 \in \Gamma(1)$, Γ is a matrix convex set if and only if it is closed under direct sums and simultaneous conjugation by contractions. It is straightforward to see that a matrix convex set is **levelwise** convex; i.e., each $\Gamma(n)$ is a convex set in $M_n(\mathbb{C})^g$. The converse is true if Γ, in addition to being a free set, is closed with respect to restrictions to reducing subspaces.

A distinguished class of matrix convex domains are those described by a linear matrix inequality. Given a positive integer d and $A_1, \dots, A_g \in M_d(\mathbb{C})$, the linear matrix-valued free polynomial

$$\Lambda_A(x) = \sum_{j=1}^{g} A_j x_j \in M_d(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \dots, x_g\rangle$$

is a **(homogeneous) linear pencil**. Its adjoint is, by definition, $\Lambda_A(x)^* = \sum_{j=1}^{g} A_j^* x_j^*$. Thus

$$L_A(x) = I_d + \Lambda_A(x) + \Lambda_A(x)^*.$$

In particular, $\mathcal{D}_A = \mathcal{D}_{\Lambda_A}$ and it is immediate that the free spectrahedron $\mathcal{D}_A$ is a matrix convex set that contains a neighborhood of 0.

1.1.3. *Free functions.* Let $\mathcal{D} \subseteq M(\mathbb{C})^g$. A **free function** f from $\mathcal{D}$ into $M(\mathbb{C})^1$ is a sequence of functions $f[n] : \mathcal{D}(n) \to M_n(\mathbb{C})$ that **respects intertwining**; i.e., if $X \in \mathcal{D}(n)$, $Y \in \mathcal{D}(m)$, $\Gamma : \mathbb{C}^m \to \mathbb{C}^n$, and

$$X\Gamma = (X_1\Gamma, \dots, X_g\Gamma) = (\Gamma Y_1, \dots, \Gamma Y_g) = \Gamma Y,$$

then $f[n](X)\Gamma = \Gamma f[m](Y)$. Equivalently, f respects direct sums and similarity. The definition of a free function naturally extends to vector-valued functions $f : \mathcal{D} \to M(\mathbb{C})^h$, matrix-valued functions $f : \mathcal{D} \to M_e(\mathbb{C})$ and even operator-valued functions. We refer the reader to [KVV14, Voi10] for a comprehensive study of free function theory.

sec:formal

1.1.4. *Formal power series and free analytic functions.* Here, assuming, as we always will, its domain $\Gamma \subseteq M(\mathbb{C})^g$ is a free open set (meaning each $\Gamma(n) \subseteq M_n(\mathbb{C})^g$ is open), a free function $f = (f[n])_n : \Gamma \to M(\mathbb{C})$ is **free analytic** if each $f[n]$ is analytic. Very weak additional hypotheses (e.g. continuity [HKM11] or even local boundedness [KVV14, AM14]) on a free function imply it is analytic.

An important fact for us is that a formal power series with positive radius of convergence determines a free analytic function within its radius of convergence and (under a mild local boundedness assumptions) vice versa, cf. [KVV14, Chapter 7] or [HKM12b, Proposition 2.24]. Given a positive integer d and Hilbert space H, an operator-valued **formal power series** f in x is an expression of the form

$$f = \sum_{m=0}^{\infty} \sum_{\substack{w \in \langle x \rangle \\ |w|=m}} f_w w = \sum_{m=0}^{\infty} f^{(m)},$$

where $f_w : \mathbb{C}^d \to H$ are linear maps and $f^{(m)}$ is the **homogeneous component** of degree m of f; that is, the sum of all monomials in f of degree m. Given $X \in M_n(\mathbb{C})^g$, define

$$f(X) = \sum_{m=0}^{\infty} \sum_{\substack{w \in \langle x \rangle \\ |w|=m}} f_w \otimes w(X),$$

provided the series converges (summed in the indicated order). If the norms of the coefficients of f grow slowly enough, then, for $\|X_j\|$ sufficiently small, the series $f(X)$ will converge. For the purposes of this article, the **formal radius of convergence** $\tau(f)$ of a formal power series $f(x) = \sum f_\alpha \alpha$ with operator coefficients is

$$\tau(f) = \frac{1}{\limsup_N \left(\sum_{|\alpha|=N} \|f_\alpha\| \right)^{\frac{1}{N}}},$$

with the obvious interpretations in the cases that the limit superior is either zero or infinity. Similarly, the **spectral radius** of a tuple $X \in M_n(\mathbb{C})^g$ is

$$\rho(X) = \limsup_N \max\{\|X^\alpha\|^{\frac{1}{N}} : |\alpha| = N\}.$$

A tuple of matrices $E \in (\mathbb{C}^{n\times n})^g$ is (jointly) **nilpotent** if there exists an N such that $E^w = 0$ for all words w of length $|w| \geq N$. The smallest such N is the **order of nilpotence** of E. In particular, if X is (jointly) nilpotent, then $\rho(X) = 0$. In any event, if $X \in M(\mathbb{C})^g$ and $\rho(X) < \tau(f)$, then the series

$$f(X) = \sum_{N=0}^{\infty} \sum_{|\alpha|=N} f_\alpha \otimes X^\alpha$$

converges. Let $\Delta_\tau = \{X \in M(\mathbb{C})^g : \rho(X) < \tau\}$.

ec:freerats

1.1.5. *Free Rational Functions.* Free rational functions regular at 0 (in the free variables $x = (x_1, \dots, x_g)$) appear in many areas of mathematics and its applications including automata theory and systems engineering. There are several different, but equivalent definitions. Based on the results of [KVV09, Theorem 3.1] and [Vol17, Theorem 3.5]) a **free rational functions regular at** 0 can, for the purposes of this article, be defined with minimal overhead as an expression of the form

eq:ratr

$$r(x) = c^*\big(I - \Lambda_E(x)\big)^{-1} b \tag{1.1}$$

where e is a positive integer, $E \in M_e(\mathbb{C})^g$ and $b, c \in \mathbb{C}^e$ are vectors. The expression r is evaluated in the obvious fashion for a tuple $X \in M_n(\mathbb{C})^g$ so long as $I - \Lambda_E(X)$ is invertible. In particular, this natural domain of r contains a free neighborhood of 0. Often in the sequel by *rational function* it will be clear from the context that we mean *free rational function regular at* 0. An exercise shows that free polynomials are (free) rational functions. Moreover, it is true that the sum and product of rational functions are again rational. Likewise a free rational function r as in equation (1.1) is free analytic. It is a fundamental result that the singularity set of r coincides with the singularity set (i.e., the free locus [KV17]) $\mathcal{Z}_E$ of $I - \Lambda_E$ (see [KVV09, Theorem 3.1] and [Vol17, Theorem 3.5]) if E is of minimal size among all representations of the form (1.1) for r. That is, r can not be extended analytically to a (open) set strictly containing the free locus $\mathcal{Z}_E$.

sec:ct

1.2. Bianalytic maps between free spectrahedra.

A **free analytic mapping** (or simply an analytic mapping) p is, for some pair of positive integers $g, \tilde{g}$, an expression of the form

$$p = (p^1, \dots, p^{\tilde{g}}),$$

where each p^j is an analytic function in the free variables $x = (x_1, \dots, x_g)$. Given free domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$, we write $p : \mathcal{D} \to \tilde{\mathcal{D}}$ to indicate $\mathcal{D}$ is a subset of the domain of p and p maps $\mathcal{D}$ into $\tilde{\mathcal{D}}$. The domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are **bianalytic** if there exist free analytic mappings $p : \mathcal{D} \to \tilde{\mathcal{D}}$ and $q : \tilde{\mathcal{D}} \to \mathcal{D}$ such that $p \circ q$ and $q \circ p$ are the identity mappings on $\tilde{\mathcal{D}}$ and $\mathcal{D}$ respectively. To emphasize the role of p (and q), we say that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are p-**bianalytic**.

In this paper we introduce a small and highly structured class of birational maps we call **convexotonic** and to each such map p describe the pairs of spectrahedra $(\mathcal{D}, \tilde{\mathcal{D}})$ bianalytic via p. We conjecture these triples $(p, \mathcal{D}, \tilde{\mathcal{D}})$ account for all bianalytic free spectrahedra (up to affine linear equivalence) and establish the result under certain irreducibility hypotheses on $\mathcal{D}$ and $\tilde{\mathcal{D}}$. We start with the definition of the convexotonic maps.

c:contonics

1.2.1. *Convexotonic maps.* A tuple $\Xi = (\Xi_1, \dots, \Xi_g) \in M_g(\mathbb{C})^g$ satisfying

eq:cttuple

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} \Xi_s \tag{1.2}$$

for each $1 \le j, k \le g$ is **convexotonic**. We say the rational mappings p and q whose entries have the form

$$p^i(x) = \sum_j x_j \left(I - \Lambda_\Xi(x)\right)^{-1}_{j,i} \qquad \text{and} \qquad q^i(x) = \sum_j x_j \left(I + \Lambda_\Xi(x)\right)^{-1}_{j,i},$$

that is, in row form,

eq:tropic

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \text{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1} \tag{1.3}$$

are **convexotonic**. It turns out (see Proposition 6.2) the mappings p and q are inverses of one another, hence they are birational maps.

Given a g-tuple $R = (R_1, \dots, R_g)$ of $n \times n$ matrices that spans a g-dimensional algebra $\mathcal{R}$, we call the g-tuple of $g \times g$ matrices $\Xi = (\Xi_1, \dots, \Xi_g)$ uniquely determined by

$$R_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} R_s,$$

the **structure matrices** for $\mathcal{R}$ (suppressing the obvious dependence on the choice of basis R). By Proposition 6.3, Ξ is convexotonic. Moreover, if Ξ is convexotonic, then $\mathcal{X}$ equal the span of Ξ is an algebra of dimension at most g for which Ξ_j are the structure matrices. See Proposition 6.2.

Conversely, each convexotonic g-tuple Ξ as in (1.2) (even if linearly dependent) arises as the set of structure matrices for a g-dimensional algebra. For instance, letting R_j denote the $(g+1) \times (g+1)$

matrices

$$R_j = \begin{pmatrix} 0 & e_j^* \\ 0 & \Xi_j \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathbb{C} \oplus \mathbb{C}^g$ (here e_j is the j-th standard basis vector for $\mathbb{C}^g$) as an easy computation reveals.

Convexotonic maps are fundamental objects and to each are attached pairs of bianalytic spectrahedra. Let $\mathcal{R} = \operatorname{span}\{R_1, \dots, R_g\} \subseteq M_d(\mathbb{C})$ be a g-dimensional algebra with structure matrices Ξ, and suppose that C a $d \times d$ is a unitary matrix and a tuple $A \in M_d(\mathbb{C})^g$, such that $R_j = (C - I)A_j$ for $1 \le j \le g$, and

:Astructure
$$A_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s. \tag{1.4}$$

In particular, the span $\mathcal{A}$ of the A_j is a right $\mathcal{R}$-module and if $C - I$ is invertible then (1.4) holds automatically. We call the so constructed $(\mathcal{D}_A, \mathcal{D}_{CA})$ a **spectrahedral pair** associated to the algebra $\mathcal{R}$.

1.2.2. *Overview of free bianalytic maps between free spectrahedra.*

thm:ctok **Theorem 1.1.** *If $(\mathcal{D}_A, \mathcal{D}_{CA})$ is a spectrahedral pair associated to a g-dimensional algebra $\mathcal{R}$ and C is unitary, then $\mathcal{D}_A$ is bianalytic to $\mathcal{D}_{CA}$ under the convexotonic map p whose structure matrices Ξ are associated to the algebra $\mathcal{R}$.*

Proof. A proof appears immediately after Theorem 6.7. ■

We conjecture that convexotonic maps are the only bianalytic maps between free spectrahedra.

conj:main **Conjecture 1.2.** *Up to conjugation with affine linear maps, the only bounded free spectrahedra $\mathcal{D}_A$, $\mathcal{D}_B$ that are p-bianalytic arise as spectrahedral pairs associated to an algebra $\mathcal{R}$ with p as the corresponding convexotonic map.*

A weaker version of the conjecture adds the hypothesis that the the ranges of the A_j and B_k span their respective spaces.

Theorem 1.5 below says the conjecture is true in a generic sense. An unusual feature of Conjecture 1.2 from the viewpoint of traditional several complex variables is that typical bianalytic mapping results would be stated up to conjugation with automorphisms of $\mathcal{D}_A$ and $\mathcal{D}_B$. Here we are actually asserting conjugation up to affine linear equivalence. See also Subsection 9.3.

We emphasize there are few indecomposable g-dimensional complex algebras. To give a clear picture we have calculated the convexotonic maps for these algebras explicitly for $g = 2$ and $g = 3$. These calculations were done in Mathematica using NCAlgebra in a notebook you can use after downloading from `https://github.com/NCAlgebra/UserNCNotebooks` [HOMS].

prop:g2g3 **Proposition 1.3.** *We list a basis R_1, R_2 for each of the four 2-dimensional indecomposable algebras over $\mathbb{C}$. Then we give the associated "indecomposable" convexotonic map and its (convexotonic) inverse.*

(1) *R_1 is nilpotent of order 3 and $R_2 = R_1^2$*

$$p(x_1, x_2) = \begin{pmatrix} x_1 & x_2 + x_1^2 \end{pmatrix} \qquad q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 - x_1^2 \end{pmatrix}.$$

(2) $R_1^2 = R_1, \ R_1 R_2 = R_2$

$$p(x) = \begin{pmatrix} (1 - x_1)^{-1} x_1 & (1 - x_1)^{-1} x_2 \end{pmatrix} \qquad q(x) = \begin{pmatrix} (1 + x_1)^{-1} x_1 & (1 + x_1)^{-1} x_2 \end{pmatrix}.$$

(3) $R_1^2 = R_1,\ R_2R_1 = R_2$

$$p(x) = \begin{pmatrix} x_1(1-x_1)^{-1} & x_2(1-x_1)^{-1} \end{pmatrix} \qquad q(x) = \begin{pmatrix} x_1(1+x_1)^{-1} & x_2(1+x_1)^{-1} \end{pmatrix}.$$

(4) $R_1^2 = R_1,\ R_1R_2 = R_2,\ R_2R_1 = R_2$

$$p(x) = \begin{pmatrix} x_1(1-x_1)^{-1} & (1-x_1)^{-1}x_2(1-x_1)^{-1} \end{pmatrix}$$
$$q(x) = \begin{pmatrix} x_1(1+x_1)^{-1} & (1+x_1)^{-1}x_2(1+x_1)^{-1} \end{pmatrix}.$$

For $g = 3$ there are exactly ten plus a one parameter family of indecomposable convexotonic maps, since there are exactly this many corresponding indecomposable 3-dimensional algebras, see Appendix A to the arXiv posting https://arxiv.org/abs/1604.04952 *of this paper.*

Proof. See Section 9. ■

Remark 1.4. All g variable convexotonic maps are direct sums of convexotonic maps associated to indecomposable algebras. See Subsection 9.2.

The composition of two convexotonic maps may not be convexotonic (see Subsection 8.4), a further indication of the very restrictive nature of convexotonic maps. ♢

1.2.3. *Results on free bianalytic maps under a genericity assumption.* The main result of this paper supporting Conjecture 1.2 is Theorem 1.5 below. It says, in part, under certain irreducibility conditions on A and B, if $\mathcal{D}_A$ and $\mathcal{D}_B$ are p-bianalytic, then p and its inverse q are in fact convexotonic.

Let d be a positive integer. A set $\{u^1, \dots, u^{d+1}\}$ is a **hyperbasis** for $\mathbb{C}^d$ if each d element subset is a basis. The tuple $A \in M_d(\mathbb{C})^g$ is **sv-generic** if there exists $\alpha^1, \dots, \alpha^{d+1}$ and $\beta^1, \dots, \beta^d$ in $\mathbb{C}^g$ such that $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ is positive semidefinite and has a one-dimensional kernel spanned by u^j and the set $\{u^1, \dots, u^{d+1}\}$ is a hyperbasis for $\mathbb{C}^d$; and $I - \Lambda_A(\beta^k)\Lambda_A(\beta^k)^*$ is positive semidefinite, its kernel spanned by v^k and the set $\{v^1, \dots, v^d\}$ is a basis for $\mathbb{C}^d$. Generic tuples A satisfy this property, see Remark 7.5. Weaker (but still sufficient) versions of the sv-generic condition are given in the body of the paper, see Subsection 7.1.2.

Theorem 1.5. *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$ are sv-generic and $\mathcal{D}_A$ is bounded. If p is a birational map between $\mathcal{D}_A$ and $\mathcal{D}_B$ with $p(0) = 0$ and $p'(0) = I_g$, then*

(1) *$d = e$;*
(2) *there exists a $d \times d$ matrix C such that B is unitarily equivalent to CA;*
(3) *the tuple $R = (C - I)A$ spans an algebra $\mathcal{R}$;*
(4) *the span of A is a right $\mathcal{R}$-module; and*
(5) *letting Ξ denote the structure matrices for this module, p has the convexotonic form of (1.3); that is,*

$$p(x) = x(I - \Lambda_\Xi(x))^{-1}.$$

Proof. A proof appears at the end of Section 7. ■

We point out the normalization conditions $p(0) = 0$ and $p'(0) = I$ can be enforced e.g. by an affine linear change of variables on the range of p, see Section 8 for details.

The proof of Theorem 1.5 is based on several intermediate results of independent interest. Subsection 1.3 contains results approximating free spectrahedra by more tractable free sets. Subsection 1.4 describes Positivstellensatz for (matrix-valued) free analytic functions with positive real part on a free spectrahedron.

1.3. **Approximating free spectrahedra and free analytic functions.** This subsection concerns approximation of functions analytic on free spectrahedra by analytic polynomials; that is, a free Oka-Weil theorem. An example is the remarkable theorem of Agler and McCarthy [AM14] (see also [BMV]) stated below as Theorem 1.6.

Given a matrix-valued free analytic polynomial Q, the set

$$\mathcal{G}_Q = \{X \in M(\mathbb{C})^g : \|Q(X)\| < 1\}$$

is a (semialgebraic) **free pseudoconvex** set. Given $t > 1$, let

$$K_{tQ} = \{X : t\|Q(X)\| \leq 1\} \subseteq \mathcal{G}_Q.$$

A (matrix-valued) free analytic function f on a free domain $\mathcal{D} \subseteq M(\mathbb{C})^g$ is **uniformly approximable by polynomials** on a subset $\mathcal{E} \subseteq \mathcal{D}$ if for each $\epsilon > 0$ there is a polynomial q such that $\|f(X) - q(X)\| < \epsilon$ for each n and $X \in \mathcal{E}(n)$.

Theorem 1.6. *If f is a bounded free analytic function on a free pseudoconvex set $\mathcal{G}_Q$, then f can be uniformly approximated by analytic free polynomials on each smaller set K_{tQ}, $t > 1$.*

Proof. This result is proved, though not stated in this form, in Section 9 of [AM14] (cf. their proof of Corollary 9.7; see also [AM14, Corollary 8.13]). ■

Free spectrahedra are approximable by free pseudoconvex sets.

Proposition 1.7. *If $\mathcal{D}_A$ is bounded and $t > 1$, then there exists free analytic polynomial Q such that*

$$\mathcal{D}_A \subseteq \mathcal{G}_Q \subseteq t\mathcal{D}_A.$$

Moreover, if $\mathcal{G}_Q$ is a free pseudoconvex set and $\mathcal{D}_A \subseteq \mathcal{G}_Q$, then there is an $s > 1$ such that $\mathcal{D}_A \subseteq K_{sQ}$. Finally, if p is a free rational function analytic on $\mathcal{D}_A$, then there is a $t > 1$ such that p is analytic and bounded on $t\mathcal{D}_A$.

Proof. A proof is given near the end of Section 2.2. ■

Theorem 1.8. *Suppose $A \in M_d(\mathbb{C})^g$ and $\mathcal{D}_A$ is bounded. If f is analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$, then f is uniformly approximable by polynomials on $\mathcal{D}_A$.*

Proof. A proof is given at the end of Section 2.2. ■

1.4. **Positivstellensätze and representations for analytic functions.** We begin this section with Positivstellensätze and then turn to representations they imply. We use **nonnegative** and **positive** as synonyms for positive semidefinite and positive definite respectively.

Theorem 1.9 (Analytic convex Positivstellensatz)**.** *Let $A \in M_d(\mathbb{C})^g$ and e be a positive integer. Assume $\mathcal{D}_A$ is bounded and $G : \mathcal{G}_Q \to M_e(M(\mathbb{C}))$ is a matrix-valued free function analytic on a free pseudoconvex set $\mathcal{G}_Q$ containing the free spectrahedron $\mathcal{D}_A$. If $G(0) = 0$ and $I + G + G^*$ is nonnegative on $\mathcal{D}_A$, then there exists*

(1) *a Hilbert space H;*
(2) *a formal power series $W = \sum_{\alpha \in \langle x \rangle} W_\alpha \alpha$ with coefficients $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$;*
(3) *a unitary mapping $C : H \otimes \mathbb{C}^d \to H \otimes \mathbb{C}^d$ and an isometry $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$,*

such that the identity

$$I + G(x) + G(x)^* = W(x)^* L_{I_H \otimes A}(x) W(x) \tag{1.5}$$

holds in the ring of $e \times e$ matrices over formal power series in x, x^* and there exists a $\tau > 0$ such that equation (1.5) holds at each $X \in \Delta_\tau$.

Moreover, letting $\mathscr{E} = H \otimes \mathbb{C}^d$, $\mathbf{A} = I_H \otimes A$ *and* $R = (C - I)\mathbf{A}$, *the functions* G *and* W *are given by*

$$G(x) = \mathscr{W}^* C\Big(\sum_{j=1}^{g} \mathbf{A}_j x_j\Big)\, W(x) \tag{1.6}$$

and

$$W(x) = \Big(I_{\mathcal{E}} - \sum_{j=1}^{g} R_j x_j\Big)^{-1} \mathscr{W} \tag{1.7}$$

and the coefficients $G_{x_j\alpha}$ *of* G *are given by*

$$G_{x_j\alpha} = \mathscr{W}^* C \mathbf{A}_j R^\alpha \mathscr{W}; \tag{1.8}$$

for all words α.

Proof. See Subsection 5.1. ∎

An analytic (not necessarily bianalytic) map p maps $\mathcal{D}_A$ into $\mathcal{D}_B$ if and only if $L_A(X) \succeq 0$ implies $L_B(p(X)) \succeq 0$. Theorem 1.9 thus provides a representation for $G(x) = \Lambda_B(p(x))$ with a state space realization flavor.

Corollary 1.10 (Rational convex Positivstellensatz)**.** *Let* $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde g}$, *assume that* $\mathcal{D}_A$ *is bounded and* $p : \mathcal{D}_A \to \mathcal{D}_B$ *satisfies* $p(0) = 0$. *Let* $G(x) = \Lambda_B(p(x))$. *If* p *is either a rational function or a free function analytic and bounded on a free pseudoconvex set* $\mathcal{G}_Q$ *containing* $\mathcal{D}_A$, *then there exists a Hilbert space* H, *a formal power series* $W = \sum_{\alpha\in\langle x\rangle} W_\alpha \alpha$ *with coefficients* $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$, *a unitary* $C : H \otimes \mathbb{C}^d \to H \otimes \mathbb{C}^d$ *and an isometry* $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ *such that* $L_B(p(x)) = I + G(x)^* + G(x)$ *and the conclusions* (1.5) *–* (1.8) *of Theorem 1.9 hold.*

Proof. By Proposition 1.7, in any case we may assume p is analytic on a pseudoconvex set containing $\mathcal{D}_A$. Since p is analytic in a pseudoconvex neighborhood of $\mathcal{D}_A$ so is G; and since p maps $\mathcal{D}_A$ into $\mathcal{D}_B$, it follows that $I + G + G^*$ is nonnegative on $\mathcal{D}_A$. An application of Theorem 1.9 completes the proof. ∎

A key ingredient in the proof of Theorem 1.9 is Proposition 4.3. It shows that a Positivstellensatz certificate like that of equation (1.5) suffices to deduce the remaining conclusions of Theorem 1.9.

For hereditary polynomials positive on a free spectrahedron, the conclusion of Theorem 1.9 is stronger. The weight(s) W in the positivity certificate (1.9) are polynomial, still analytic and we get optimal degree bounds.

Theorem 1.11 (Hereditary Convex Positivstellensatz)**.** *Let* $A \in M(\mathbb{C})^g$, *and let* $h \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle$ *be an hereditary matrix polynomial of degree* d. *Then* $h|_{\mathcal{D}_A} \succeq 0$ *if and only if*

$$h = \sum_{k}^{\text{finite}} h_k^* h_k + \sum_{j}^{\text{finite}} f_j^* L_A f_j \tag{1.9}$$

for some analytic polynomials $h_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_{d+1}$, $f_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_d$. *Moreover, if* $\mathcal{D}_A$ *is bounded, then the pure sum of squares term in* (1.9) *may be omitted, provided the* f_j *are allowed to have degree* $\le d + 1$.

Proof. See Section 3 and in particular Theorem 3.1. ∎

Theorem 1.12 shows, under the assumption of a square **one term Positivstellensatz certificate** (1.10) for a mapping $p : \mathcal{D}_A \to \mathcal{D}_B$ between free spectrahedra, that p is convexotonic and in particular birational between $\mathcal{D}_A$ and $\mathcal{D}_B$.

thm:all **Theorem 1.12.** *Suppose $A, B \in M_d(\mathbb{C})^g$, the set $\{A_1, \dots, A_g\}$ is linearly independent and $p = (p^1, \dots, p^g)$ is a free formal power series map in x (no x_j^*) with $p(0) = 0$ and $p'(0) = I$. If there exists a $d \times d$ free formal power series W such that*

eq:1term
$$L_B(p(x)) = W(x)^* L_A(x) W(x), \tag{1.10}$$

then p is a convexotonic map

$$p(x) = x(I - \Lambda_\Xi(x))^{-1}$$

as in (1.3), determined by a module spanned by the set $\{A_1, \dots, A_g\}$ over an algebra of dimension at most g with structure matrices Ξ.

Proof. See Theorem 6.7. ∎

In the context of Theorem 1.5, the sv-generic condition is used to show the one term Positivstellensatz hypothesis of Theorem 1.12 holds.

sec:guide 1.5. **Readers guide.** The paper is organized as follows. The polynomial approximation results of Subsection 1.3 are proved in Section 2. Section 3 contains the proof of Theorem 1.11. In Section 4, key algebraic consequences of an Hereditary Positivstellensatz representation are collected for use in the following sections. The proof of Theorem 1.9 appearing in Section 5 uses the results of the previous three sections. Theorem 1.12 is proved in Section 6. A somewhat more general version of Theorem 1.5 is the topic of Section 7. Throughout much of the article the (bi)analytic maps are assumed to satisfy the normalization $p(0) = 0$ and $p'(0)$ a projection. Section 8 describes the consequences of relaxing this assumption. Section 9 provides examples of convexotonic maps. In the several complex variables spirit of classifying domains up to affine linear equivalence, it is natural to ask if there exist matrix convex domains that are polynomially, but not affine linearly, bianalytic. The hard won answer is yes. A class of examples appears in Section 10.

sec:approx

2. Approximating Free Analytic Functions by Polynomials

In this section we prove Proposition 1.7 and Theorem 1.8 approximating free spectrahedra with free pseudoconvex sets and approximating free mappings analytic on free spectrahedra by free polynomials, respectively.

2.1. **Approximating free spectrahedra and free analytic functions using free polynomials.** For $C > 0$, let $\mathfrak{F}_C$ denote the free set of matrices T such that $C - (T + T^*) \succeq 0$ and for $M > 0$, let $\mathfrak{F}_{C,M}$ denote those $T \in \mathfrak{F}_C$ such that $\|T\| < M$. Let φ denote the linear fractional mapping $\varphi(z) = z(1-z)^{-1}$. In particular, φ maps the region $\{z : \operatorname{Re} z \le \frac{1}{2}\}$ in the complex plane to the set $\{z : |z| \le 1, z \neq -1\}$. The inverse of φ is $\psi(w) = w(1+w)^{-1}$. Given $\epsilon > \delta > 0$ sufficiently small, the ball $\mathbb{B}_\delta(\epsilon) = \{z : |z - \epsilon| \le 1 + \delta\}$ does not contain -1 and there exists a $K \in (1, 2)$ such that $\psi(\mathbb{B}_\delta(\epsilon)) \subseteq \{z : \operatorname{Re} z < \frac{K}{2}\}$.

lem:reAhalf **Lemma 2.1.** *If $2 > C$ and $T \in \mathfrak{F}_C$, then $I - T$ is invertible. Moreover, given $M > 0$ and $t > 0$, there exists $2 > C > 1$, and $t > \epsilon > \delta > 0$ such that if $T \in \mathfrak{F}_{C,M}$, then*

$$\|\varphi(T) - \epsilon\| \le 1 + \delta.$$

Proof. A routine argument establishes the first part of the lemma. To prove the moreover part, fix $M > 0$ and $t > 0$ and suppose $\min\{1, t\} > \epsilon > 0$. Choose $0 < \rho < 1$ such that both

$$\big(2(1-\rho) + \epsilon(1-\rho^2)\big)M^2 < \frac{1}{2},$$

$$1 < C := \frac{1 + 2(\rho - \frac{1}{4})\epsilon - (1-\rho^2)\epsilon^2}{1 + 2(\rho - \frac{1}{2})\epsilon - (1-\rho^2)\epsilon^2} < 2.$$

Let $T \in \mathfrak{F}_{C,M}$ be given. It follows that

eq:epsrho

$$\begin{aligned} \epsilon\Big(2(1-\rho)+\epsilon(1-\rho^2)\Big)T^*T &\preceq \frac{\epsilon}{2} \\ &\preceq \frac{\epsilon}{2} + \Big(1+2(\rho-\frac{1}{2})\epsilon-(1-\rho^2)\epsilon^2\Big)\big(C-(T+T^*)\big) \qquad (2.1) \\ &\preceq \frac{\epsilon}{2} + \Bigg(\Big(1+2(\rho-\frac{1}{4})\epsilon-(1-\rho^2)\epsilon^2\Big) - \Big(1+2(\rho-\frac{1}{2})\epsilon-(1-\rho^2)\epsilon^2\Big)(T+T^*)\Bigg). \end{aligned}$$

Let $\delta = \rho\epsilon$ and observe,

eq:epsdel1

$$\epsilon\Big(2(1-\rho)+\epsilon(1-\rho^2)\Big) = (1+\epsilon)^2-(1+\delta)^2, \qquad (2.2)$$

eq:epsdel2

$$\begin{aligned} \frac{\epsilon}{2}+1+2(\rho-\frac{1}{4})\epsilon-(1-\rho^2)\epsilon^2 &= \frac{\epsilon}{2}+1+2\rho\epsilon-\frac{\epsilon}{2}-\epsilon^2+(\rho\epsilon)^2 \qquad (2.3) \\ &= 1+2\delta+\delta^2-\epsilon^2 = (1+\delta)^2-\epsilon^2, \end{aligned}$$

and

eq:epsdel3

$$1+2(\rho-\frac{1}{2})\epsilon-(1-\rho^2)\epsilon^2 = 1+2\rho\epsilon-\epsilon-\epsilon^2+(\rho\epsilon)^2 = (1+\delta)^2-\epsilon(1+\epsilon). \qquad (2.4)$$

Thus, substituting $\delta = \rho\epsilon$ into equation (2.1) and using equations (2.2), (2.3), (2.4) yields,

$$\Big((1+\epsilon)^2-(1+\delta)^2\Big)T^*T \preceq \Big((1+\delta)^2-\epsilon^2\Big) - \Big((1+\delta)^2-\epsilon(1+\epsilon)\Big)(T+T^*).$$

Rearranging gives,

$$(1+\epsilon)^2T^*T-\epsilon(1+\epsilon)(T+T^*)+\epsilon^2 \preceq (1+\delta)^2\big(I-(T+T^*)+T^*T\big)$$

and hence

$$\big((1+\epsilon)T-\epsilon\big)^*\big((1+\epsilon)T-\epsilon\big) \preceq (1+\delta)^2\,(I-T)^*\,(I-T).$$

The lemma follows from this last inequality together with $(1+\epsilon)T-\epsilon = T-\epsilon(I-T)$. ■

m:uniffrakT

Proposition 2.2. *For each $M > 0$ and $\rho > 0$ there exists $2 > C_0 > 1$ such that for each $1 < C < C_0$ there exists $\rho > \epsilon > \delta > 0$ such that for each $\eta > 0$ there is an analytic polynomial q in one variable such that for all $T \in \mathfrak{F}_{C,M}$,*

(1) $\|\varphi(T)-\epsilon\| < 1+\delta$;
(2) $\|q(T)-\varphi(T)\| < \eta$.

Proof. The linear fractional map $\varphi(z) = z(1-z)^{-1}$ is analytic on $\mathfrak{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$. By Lemma 2.1 with $t = \rho$, given $M > 0$ and $\rho > 0$ there exists a $2 > C > 1$ and $\rho > \epsilon > \delta > 0$ such that if $T \in \mathfrak{F}_{C,M}$, then $\|\varphi(T)-\epsilon\| < 1+\delta$. Let $\varphi_*(z) = \varphi(z)-\epsilon$. Its inverse is $\psi_*(w) = \psi(w+\epsilon)$. In particular, ψ_* is analytic in a neighborhood of the closed ball $\mathbb{B}_\delta(0) = \{z \in \mathbb{C} : |z| \le 1+\delta\}$ and for $\delta > 0$ sufficiently small, $\psi_*(\mathbb{B}_\delta(0))$ is a compact subset of $\mathfrak{H}$. Thus, by Runge's Theorem, there exists a polynomial p such that

$$\|p-\varphi_*\|_{\psi_*(\mathbb{B}_\delta(0))} := \sup\{|p(z)-\varphi_*(z)| : z \in \psi_*(\mathbb{B}_\delta)\} < \eta.$$

Hence,

$$\|p\circ\psi_* - z\|_{\mathbb{B}_\delta(0)} < \eta.$$

Now let $T \in \mathfrak{F}_{C,M}$ be given. The matrix $S = \varphi(T)-\epsilon$ has norm at most $1+\delta$ and hence

$$\|(p\circ\psi_*)(S)-S\| \le \eta.$$

Equivalently,

$$\|p(T)-\varphi_*(T)\| \le \eta.$$

Choosing $q = p+\epsilon$, completes the proof. ■

varphibound **Corollary 2.3.** *There exists a* $2 > C_0 > 1$ *such that for each* $M > 0$ *and* $C_0 > C > 1$*, the set* $\{\|\varphi(T)\| : T \in \mathfrak{F}_{C,M}\}$ *is bounded.*

Proof. By Proposition 2.2, there exists $\epsilon, \delta > 0$ such that $\|\varphi(T) - \epsilon\| < 1 + \delta$ for $T \in \mathfrak{F}_{C,M}$. Hence,

$$\|\varphi(T)\| \leq 1 + \epsilon + \delta$$

for $T \in \mathfrak{F}_{C,M}$. ■

lem:FGF **Lemma 2.4.** *For each* $M > 0$ *and* $2 > C > 1$ *there exists an analytic* (2×2 *matrix) polynomial* s *in one variable such that*

$$\mathfrak{F}_{1,M} \subseteq \mathcal{G}_s = \{T : \|s(T)\| < 1\} \subseteq \mathfrak{F}_{C,M}.$$

Proof. Choose $\rho > 0$ such that

$$\frac{M^2 + 1}{\rho} < C - 1$$

and let $R^2 = M^2 + 1 + \rho + \rho^2$. In particular,

eq:rhoR
$$\frac{R^2 - \rho^2}{\rho} = \frac{M^2 + 1 + \rho}{\rho} < C. \tag{2.5}$$

Let $s_1(x) = \frac{x+\rho}{R}$, $s_2(x) = \frac{x}{M}$ and $s = s_1 \oplus s_2$. Thus, $\|s(T)\| < 1$ if and only if $\|T + \rho\| < R$ and $\|T\| < M$. Suppose $T \in \mathfrak{F}_{1,M}$. Then automatically $\|s_2(T)\| < 1$. Using $T + T^* \preceq I$, estimate

$$(T + \rho)^*(T + \rho) = T^*T + \rho(T + T^*) + \rho^2 \preceq M^2 + \rho + \rho^2 < R^2.$$

Thus $\|s_1(T)\| < 1$ and the first inclusion of the lemma is proved.

Now suppose $\|s(T)\| < 1$. Equivalently $\|T\| < M$ and $\|T + \rho\| < R$. Hence,

$$\begin{aligned} 0 \preceq R^2 - (T + \rho)^*(T + \rho) &= R^2 - T^*T - \rho(T + T^*) - \rho^2 \\ &\preceq \rho\Big(\frac{R^2 - \rho^2}{\rho} - (T + T^*)\Big) \preceq \frac{1}{\rho}\big(C - (T^* + T)\big), \end{aligned}$$

where equation (2.5) was used to obtain the last inequality. Thus, $\|s(T)\| < 1$ implies $T \in \mathfrak{F}_{C,M}$ and the proof is complete. ■

lem:approx **Lemma 2.5.** *If* $\mathcal{D}_A$ *is bounded and* $t > 1$*, then there exists a matrix-valued free polynomial* Q *such that*

$$\mathcal{D}_A \subseteq \mathcal{G}_Q \subseteq t\mathcal{D}_A.$$

Proof. Since $\mathcal{D}_A$ is bounded, there is an $M > 0$ such that $t\|\Lambda_A(X)\| \leq M$ for all $X \in \mathcal{D}_A$. By Proposition 2.2, there exists a $2 > C > 1$ and a sequence of q_k polynomials converging uniformly to $\varphi(z)$ on $\mathfrak{F}_{C,M}$. Passing to a subsequence if needed, we can assume

$$\|q_k(T) - \varphi(T)\| < \frac{1}{k}$$

for $T \in \mathfrak{F}_{C,M}$. Writing

eq:unnamed
$$\begin{aligned} &2\big(q_k(T)^* q_k(T) - \varphi(T)^*\varphi(T)\big) \\ &\qquad = \big(q_k(T) - \varphi(T)\big)^*\big(q_k(T) + \varphi(T)\big) + \big(q_k(T) + \varphi(T)\big)^*\big(q_k(T) - \varphi(T)\big), \end{aligned} \tag{2.6}$$

and using $\varphi(T)$ is uniformly bounded on $\mathfrak{F}_{C,M}$ (see Corollary 2.3), there is a constant κ (independent of k and T) such that

$$q_k(T)^* q_k(T) - \varphi(T)^*\varphi(T) \preceq \frac{\kappa}{k}.$$

Hence,

$$I + \frac{\kappa}{k} - q_k(T)^* q_k(T) \succeq I - \varphi(T)^*\varphi(T).$$

Thus, if $T \in \mathfrak{F}_{C,M}$ and $I - \varphi(T)^*\varphi(T) \succeq 0$, then $I - (1+\frac{\kappa}{k})^{-1} q_k(T)^* q_k(T) \succeq 0$.

Now, given a monic linear pencil $L_A = I + \Lambda_A + \Lambda_A^*$, let

$$Q_k = \left(1 + \frac{\kappa}{k}\right)^{-\frac{1}{2}} q_k \circ \Lambda_A.$$

If $X \in \mathcal{D}_A$, then $T = \Lambda_A(X) \in \mathfrak{F}_{1,M}$. Hence $I - Q_k(X)^* Q_k(X) \succeq 0$; that is, $\mathcal{D}_A \subseteq K_{Q_k}$, in the notation $K_{Q_k} := \{X : \|Q_k(X)\| \le 1\}$ of [AM14]. Moreover, since $q_k(T)$ converges to $\varphi(T)$,

$$\mathcal{D}_A = \bigcap_k^\infty K_{Q_k}.$$

Choose s as in Lemma 2.4 so that $\mathfrak{F}_{1,M} \subseteq \{T : \|s(T)\| \le 1\} \subseteq \mathfrak{F}_{C,M}$. Thus,

$$\mathcal{D}_A \subseteq \{X : \|s(\Lambda_A(X))\| < 1\}.$$

Consequently, letting

$$\hat{Q}_k = \begin{pmatrix} Q_k & 0 \\ 0 & s \circ \Lambda_A \end{pmatrix},$$

we have

$$\mathcal{D}_A \subseteq \{X : \|\hat{Q}_k(X)\| \le 1\}.$$

We now turn to showing, given $t > 1$, there is a k such that $\{X : \|\hat{Q}_k(X)\| \le 1\} \subseteq t\mathcal{D}_A$. The estimate (2.6) works reversing the roles of q_k and φ giving the inequality

$$\varphi(T)^*\varphi(T) - q_k(T)^* q_k(T) \preceq \frac{\kappa}{k} \tag{2.7}$$

for $T \in \mathfrak{F}_{C,M}$. Now suppose $X \in M(\mathbb{C})^g$ and $I - \hat{Q}_k(X)^*\hat{Q}_k(X) \succeq 0$. Let, as before $T = \Lambda_A(X)$. It follows that $\|s(T)\| \le 1$ and hence $T \in \mathfrak{F}_{C,M}$. We can thus apply (2.7) to conclude

$$\begin{aligned} 0 \preceq 1 - Q_k(X)^* Q_k(X) &= 1 - \left(1 + \frac{\kappa}{k}\right)^{-1} q_k(T)^* q_k(T) \\ &\preceq 1 + \frac{\kappa}{k}\left(1 + \frac{\kappa}{k}\right)^{-1} - \left(1 + \frac{\kappa}{k}\right)^{-1} \varphi(T)^*\varphi(T). \end{aligned}$$

Let $\tau_k = 1 + \frac{2\kappa}{k}$. This last inequality implies

$$(I - T)^{-*} T^* T (I - T)^{-1} \le \tau_k.$$

A bit of algebra shows this inequality is equivalent to

$$\tau_k - \tau_k(T + T^*) + (\tau_k - 1)T^*T \succeq 0.$$

Since $\tau_k \to 1$, for all sufficiently large k,

$$t > 1 + \frac{\tau_k - 1}{\tau_k} M^2.$$

Using $T^*T \preceq M^2$, it follows that

$$\tau_k(t - (T + T^*)) \succeq \tau_k + (\tau_k - 1)M^2 - \tau_k(T + T^*) \succeq \tau_k\big(I - (T + T^*)\big) + (\tau_k - 1)T^*T \succeq 0.$$

Thus $X \in t\mathcal{D}_A$. Summarizing, for sufficiently large k,

$$\mathcal{D}_A \subseteq \{X : \|\hat{Q}_k(X)\| \le 1\} \subseteq t\mathcal{D}_A. \qquad \blacksquare$$

Lemma 2.6. *Let $A \in M_d(\mathbb{C})^g$. If $\mathcal{D}_A$ is bounded and $\mathcal{G}_Q$ is a free pseudoconvex set such that $\mathcal{D}_A \subseteq \mathcal{G}_Q$, then there is $s > 1$ such that*

$$\mathcal{D}_A \subseteq K_{sQ} \subseteq \mathcal{G}_Q. \tag{2.8}$$

Proof. By definition, $\mathcal{D}_A \subseteq \mathcal{G}_Q$ is equivalent to $\|Q(X)\| < 1$ on $\mathcal{D}_A$. For each M the set $\mathcal{D}_A(M)$ is compact (as $\mathcal{D}_A$ is bounded and closed). Thus, for each M there is an $0 < r_M < 1$ such that $\|Q(X)\| \le r_M$ on $\mathcal{D}_A(M)$.

For $C \in \mathbb{R}_{>0}$, we have $\|Q(X)\| \le C$ on $\mathcal{D}_A$ if and only if $C^2 - Q^*Q \succeq 0$ on $\mathcal{D}_A$ if and only if $C^2 - Q^*Q \succeq 0$ on $\mathcal{D}_A(N)$ for $N := N(\deg Q, g, d)$ large enough ([HKM12a, Remark 1.2]) if and only if $\|Q(X)\| \le C$ on $\mathcal{D}_A(N)$. Since $\|Q(X)\| \le r_N < 1$ on $\mathcal{D}_A(N)$, it follows that $\|Q(X)\| \le r_N < 1$ on $\mathcal{D}_A$. So (2.8) holds with $s = \frac{1}{r_N}$. ■

2.2. **Rational functions analytic on $\mathcal{D}_A$.** In this subsection we show a rational function p without singularities on $\mathcal{D}_A$ is analytic and bounded on $t\mathcal{D}_A$ for some $t > 1$. Hence, by Lemma 2.5, p is analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$.

Lemma 2.7. *Suppose $\mathcal{D}_A$ is bounded and let r be an analytic noncommutative rational function with no singularities on $\mathcal{D}_A$. Then there is a $t > 1$ such that r is bounded with no singularities on $t\mathcal{D}_A$.*

Proof. Since r is analytic on $\mathcal{D}_A$ and $\mathcal{D}_A$ contains 0, we can consider its minimal realization,

$$r(x) = c^*\big(I - \Lambda_E(x)\big)^{-1} b$$

for some $e \times e$ tuple $E \in M_e(\mathbb{C})^g$ and vectors $b, c \in \mathbb{C}^e$. The singularity set of r coincides with the singularity set (i.e., the free locus [KV17]) $\mathcal{Z}_E$ of $I - \Lambda_E$ (see [KVV09, Theorem 3.1] and [Vol17, Theorem 3.5]).

We claim that $\mathcal{Z}_E \cap \mathcal{D}_A = \varnothing$ if and only if $\cup_{1 \le e' \le e} \left(\mathcal{Z}_E(e) \cap \mathcal{D}_A(e)\right) = \varnothing$. To prove the claim, suppose $X \in \mathcal{Z}_E \cap \mathcal{D}_A(m)$. Then for some nonzero $v = \sum_{j=1}^e e_j \otimes v_j$, where the e_j are standard unit vectors in $\mathbb{C}^e$ and $v_j \in \mathbb{C}^m$,

$$0 = (I - \Lambda_E)(X)v = (I \otimes I - \sum_k E_k \otimes X_k)(\sum_{j=1}^e e_j \otimes v_j) = \sum_{j=1}^e e_j \otimes v_j - \sum_{j,k} E_k e_j \otimes X_k v_j.$$

Let P denote the orthogonal projection $\mathbb{C}^m \to \mathcal{V} = \operatorname{span}\{v_1, \dots, v_e\}$ and let $e' = \dim \mathcal{V}$. Then $P^*XP \in \mathcal{Z}_E$. Indeed, for any $u = \sum_i e_i \otimes u_i \in \mathbb{C}^d \otimes \mathcal{V}$,

$$\begin{aligned} u^*(I - \Lambda_E)(P^*XP)v &= (\sum_i e_i \otimes u_i)^*(I \otimes I - \sum_k E_k \otimes P^*X_kP)(\sum_{j=1}^e e_j \otimes v_j) \\ &= \sum_i u_i^* v_i - (\sum_i e_i \otimes u_i)^*(\sum_k E_k \otimes P^*X_kP)(\sum_{j=1}^e e_j \otimes v_j) \\ &= u^*v - \sum_{i,j,k} e_i^* E_k e_j u_i^* P^* X_k P v_j = u^*v - \sum_{i,j,k} e_i^* E_k e_j u_i^* X_k v_j \\ &= (\sum_i e_i \otimes u_i)^*(I \otimes I - \sum_k E_k \otimes X_k)(\sum_{j=1}^e e_j \otimes v_j) \\ &= u^*(I - \Lambda_E)(X)v = 0. \end{aligned}$$

This calculation shows $P^*XP \in \mathcal{Z}_E(e') \cap \mathcal{D}_A(e')$. Hence $\cup_{1 \le e' \le e} \left(\mathcal{Z}_E(e') \cap \mathcal{D}_A(e')\right) \ne \varnothing$. The reverse implication is evident and so the claim is proved.

Since each $\mathcal{D}_A(e')$ is compact and disjoint from the closed set $\mathcal{Z}_E(e')$, there exists $t > 1$ such that $t\mathcal{D}_A(e') \cap \mathcal{Z}_E(e') = \varnothing$ for each $1 \le e' \le e$. But now using $t\mathcal{D}_A = \mathcal{D}_{\frac{1}{t}A}$, the above claim proves there is a $t > 1$ such that r has no singularities on $t\mathcal{D}_A$; that is $\mathcal{Z}_E \cap \mathcal{D}_{tA} = \varnothing$.

We now argue that in fact r is bounded on $t\mathcal{D}_A$. First observe that if $X_n \in t\mathcal{D}_A$ and $\|r(X_n)\|$ grows without bound, then so does $\|(I - \Lambda_E(X_n))^{-1}\|$. Hence, there is a sequence γ_n of unit vectors such that $(\|(I - \Lambda_E(X_n))\gamma_n\|)_n$ tends to zero. By the argument above, we can replace X_n with $Y_n = V_n^* X_n V_n$ where V_n includes an e-dimensional space containing γ_n and assume that the $Y_n \in \mathcal{D}_{tA}(e)$. By compactness of $\mathcal{D}_{tA}(e) = t\mathcal{D}_A(e)$, and passing to a subsequence if needed, without loss of generality Y_n converges to some $Y \in \mathcal{D}_{tA}(e)$ and γ_n to some unit vector γ. It follows that $(I - \Lambda_E(Y))\gamma = 0$, and we have arrived at the contradiction that $Y \in t\mathcal{D}_{tA}$ and Y is a singularity of $I - \Lambda_E(x)$. ■

The ingredients are now in place to prove Propositions 1.7 and 1.8.

Proof of Proposition 1.7. The first statements are immediate from Lemmas 2.5 and 2.6. Lemma 2.7 finishes off the proof. ■

Proof of Theorem 1.8. Suppose f is analytic and bounded on some $\mathcal{G}_Q$ containing $\mathcal{D}_A$. By Proposition 1.7, there is $s > 1$ with $\mathcal{D}_A \subseteq K_{sQ} \subseteq \mathcal{G}_Q$. By the free Oka-Weil Theorem 1.6, f can be uniformly approximated by polynomials on K_{sQ} and thus on $\mathcal{D}_A$. ■

sec:null

3. Hereditary Convex Positivstellensatz

In this section we present a strengthening of the Convex Positivstellensatz [HKM12a], characterizing hereditary polynomials nonnegative on free spectrahedra. In the obtained sum of squares certificate all weights will be analytic.

Fix a symmetric $q \in \mathbb{C}^{\ell\times\ell}\langle x, x^*\rangle$, let

$$\mathcal{D}_q(n) := \{X \in M_n(\mathbb{C})^g : q(X) \succeq 0\}$$

for positive integers n and let $\mathcal{D}_q = (\mathcal{D}_q(n))_n$. Given $\alpha, \beta \in \mathbb{N}$, set

eq:Malbeta (3.1)

$$M^{\nu,\text{her}}_{\alpha,\beta}(q) := \Big\{ \sum_j^{\text{finite}} \varphi_j^*\varphi_j + \sum_i^{\text{finite}} \psi_i^* q \psi_i : \ \psi_i \in \mathbb{C}^{\ell\times\nu}\langle x\rangle_\beta, \ \varphi_j \in \mathbb{C}^{\nu\times\nu}\langle x\rangle_\alpha \Big\} \subseteq \mathbb{R}^{\nu\times\nu}\langle x, x^*\rangle_{\max\{2\alpha, 2\beta + a\}},$$

where $a = \deg(q)$. Observe that $M^{\nu,\text{her}}_{\alpha,\beta}(q)$ is a proper subset of the quadratic module $M^{\nu}_{\alpha,\beta}(q)$ as defined in [HKM12a]. We emphasize that φ_j, ψ_i are assumed to be analytic in (3.1) defining $M^{\nu,\text{her}}_{\alpha,\beta}(q)$. Obviously, if $f \in M^{\nu,\text{her}}_{\alpha,\beta}(q)$ then $f|_{\mathcal{D}_q} \succeq 0$.

For notational convenience, let $\Sigma^{\nu,\text{her}}_\alpha$ denote the cone of sum of squares obtained from $M^{\nu,\text{her}}_{\alpha,\alpha}(q)$ with $q = 1$.

We call $M^{\nu,\text{her}}_{\alpha,\beta}(q)$ the **truncated hereditary quadratic module** defined by q. We often abbreviate $M^{\nu,\text{her}}_{\alpha,\beta}(q)$ to $M^{\nu}_{\alpha,\beta}$. If $q(0) = I$ (q is **monic**), then $\mathcal{D}_q$ contains an **free neighborhood of** 0; i.e., there exists $\epsilon > 0$ such that for each $n \in \mathbb{N}$, if $X \in M_n(\mathbb{C})^g$ and $\|X\| < \epsilon$, then $X \in \mathcal{D}_q$. Likewise $\mathcal{D}_q$ is called **bounded** provided there is a number $N \in \mathbb{N}$ for which all $X \in \mathcal{D}_q$ satisfy $\|X\| < N$. The following theorem is, using the notations above, a restatement of Theorem 1.11.

:heredposSS **Theorem 3.1** (Hereditary Convex Positivstellensatz). *Suppose $L \in \mathbb{C}^{\ell\times\ell}\langle x, x^*\rangle$ is a monic linear pencil and $h \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle$ is a symmetric hereditary matrix polynomial. If $\deg(h) = d$, then*

eq:posss1 (3.2)

$$h(X) \succeq 0 \text{ for all } X \in \mathcal{D}_L \quad \iff \quad h \in M^{\nu,\text{her}}_{d+1,d}(L).$$

If, in addition, the set $\mathcal{D}_L$ is bounded, then the right-hand side of this equivalence is further equivalent to

eq:posss2 (3.3)

$$h \in \Big\{ \sum_j^{\text{finite}} \psi_j^* L \psi_j : \psi_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_{d+1} \Big\} =: \mathring{M}^{\nu,\text{her}}_{d+1}(L).$$

3.1. **Proof of Theorem 3.1.** The proof consists of two main steps: a separation argument together with a partial Gelfand-Naimark-Segal (GNS) construction.

3.1.1. *Step 1: Towards a separation argument.* Let $\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}$ denote the vector space of all hereditary $\nu\times\nu$ matrix polynomials.

Lemma 3.2. *$M^{\nu,\mathrm{her}}_{\alpha,\beta}(L)$ is a closed convex cone in $\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}_{\max\{2\alpha,2\beta+1\}}$.*

Proof. The proof is the same as for the corresponding free non-hereditary setting [HKM12a, Proposition 3.1]; its main ingredient is Carathéodory's theorem on convex hulls [Bar02, Theorem I.2.3]. ■

3.1.2. *Step 2: A GNS construction.* Proposition 3.3 below, embodies the well known connection, through the Gelfand-Naimark-Segal (GNS) construction, between operators and positive linear functionals. It is adapted here to hereditary matrix polynomials.

Given a Hilbert space $\mathscr{H}$ and a positive integer ν, let $\mathscr{H}^{\oplus\nu}$ denote the orthogonal direct sum of $\mathscr{H}$ with itself ν times. Let L be a monic $\ell\times\ell$ linear pencil and abbreviate

$$M^{\nu}_{k+1} := M^{\nu,\mathrm{her}}_{k+1,k}(L).$$

Proposition 3.3. *If $\lambda:\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}_{2k+2}\to\mathbb{C}$ is a symmetric linear functional that is nonnegative on $\Sigma^{\nu,\mathrm{her}}_{k+1}$ and positive on $\Sigma^{\nu,\mathrm{her}}_{k}\setminus\{0\}$, then there exists a tuple $X=(X_1,\dots,X_g)$ of operators on a Hilbert space $\mathscr{H}$ of dimension at most $\nu\sigma_{\#}(k)=\nu\dim\mathbb{R}\langle x\rangle_k$ and a vector $\gamma\in\mathscr{H}^{\oplus\nu}$ such that*

$$\lambda(f)=\langle f(X)\gamma,\gamma\rangle \tag{3.4}$$

for all $f\in\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}_k$, where $\langle\_,\_\rangle$ is the inner product on $\mathscr{H}$. Further, if λ is nonnegative on M^{ν}_{k+1}, then $X\in\mathcal{D}_L$.

Conversely, if $X=(X_1,\dots,X_g)$ is a tuple of operators on a Hilbert space $\mathscr{H}$ of dimension N, γ is a vector in $\mathscr{H}^{\oplus\nu}$, and k is a positive integer, then the linear functional $\lambda:\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}_{2k+2}\to\mathbb{C}$ defined by

$$\lambda(f)=\langle f(X)\gamma,\gamma\rangle$$

is nonnegative on Σ^{ν}_{k+1}. Further, if $X\in\mathcal{D}_L$, then λ is nonnegative also on M^{ν}_{k+1}.

Proof. First suppose that $\lambda:\mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle^{\mathrm{her}}_{2k+2}\to\mathbb{C}$ is nonnegative on $\Sigma^{\nu,\mathrm{her}}_{k+1}$ and positive on $\Sigma^{\nu,\mathrm{her}}_{k}\setminus\{0\}$. Consider the symmetric bilinear form, defined on the vector space $K=\mathbb{C}^{\nu\times 1}\langle x\rangle_{k+1}$ (row vectors of length ν whose entries are analytic polynomials of degree at most $k+1$) by,

$$\langle f,h\rangle=\lambda(h^*f). \tag{3.5}$$

From the hypotheses, this form is positive semidefinite.

A standard use of Cauchy-Schwarz inequality shows that the set of null vectors

$$\mathcal{N}:=\{f\in K:\langle f,f\rangle=0\}$$

is a vector subspace of K. Whence one can endow the quotient $\tilde{\mathscr{H}}:=K/\mathcal{N}$ with the induced positive definite bilinear form making it a Hilbert space. Further, because the form (3.5) is positive definite on the subspace $\mathscr{H}=\mathbb{C}^{\nu\times1}\langle x\rangle_k$, each equivalence class in that set has a unique representative which is a ν-row of analytic polynomials of degree at most k. Hence we can consider $\mathscr{H}$ as a subspace of $\tilde{\mathscr{H}}$ with dimension $\nu\sigma_{\#}(k)$.

Each x_j determines a multiplication operator on $\mathscr{H}$. For $f=\begin{pmatrix}f_1 & \cdots & f_\nu\end{pmatrix}\in\mathscr{H}$, let

$$x_jf=\begin{pmatrix}x_jf_1 & \cdots & x_jf_\nu\end{pmatrix}\in\tilde{\mathscr{H}}$$

and define $X_j:\mathscr{H}\to\mathscr{H}$ by

$$X_jf=Px_jf,\quad f\in\mathscr{H},\ 1\le j\le g,$$

where P is the orthogonal projection from $\tilde{\mathscr{H}}$ onto $\mathscr{H}$ (which is only needed on the degree $k+1$ part of $x_j f$). From the positive definiteness of the bilinear form (3.5) on $\mathscr{H}$, one easily sees that each X_j is well defined.

Let $\gamma \in \mathscr{H}^{\oplus\nu}$ denote the vector whose j-th entry, γ_j has the empty word (the monomial 1) in the j-th entry and zeros elsewhere. Finally, given words $v_{s,t} \in \langle x\rangle_k$ and $w_{s,t} \in \langle x\rangle_k$ for $1 \le s,t \le \nu$, choose $f \in \mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle_k^{\text{her}}$ to have (s,t)-entry $w_{s,t}^* v_{s,t}$. In particular, with $e_1,\dots,e_\nu$ denoting the standard orthonormal basis for $\mathbb{R}^\nu$,

$$f = \sum_{s,t=1}^{\nu} w_{s,t}^* v_{s,t} e_s e_t^*.$$

Thus,

$$\begin{aligned}\langle f(X)\gamma,\gamma\rangle &= \sum \langle f_{s,t}(X)\gamma_t,\gamma_s\rangle = \sum\langle w_{s,t}^*(X)v_{s,t}(X)\gamma_t,\gamma_s\rangle = \sum\langle v_{s,t}(X)\gamma_t, w_{s,t}(X)\gamma_s\rangle \\ &= \sum\langle v_{s,t}e_t^*, w_{s,t}e_s^*\rangle = \sum \lambda(w_{s,t}^* v_{s,t} e_s e_t^*) = \lambda\big(\sum (w_{s,t}^* v_{s,t} e_s e_t^*)\big) = \lambda(f).\end{aligned}$$

Since any $f \in \mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle_k^{\text{her}}$ can be written as a linear combination of words of the form w^*v with $v,w \in \langle x\rangle_k$ as was done above, equation (3.4) is established.

To prove the further statement, suppose λ is nonnegative on M_{k+1}^{ν}. Write $L = I + \Lambda + \Lambda^*$, where Λ is the homogeneous linear analytic part of L. Given

$$\psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_\ell\end{pmatrix} \in \mathscr{H}^{\oplus\ell},$$

note that

$$\begin{aligned}\langle L(X)\psi,\psi\rangle &= \langle (I+\Lambda(X)+\Lambda(X)^*)\psi,\psi\rangle = \langle (I+\Lambda(X))\psi,\psi\rangle + \langle\psi,\Lambda(X)\psi\rangle \\ &= \langle \psi + \sum A_j P x_j\psi,\psi\rangle + \langle\psi,\sum A_j P x_j \psi\rangle = \langle\psi+\sum A_j x_j\psi,\psi\rangle + \langle\psi,\sum A_j x_j\psi\rangle \\ &= \langle (I+\Lambda(x))\psi,\psi\rangle + \langle\psi,\Lambda(x)\psi\rangle = \lambda\big(\psi^*(I+\Lambda_A(x))\psi\big) + \lambda(\psi^*\Lambda(x)^*\psi) \\ &= \lambda\big(\psi^*(I+\Lambda(x)+\Lambda(x)^*)\psi\big) = \lambda(\psi^* L\psi) \ge 0.\end{aligned}$$

Hence, $L(X) \succeq 0$.

The proof of the converse is routine and is not used in the sequel. ∎

3.1.3. *Step 3: Conclusion.* Let us first prove (3.2). The reverse implication is obvious. To prove the forward implication assume $h \notin M_{d+1,d}^{\nu,\text{her}}(L)$. By Lemma 3.2 then there is a linear functional $\lambda : \mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle_{2d+2}^{\text{her}} \to \mathbb{C}$ satisfying

$$\lambda(h) < 0, \qquad \lambda(M_{d+1,d}^{\nu,\text{her}}) \subseteq \mathbb{R}_{\ge 0}.$$

By adding a small multiple of a linear functional that is strictly positive on $\Sigma_{d+1}^{\nu,\text{her}} \setminus \{0\}$ (see e.g. [HKM12a, Lemma 3.2] for its existence), we may assume moreover that

$$\lambda\big(\Sigma_{d+1}^{\nu,\text{her}}\setminus\{0\}\big) \subseteq \mathbb{R}_{>0}.$$

Now Proposition 3.3 applies: there exists a tuple $X = (X_1,\dots,X_g) \in \mathcal{D}_L$ of $\nu\sigma_\#(d)\times\nu\sigma_\#(d)$ matrices, and a vector γ such that (3.4) holds for all $f \in \mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle_d^{\text{her}}$. Hence

$$0 > \lambda(h) = \langle h(X)\gamma,\gamma\rangle.$$

Thus $h(X) \not\succeq 0$.

Finally, (3.3) follows from the Hahn-Banach theorem. Namely, if $\mathcal{D}_L$ is bounded, then $I = \sum_j V_j^* L(X) V_j$ for some V_j, see e.g. [HKM12a, Section 4.1]. ∎

4. Positivity Certificates for Analytic Mappings

This section chronicles consequences of a Positivstellensatz certificate like that of equation (1.5). Proposition 4.3 is the principal result.

Given a g-tuple A of operators on a Hilbert space $\mathcal{E}$, a positive integer e and a formal powers series $W(x) = \sum W_\alpha \alpha$ with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$, and $G = \sum G_\alpha \alpha$ with coefficients $G_\alpha : \mathbb{C}^e \to \mathbb{C}^e$ and $G(0) = G_\varnothing = 0$, the identity

$$I_e + G(x) + G(x)^* = W(x)^* L_A(x) W(x) \tag{4.1}$$

is interpreted as holding in the ring of (matrices over) formal power series in x, x^*. Equivalently, for words α, β and $1 \le j \le g$,

$$\begin{aligned} W_\beta^* A_k^* W_{x_j\alpha} + W_{x_k\beta}^* A_j W_\alpha + W_{x_k\beta}^* W_{x_j\alpha} &= 0 \\ W_\varnothing^*(A_k W_\alpha + W_{x_k\alpha}) &= G_{x_k\alpha} \\ W_\varnothing^* W_\varnothing &= I. \end{aligned} \tag{4.2}$$

Proposition 4.1. *Suppose e is a positive integer, G is an $e\times e$ matrix-valued free analytic function, $\mathcal{E}$ is a (not necessarily finite-dimensional) Hilbert space, W is a formal power series with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$ and A is a g-tuple of operators on $\mathcal{E}$.*

The following are equivalent.

(i) *Equation (4.1) holds in the ring of formal power series. Equivalently, the equations (4.2) hold.*

(ii) *For all nilpotent $X \in M(\mathbb{C})^g$,*

$$I + G(X) + G(X)^* = W(X)^* L_A(X) W(X). \tag{4.3}$$

In addition, if W and G have positive formal radii of convergence at least $\tau > 0$, then items (i) and (ii) are equivalent to

(iii) *Equation (4.3) holds for all $X \in \Delta_\tau$.*

Before beginning the proof of Proposition 4.1, we first state and prove a routine lemma. Fix N a positive integer. Consider the truncated Fock Hilbert space $\mathscr{F}_N$ with orthonormal basis $\{\alpha \in \langle x\rangle : |\alpha| \le N\}$. Let S (we suppress the dependence on N) denote the tuple of shifts determined by $S_j w = x_j w$ if the length of the word w is strictly less than N and $S_j w = 0$ if the length of the word w is N. In particular, S is nilpotent of order N.

Lemma 4.2. *Given Hilbert spaces H and K and operators $F_{\alpha,\beta} : H \to K$ parameterized over words α, β, of length at most N, if*

$$\sum_{|\alpha|,|\beta|\le N} F_{\alpha,\beta} \otimes S^\beta S^{*\alpha} = 0,$$

then $F_{\alpha,\beta} = 0$ for all α, β.

Proof. We argue by induction on the length of α. In the case $\alpha = \varnothing$, evaluating at vectors of the form $h \otimes \varnothing$ with $h \in H$ gives

$$0 = \sum_{|\alpha|,|\beta|\le N} F_{\alpha,\beta} h \otimes S^\beta S^{*\alpha} \varnothing = \sum_{|\beta|\le N} F_{\varnothing,\beta} h \otimes \beta.$$

Hence $F_{\varnothing,\beta} = 0$ for all $|\beta| \le N$. Now suppose $0 \le n < N$ and $F_{\alpha,\beta} = 0$ for all $|\alpha| \le n$ and $|\beta| \le N$. Let a word γ with length $n+1$ be given. Evaluating at vectors of the form $h \otimes \gamma$ and using the induction hypothesis gives,

$$\begin{aligned} 0 &= \sum_{|\alpha|,|\beta|\le N} F_{\alpha,\beta} h \otimes S^{\beta} S^{*\alpha}\gamma \\ &= \sum_{n<|\alpha|\le N, |\beta|\le N} F_{\alpha,\beta} h \otimes S^{\beta} S^{*\alpha}\gamma \\ &= \sum_{|\beta|\le N} F_{\gamma,\beta} h \otimes \beta \end{aligned}$$

Hence $F_{\gamma,\beta} = 0$ for all $|\beta| \le N$. ∎

Proof of Proposition 4.1. Suppose item (ii) holds. Thus for all nilpotent tuples X,

$$I + G(X) + G(X)^* - W(X)^* L_A(X) W(X) = 0.$$

In this case Lemma 4.2 implies the identities of equation (4.2) hold. Hence item (ii) implies item (i). That item (i) implies (ii) is evident.

Under the added hypotheses on the radii of convergence, item (iii) implies (ii). It remains to prove the converse. Accordingly, suppose $X \in \Delta_\tau$. Let R and L denote the values of the right and left hand side of (4.3) evaluated at X respectively and let $G^{(N)}$ and $W^{(N)}$ denote the N-th partial sums of the respective series $G(X)$ and $W(X)$. Given $\epsilon > 0$ there is an N such that

$$\begin{aligned} \|I + G^{(N)}(X) + G^{(N)}(X)^* - R\| &< \epsilon, \\ \|W^{(N)}(X)^* L_A(X) W^{(N)}(X) - L\| &< \epsilon. \end{aligned}$$

Use (4.2) to compute

eq:psCalc2

$$\begin{aligned} &I + G^{(N)}(X) + G^{(N)}(X)^* - W^{(N)}(X)^* L_A(X) W^{(N)}(X) \\ &\quad = -\sum_{k=1}^{g}\sum_{|\beta|=N}\sum_{|\alpha|\le N} W_\beta^* A_k^* W_\alpha \otimes X^{*\beta} X_k^* X^\alpha - \sum_{j=1}^{g}\sum_{|\beta|\le N}\sum_{|\alpha|=N} W_\beta^* A_j W_\alpha \otimes X^{*\beta} X_j X^\alpha \\ &= -\Big(\sum_{|\beta|=N} W_\beta \otimes X^\beta\Big)^* \Lambda_A(X)^* \Big(\sum_{|\alpha|\le N} W_\alpha \otimes X^\alpha\Big) - \Big(\sum_{|\beta|\le N} W_\beta \otimes X^\beta\Big)^* \Lambda_A(X) \Big(\sum_{|\alpha|=N} W_\alpha \otimes X^\alpha\Big). \end{aligned} \tag{4.4}$$

The norm of each of the two summands in the last line of (4.4) is at most

eq:psCalc

$$\sum_{k=1}^{g} \|A_k \otimes X_k\| \Big(\sum_{|\beta|=N} \|W_\beta\|\, \|X^\beta\|\Big)\Big(\sum_{|\alpha|\le N} \|W_\alpha\|\, \|X^\alpha\|\Big). \tag{4.5}$$

By hypothesis the second factor in (4.5) tends to 0 with N and the first and third factor are uniformly bounded on Δ_τ. Thus the left hand side of (4.4) tends to zero with N and the proof is complete. ∎

With the notations already introduced, let

$$\mathrm{rg}(A, W) = \{A_j W_\alpha h : 1 \le j \le g,\ \alpha,\ h \in \mathbb{C}^e\}.$$

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Proposition 4.3. *Suppose e is a positive integer and $\mathcal{E}$ is a separable Hilbert space. If*

(a) *A is a g-tuple of operators on $\mathcal{E}$;*
(b) *W is a formal power series with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$; and*
(c) *G is a formal power series with $G(0) = 0$ and $e \times e$ matrix coefficients G_α such that equation (4.1) holds,*

then $\mathscr{W} = W_\varnothing : \mathbb{C}^e \to \mathcal{E}$ *is an isometry and there exists a contraction* $C : \mathcal{E} \to \mathcal{E}$ *that is isometric on* $\operatorname{rg}(A, W)$ *such that*

$$G(x) = \mathscr{W}^* C\big(\sum_{j=1}^{g} A_j x_j\big) W(x),$$

where, letting $R = (C - I_{\mathcal{E}})A$*, the function* W *is given as in equation* (1.7)

$$W(x) = \big(I_{\mathcal{E}} - \sum_{j=1}^{g} R_j x_j\big)^{-1} \mathscr{W}.$$

Moreover, if $\mathcal{E}$ *is finite dimensional, then* C *can be chosen unitary. In any case, choosing an auxiliary separable infinite dimensional Hilbert space* $\mathcal{E}'$ *and a tuple* A' *acting on* $\mathcal{E}'$ *and letting*

$$\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{E}', \quad \tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \quad \tilde{\mathscr{W}} = \begin{pmatrix} \mathscr{W} \\ 0 \end{pmatrix} : \mathbb{C}^e \to \tilde{\mathcal{E}}, \quad \tilde{W} = \begin{pmatrix} W \\ 0 \end{pmatrix},$$

we have

$$I_e + G(x) + G(x)^* = \tilde{W}(x)^* L_{\tilde{A}}(x) \tilde{W}(x),$$

and there is a unitary mapping $\tilde{C} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ *such that, letting* $\tilde{R} = (\tilde{C} - I_{\tilde{\mathcal{E}}})\tilde{A}$,

$$\tilde{W}(x) = \big(I_{\mathcal{E}} - \sum_{j=1}^{g} \tilde{R}_j x_j\big)^{-1} \tilde{\mathscr{W}}, \quad G(x) = \tilde{\mathscr{W}}^* \tilde{C}\big(\sum_{j=1}^{g} \tilde{A}_j x_j\big) \tilde{W}(x).$$

In particular,

(1) *the following recursion formula holds,*

$$W_{x_j\alpha} = (C - I_{\mathcal{E}}) A_j W_\alpha, \quad \tilde{W}_{x_j\alpha} = (\tilde{C} - I_{\tilde{\mathcal{E}}}) \tilde{A}_j \tilde{W}_\alpha;$$

(2) *the* x_j *coefficient of* $G(x)$ *is*

$$G_{x_j} = \mathscr{W}^* C A_j \mathscr{W} = \tilde{\mathscr{W}}^* \tilde{C} \tilde{A}_j \tilde{\mathscr{W}};$$

(3) *more generally,* $G_{x_j\alpha}$*, the* $x_j\alpha$ *coefficient of* G*, is*

$$G_{x_j\alpha} = \mathscr{W}^* C A_j R^\alpha \mathscr{W} = \tilde{\mathscr{W}}^* \tilde{C} \tilde{A}_j \tilde{R}^\alpha \tilde{\mathscr{W}}. \tag{4.6}$$

Conversely, given a tuple A *on a Hilbert space* $\mathcal{E}$ *a contraction* $C : \mathcal{E} \to \mathcal{E}$ *that is isometric on the range of* A *and an isometry* $\mathscr{W} : \mathbb{C}^d \to \mathcal{E}$*, defining* $R = (C - I)A$*, and* W *and* G *as in equations* (1.7) *and* (1.6)*, the identities of* (4.2) *hold.*

Proof. Completing the square in the first equation of (4.2) gives,

$$(A_k W_\beta + W_{x_k\beta})^* (A_j W_\alpha + W_{x_j\alpha}) = W_\beta^* A_k^* A_j W_\alpha. \tag{4.7}$$

Fix, for the moment, a positive integer N. Recall $W_\alpha : \mathbb{C}^e \to \mathcal{E}$ and $A_j : \mathcal{E} \to \mathcal{E}$. Let $\mathcal{K}_N = \oplus_{|\alpha| \le N} \mathbb{C}^e$, the Hilbert space direct sum of $\mathbb{C}^e$ over the set of words of length at most N in the variables $x = (x_1, \dots, x_g)$. Finally, let $\mathcal{L}_N := \oplus_{j=1}^{g} \mathcal{K}_N$ and note that $h \in \mathcal{L}_N$ takes the form $h = \oplus_j \oplus_{|\alpha| \le N} h_{j,\alpha} = \oplus h_{j,\alpha}$. Let

$$\mathcal{E}_N = \Big\{ \sum_{j, |\alpha| \le N} A_j W_\alpha h_{j,\alpha} : h_{j,\alpha} \in \mathbb{C}^e \text{ for } 1 \le j \le g, \ \ |\alpha| \le N \Big\} \subseteq \operatorname{rg}(A) \subseteq \mathcal{E}.$$

The subspaces $\mathcal{E}_N$ are nested increasing and $\operatorname{rg}(A, W) = \cup_N \mathcal{E}_N$.

Define $X_N, Y_N : \mathcal{L}_N \to \mathcal{E}$,

$$X_N(\oplus_{j=1}^{g} \oplus_{|\alpha|\le N} h_{j,\alpha}) = \sum_{j,|\alpha|\le N} (A_j W_\alpha + W_{x_j\alpha}) h_{j,\alpha}$$

$$Y_N(\oplus_{j=1}^{g} \oplus_{|\alpha|\le N} h_{j,\alpha}) = \sum_{j,|\alpha|\le N} (A_j W_\alpha) h_{j,\alpha}.$$

Note that the range of Y_N is $\mathcal{E}_N$. Equation (4.7) implies that

$$X_N^* X_N = Y_N^* Y_N : \mathcal{L}_N \to \mathcal{L}_N. \tag{4.8}$$

In particular, if $Y_N h = 0$, then $X_N h = 0$. Hence, $Y_N h \mapsto X_N h$ is a well-defined map $C_N : \mathcal{E}_N \to \mathcal{E}$. Further, equation (4.8) implies that C_N is an isometry. Since $\mathcal{E}_N$ is finite-dimensional, C_N can be extended to a unitary $C_N : \mathcal{E} \to \mathcal{E}$. Thus, there is a unitary mapping $C_N : \mathcal{E} \to \mathcal{E}$ such that

$$X_N = C_N Y_N.$$

Moreover, for $N \ge M$,

$$X_M = C_N Y_M.$$

Since $(C_N)_N$ is a sequence of unitaries on $\mathcal{E}$, a subsequence $(C_{N_j})_j$ converges in the weak operator topology (WOT) to a contraction operator C. (In the case $\mathcal{E}$ is finite dimensional, C is unitary.) Fix M. For $N_j \ge M$, $C_{N_j} Y_M = X_M$. Hence, for a vector $h \in \mathcal{L}_M$ and a vector $e \in \mathcal{E}$,

$$\langle C Y_M h, e\rangle = \lim_j \langle C_{N_j} Y_M h, e\rangle = \langle X_M h, e\rangle.$$

Thus, $CY_M = X_M$ for all M. In particular, C is an isometry on $\operatorname{rg}(A, W)$ and, combined with the second and third identities in equation (4.2), for each j, α

$$\begin{aligned} A_j W_\alpha + W_{x_j\alpha} &= C A_j W_\alpha \\ W_\varnothing^*(A_k W_\alpha + W_{x_k\alpha}) &= G_{x_k\alpha} \\ W_\varnothing^* W_\varnothing &= I. \end{aligned} \tag{4.9}$$

The first identity in equation (4.9) is equivalent to

$$W_{x_j\alpha} = (C - I) A_j W_\alpha. \tag{4.10}$$

Now suppose $C : \mathcal{E} \to \mathcal{E}$ is a contraction that is isometric on $\operatorname{rg}(A, W)$ and the identities of equation (4.9) hold. In particular, equation (4.10) also holds. For notational ease, if not consistency, let $\mathscr{W} = W_\varnothing$ and $W_\ell = W_{x_\ell}$. In particular, from $\mathscr{W}$ is an isometry. Moreover, it follows from equation (4.10) with $\alpha = \varnothing$, that $W_j = (C - I) A_j \mathscr{W}$ for each j. Thus $W_j = R_j \mathscr{W}$, where $R_j = (C - I) A_j$.

For each k an application of equation (4.10) with $\alpha = x_j$ yields

$$W_{x_k x_j} = (C - I) A_k W_j = R_k R_j \mathscr{W},$$

for $j, k = 1, \dots g$. Induction on the length of words gives,

$$W_\alpha = R^\alpha \mathscr{W}$$

where $R = (R_1, \dots, R_g)$. Hence,

$$W(x) = (I - \sum R_j x_j)^{-1} \mathscr{W}. \tag{4.11}$$

Now using the second and third equations of (4.2) together with (4.11) gives

$$\mathscr{W}^*(I + \sum A_k x_k)(I - \sum R_\ell x_\ell)^{-1} \mathscr{W} = I + G(x).$$

Hence,

$$\begin{aligned}G(x) &= \mathscr{W}^*\Big[\big(I+\sum A_k x_k\big)\big(I-\sum_{\ell=1}^{g} R_\ell x_\ell\big)^{-1} - I\Big]\mathscr{W}\\ &= \mathscr{W}^*\big(\sum_{k=1}^{g}(A_k+R_k)x_k\big)\,\big(I-\sum_{\ell=1}^{g} R_\ell x_\ell\big)^{-1}\mathscr{W}\\ &= \mathscr{W}^* C\big(\sum A_k x_k\big)\,\big(I-(C-I)\sum A_\ell x_\ell\big)^{-1}\mathscr{W}\\ &= \mathscr{W}^* C\big(\sum A_k x_k\big)\,\big(I-\sum R_\ell x_\ell\big)^{-1}\mathscr{W}.\end{aligned}$$

At this point we have proved that if W and G solve equation (4.1), then there exists a contraction C that is isometric on $\mathcal{E}$ such that W and G have the claimed form. Further, in the case $\mathcal{E}$ is finite dimensional, C can be chosen unitary.

Now let $\tilde{\mathcal{E}}$, $\tilde{A}$, $\tilde{\mathscr{W}}$ and $\tilde{W}$ be given as in the statement of the proposition. In particular equation (4.6) holds. Further, Let $\mathrm{rg}(\tilde{A},\tilde{W}) = \mathrm{rg}(A,W)\oplus(0)\subseteq\tilde{\mathcal{E}}$. The orthogonal complements of $\mathrm{rg}(\tilde{A},\tilde{W})$ and $C(\mathrm{rg}(\tilde{A},\tilde{W}))$ in $\tilde{\mathcal{E}}$ are infinite dimensional separable Hilbert spaces. Hence there exists a unitary operator $\tilde{C}:\tilde{\mathcal{E}}\to\tilde{\mathcal{E}}$ such that $\tilde{C}(h\oplus 0) = Ch\oplus 0$ for $h\in\mathrm{rg}(A,0)$. Thus $\tilde{C}$ is unitary and $\tilde{C}$, $\tilde{W}$ and $\tilde{A}$ together satisfy the analog of equation (4.9) and hence the conclusions of the proposition.

To prove the converse, given a tuple $A=(A_1,\dots,A_g)$ of operators on the Hilbert space $\mathcal{E}$, and a contraction $C:\mathcal{E}\to\mathcal{E}$ that is isometric on the range of A and an isometry $\mathscr{W}:\mathbb{C}^d\to\mathcal{E}$, let $R=(C-I)A$ and $W(x) = (I-\Lambda_R(x))^{-1}\mathscr{W}$ and define G by equation (1.6),

$$G(x) = \mathscr{W}^* C\Lambda_A(x)(I-\Lambda_R(x))^{-1}\mathscr{W}.$$

By construction, $W_\alpha = R^\alpha\mathscr{W}$. Moreover, for each word α and $1\le j\le g$,

$$(C-I)A_j W_\alpha = R_j R^\alpha \mathscr{W} = W_{x_j\alpha}.$$

Hence,

$$CA_jW_\alpha = A_jW_\alpha + W_{x_j\alpha}.$$

Since C is isometric on the range of A, given $1\le k\le g$ and a word β,

$$W_\beta^* A_k^* A_j W_\alpha = W_\beta^* A_k^* A_j W_\alpha + W_\beta^* A_k^* W_\alpha + W_{x_k\beta}^* A_j W_\alpha + W_{x_\beta}^* W_{x_j\alpha}.$$

Thus the first of the identities of equation (4.2) hold. The third identity holds since $\mathscr{W} = W_\varnothing$ is an isometry and, as $\mathscr{W} = W_\varnothing$, the second identity holds by the choice of G and the proof is complete. ■

Remark 4.4. We note that the proof of the converse of Proposition 4.3 would, under some convergence assumptions, follow from the following formal calculation starting from the formula for G of equation (1.6). Using $R_j=(C-I)A_j$ gives

$$I-\Lambda_R(x) = I+\Lambda_A(x) - C\Lambda_A(x).$$

Let

$$H(x) = C\Lambda_A(x)(I-\Lambda_R(x))^{-1} = \Lambda_{CA}(x)(I-\Lambda_R(x))^{-1}$$

and note that

$$\mathscr{W}^*\big(I+H(x)+H^*(x)\big)\mathscr{W} = I+G(x)+G^*(x).$$

Now,

$$\begin{aligned}
&I + H(x) + H(x)^* \\
&= (I-\Lambda_R(x))^{-*}[(I-\Lambda_R(x))^*(I-\Lambda_R(x))+(I-\Lambda_R(x))^*C\Lambda_A(x)+\Lambda_A(x)^*C^*(I-\Lambda_R(x))](I-\Lambda_R(x))^{-1} \\
&= (I-\Lambda_R(x))^{-*}[(I-\Lambda_R(x)+C\Lambda_A(x))^*(I-\Lambda_R(x)+C\Lambda_A(x))-\Lambda_A(x)^*C^*C\Lambda_A(x)](I-\Lambda_R(x))^{-1} \\
&= (I-\Lambda_R(x))^{-*}[\Psi(x)^*\Psi(x)-\Lambda_A(x)^*\Lambda_A(x)](I-\Lambda_R(x))^{-1} \\
&= (I-\Lambda_R(x))^{-*}[I+\Lambda_A(x)^*+\Lambda_A(x)](I-\Lambda_R(x))^{-1},
\end{aligned}$$

from which it follows that

$$\mathscr{W}^* L_R(x)^{-*}\big(I+\Lambda_A(x)+\Lambda_A^*(x)\big)L_R(x)^{-1}\mathscr{W} = I + G(x) + G^*(x). \qquad \lozenge$$

4.1. Polynomials correspond to nilpotent R.

Corollary 4.5. *Suppose, in the context of Proposition 4.3, that $W_\varnothing$ is $e \times D$. If*

(a) *G is a polynomial;*
(b) $\operatorname{span}\{R^\omega W_\varnothing h : h \in \mathbb{C}^d, \omega \in \langle x\rangle\} = \mathbb{C}^D$*; and*
(c) $\bigcap\{\ker(W_\varnothing^* C A_j R^w) : w \in \langle x\rangle,\ 1 \le j \le g\} = (0)$,

then the tuple R is nilpotent. In particular, if $D = d$ and G is a polynomial, then R is nilpotent.

In the language of systems theory, the hypotheses of items (b) and (c) are that the system $(R, W_\varnothing, \{W_\varnothing^* C A_j\})$ is **controllable** and **observable** respectively.

Proof. Since $W_\varnothing^* C\Lambda_A(x)(I - \sum R_j x_j)^{-1} W_\varnothing$ is a polynomial, there exists a positive integer N such that

$$W_\varnothing^* C A_i R^\omega W_\varnothing = 0$$

for all words ω for which $|\omega| \ge N$. Hence, if $|\xi| \ge N$, then for words α, β,

$$0 = W_\varnothing^* C A_i R^\omega W_\varnothing = W_\varnothing^* C A_i R^\alpha R^\xi R^\beta W_\varnothing$$

Conditions (b) and (c) now imply that $R^\xi = 0$. ■

Remark 4.6. In any case, W is a polynomial if and only if $R^\alpha W_\varnothing = 0$ for $|\alpha|$ large enough. Of course if $W_\varnothing$ is square, then it is invertible. Thus, in this case, the R_j are jointly nilpotent if and only if W is a polynomial. ♢

5. Extending the Hereditary Positivstellensatz to Analytic Functions

In this section we prove Theorem 1.9, extending the Hereditary Convex Positivstellensatz (Theorem 3.1) to analytic and rational maps between free spectrahedra. The proof combines Theorems 3.1 and 1.8 and Proposition 4.3.

Lemma 5.1. *Suppose e is a positive integer, $G : \mathcal{G} \to M_e(M(\mathbb{C}))$ is analytic on a pseudoconvex set $\mathcal{G}$ containing $\mathcal{D}_A$ and $I + G + G^*$ is nonnegative on $\mathcal{D}_A$ and $G(0) = 0$. If (G_ℓ) is a sequence of $e \times e$ matrix polynomials converging uniformly to G on $\mathcal{D}_A$, then there exists a sequence of polynomials (Q_k) converging uniformly to G on $\mathcal{D}_A$ such that $Q_k(0) = 0$ and $I + Q_k + Q_k^*$ is nonnegative on $\mathcal{D}_A$.*

Proof. Note that $(G_\ell(0))_\ell$ converges to 0 since $0 \in \mathcal{D}_A$ and $G(0) = 0$. Let $H_\ell = G_\ell - G_\ell(0)$. In particular, H_ℓ converges uniformly to G on $\mathcal{D}_A$ and $H_\ell(0) = 0$. Choose a sequence $(t_k)_k$ such that $0 < t_k < 1$ and $\lim t_k = 1$. Note that, for $X \in \mathcal{D}_A$,

$$I + t_k(G(X) + G(X)^*) = (1-t_k)I + t_k(I + G(X) + G(X)^*) \succeq (1-t_k)I.$$

For each k there is an ℓ_k such that $H_{\ell_k}(X)$ is uniformly sufficiently close to G so that

$$I + t_k(H_{\ell_k}(X) + H_{\ell_k}(X)^*) \succeq I + t_k(G(X) + G(X)^*) - (1-t_k)I \succeq 0.$$

Hence, the sequence $(Q_k = t_k H_{\ell_k})$, converges uniformly to G on $\mathcal{D}_A$ and satisfies $Q_k(0) = 0$ and $I + Q_k + Q_k^*$ is nonnegative on $\mathcal{D}_A$. ■

5.1. **Proof of Theorem 1.9.** By Lemma 5.1 and Theorem 1.8 (using, in particular, the boundedness assumption on $\mathcal{D}_A$), without loss of generality there is a sequence $(G_\ell)_\ell$ of polynomials converging uniformly to G on $\mathcal{D}_A$ and such that $I + G_\ell + G_\ell^*$ is nonnegative on $\mathcal{D}_A$ and $G_\ell(0) = 0$. By Theorem 3.1 (again using the boundedness of $\mathcal{D}_A$), there is a separable infinite-dimensional Hilbert space H such that for each ℓ there exists a polynomial W_ℓ with coefficients $W_{\ell,\alpha} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that

$$I + G_\ell(x) + G_\ell(x)^* = {W_\ell}^*(x) L_{I_H \otimes A}(x) W_\ell(x). \tag{5.1}$$

Applying Proposition 4.3, there exists a contraction C_ℓ on $H \otimes \mathbb{C}^d$ and an isometry $\mathscr{W}_\ell : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that, with $R_\ell = (C_\ell - I)[I_H \otimes A]$,

$$W_\ell(x) = (I - \Lambda_{R_\ell}(x))^{-1} \mathscr{W}_\ell.$$

Moreover, from the identity $W_{\ell, x_j \alpha} = (C_\ell - I)[I_H \otimes A_j] W_{\ell,\alpha}$ of item (1) of Proposition 4.3,

$$\|W_{\ell, x_j\alpha}\| \le 2 \max\{\|A_1\|, \dots, \|A_g\|\} \, \|W_{\ell,\alpha}\|.$$

Thus, using the fact that $\mathscr{W}_\ell = W_{\ell,\varnothing}$ is an isometry and hence has norm one, $\|W_{\ell,\alpha}\|$ has a uniform bound depending only on the length of the word α (independent of ℓ).

Observe that for each α, the dimension of the range of $W_{\ell,\alpha}$ is at most e. Hence, for a fixed N, there is a constant D_N such that for each ℓ the dimension of the span of

$$H_{N,\ell} = \bigvee_{|\alpha| \le N} \operatorname{rg}(W_{\ell,\alpha})$$

is at most D_N. (Indeed one can take D_N to be de times the number of words of length at most N.) It follows that, given N, for each ℓ there exists a subspace $H_{N,\ell}$ of H of dimension D_N such that the ranges of $W_{\ell,\alpha}$ all lie in $H_{N,\ell} \otimes \mathbb{C}^d$. For technical reasons that will soon be apparent, choose a basis $\{e_1, e_2, \dots\}$ for H and inductively construct subspaces $\mathcal{H}_{N,\ell}$ of H of dimension $2D_N$ such that $\mathcal{H}_{N,\ell}$ contains both $H_{N,\ell}$ and $\operatorname{span}(\{e_1, \dots, e_{D_N}\})$ and such that $\mathcal{H}_{N,\ell} \subseteq \mathcal{H}_{N+1,\ell}$. In particular, $H = \oplus_{N=-1}^{\infty} (\mathcal{H}_{N+1,\ell} \ominus \mathcal{H}_{N,\ell})$, where $\mathcal{H}_{-1,\ell} = \{0\}$. Set $D_{-1} = 0$ and let $E_m = 2(D_m - D_{m-1})$. Letting $K_m = \mathbb{C}^{E_m}$ and K denote the Hilbert space $\oplus_{m=0}^{\infty} K_m$, it follows that for each ℓ there is a unitary mapping $\rho_\ell : H \to K$ such that $\rho_\ell(\mathcal{H}_{N,\ell}) = \oplus_{m=0}^{N} K_m$. We have,

$$W_\ell(x)^* (\rho_\ell \otimes I_d)^* [I_K \otimes L_A(x)] (\rho_\ell \otimes I_d) W_\ell(x) = W_\ell(x)^* [I_H \otimes L_A(x)] W_\ell(x).$$

Hence, we can replace $W_\ell(x)$ with $(\rho_\ell \otimes I) W_\ell(x)$ in (5.1) and thus, given a word α of length N, assume that $W_{\ell,\alpha}$ maps into $\oplus_{m=0}^{N} K_m$ independent of ℓ.

For a fixed word α, the set $\{W_{\ell,\alpha} : \ell\}$ maps into a common finite-dimensional Hilbert space and is, in norm, uniformly bounded. Hence, by passing to a subsequence, we can assume for each word α the sequence $W_{\ell,\alpha}$ converges to some W_α in norm. Let W denote the corresponding formal power series. We will argue that

$$I + G(x) + G(x)^* = W(x)^* L_{I_H \otimes A}(x) W(x)$$

in the sense explained as follows. Since $G_\ell(0) = 0$, it follows that $W_{\ell,\varnothing}^* W_{\ell,\varnothing} = I$ for each ℓ. Hence

$$W_\varnothing^* W_\varnothing = I. \tag{5.2}$$

Likewise, given α and j, for every ℓ,

$$W_{\ell,\varnothing}^* (I_H \otimes A_j) W_{\ell,\alpha} + W_{\ell, x_j\alpha} = (G_\ell)_{x_j\alpha},$$

the coefficient of the $x_j\alpha$ term of G_ℓ. From what has already been proved, the left hand side above converges to $(I_H\otimes A_j)W_\alpha + W_{x_j\alpha}$. Since G_ℓ converges uniformly to G on $\mathcal{D}_A$, the sequence $((G_\ell)_{x_j\alpha})$ converges to $G_{x_j\alpha}$, the $x_j\alpha$ coefficient of G. Thus,

$$W_\varnothing^*(I_H\otimes A_j)W_\alpha + W_{x_j\alpha} = G_{x_j\alpha}. \tag{5.3}$$

Moreover, also by construction, for each α,β and j,k,

$$W_{\ell,\beta}^*(I_H\otimes A_k)^* W_{\ell,x_j\alpha} + W_{\ell,x_k\beta}^*(I_H\otimes A_j)W_{\ell,\alpha} + W_{\ell,x_k\beta}^* W_{\ell,x_j\alpha} = 0.$$

Hence,

$$W_\beta^*(I_H\otimes A_k)^* W_{x_j\alpha} + W_{x_k\beta}^*(I_H\otimes A_j)W_\alpha + W_{x_k\beta}^* W_{x_j\alpha} = 0. \tag{5.4}$$

Equations (5.2), (5.3) and (5.4) together show the equations of (4.2) holds in the ring of formal power series. Thus, equation (5.1) holds. Hence Proposition 4.3 applies and there exists a contraction $C : H\otimes\mathbb{C}^d \to H\otimes\mathbb{C}^d$ that is isometric on $\operatorname{rg}(A,W)$ such that equations (1.6) and (1.7) hold.

To complete the proof, in the notation of Proposition 4.3, choose $\mathcal{E}' = H\otimes\mathbb{C}^d$ and make the identification $\tilde{\mathcal{E}} = \mathcal{E}\oplus\mathcal{E}' = (\mathbb{C}^2\otimes H)\otimes\mathbb{C}^d$. Likewise, let $A' = A$ and make the identification $\tilde{A} = I_{\mathbb{C}^2\otimes H}\otimes A$. The moreover portion of Proposition 4.3 produces a unitary $\tilde{C}$ and isometry $\tilde{\mathscr{W}}$ satisfying equations (1.5) and (1.6). Finally, from the formulas for G and $\tilde{W}$, there series have positive radii of convergence say both at least $\tau > 0$. Hence equation (1.5) holds for $X\in\Delta_\tau$ by Proposition 4.1.

6. Consequences of a One Term Positivstellensatz

In this section, we consider the consequences of a one term square Positivstellensatz. In particular, a one term Positivstellensatz produces a convexotonic map. Accordingly, suppose $p = (p^1,\dots,p^g)$ where each p^j is a free formal power series in $x = (x_1,\dots,x_g)$ such that $p(0) = 0$ and $p'(0) = I$. Further assume $A, B\in M_d(\mathbb{C}^g)$ and W is a formal power series with coefficients in $M_d(\mathbb{C})$ (square matrices) satisfying

$$L_B(p(x)) = W(x)^* L_A(x) W(x) \tag{6.1}$$

in the sense that the relations of equation (4.2) hold with $G(x) = \Lambda_B(p(x))$. Thus the sizes of A and B are the same and both L_A and L_B are pencils in g variables. As we will see, under this assumption (that W is square), equation (6.1) implies p is a convexotonic map and imposes rigid structure on the triple (p, A, B).

Proposition 4.3 produces $d\times d$ unitary matrices C and $\mathscr{W}$ such that, with $R = (C-I)A$,

$$W(x) = (I-\Lambda_R(x))^{-1}\mathscr{W}$$

$$\Lambda_B(p(x)) = \mathscr{W}^* C\Big(\sum_{j=1}^g A_j x_j\Big) W(x). \tag{6.2}$$

Before continuing, we pause to collect some consequences of these relations.

Lemma 6.1. *Let d, e and $g\le\tilde{g}$ denote positive integers. Suppose $p = (p^1,\dots,p^{\tilde{g}})$ and each p^t is a formal power series. Further suppose $p(0) = 0$ and $p(x) = (x,0) + h(x)$, where h consists of higher (two and larger) degree terms. Write*

$$p^t(x) = \sum_j\sum_\alpha p^t_{x_j\alpha} x_j\alpha.$$

If

(a) *$A\in M_d(\mathbb{C})^g$ and $B\in M_e(\mathbb{C})^{\tilde{g}}$;*
(b) *C is a $d\times d$ unitary matrix and $\mathscr{W} : \mathbb{C}^e\to\mathbb{C}^d$ is an isometry;*

(c) *with* $R = (C - I)A$, W *and* $\Lambda_B(p(x))$ *are as in equation* (6.2);

then

it:Bjagain (1) $B_j = \mathscr{W}^* C A_j \mathscr{W}$ *for* $1 \le j \le g$;

it:LambdaBp (2) $\Lambda_B(p(x)) = \sum_t B_t p^t(x) = \sum_{k=1}^{g} \sum_\alpha \mathscr{W}^* C A_k R^\alpha \mathscr{W} x_k \alpha$;

it:bill (3) *for each word* ω *and* $1 \le k \le g$,

$$\mathscr{W}^* C A_k R^\omega \mathscr{W} = \sum_{j=1}^{\tilde g} p^j_{x_k\omega} B_j;$$

t:billtgisg (4) *in the case* $\tilde g = g$, *for all words* ω *and* $1 \le k \le g$,

q:billgtisg
$$\mathscr{W}^* C A_k R^\omega \mathscr{W} = \sum_{j=1}^{g} p^j_{x_k\omega} B_j = \sum_{j=1}^{g} p^j_{x_k\omega} \mathscr{W}^* C A_j \mathscr{W}. \tag{6.3}$$

Proof. The result follow by comparing power series expansions terms and using the normalization hypotheses on p. ■

In the case $e = d$ and $\tilde g = g$, Lemma 6.1 implies $B = \mathscr{W}^* C A \mathscr{W}$, where $\mathscr{W} = W(0) = W_\varnothing$ is unitary. Further, since $\mathscr{W}$ is unitary, equation (6.3) of Lemma 6.1 gives $A_k(C - I)A_j$ is in the span of $A_1, \dots, A_g$ for all j, k; that is, for each $1 \le j \le g$ there is a $g \times g$ matrix Ξ_j (described explicitly in terms of the coefficients of p) such that for all $1 \le k \le g$,

eq:AZA
$$A_k(C - I)A_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s. \tag{6.4}$$

The structure inherent in equation (6.4) is analyzed in the next subsection.

ecretdevils

6.1. **Lurking algebras.**

prop:con **Proposition 6.2.** *If* $\Xi = (\Xi_1, \dots, \Xi_g)$ *is a convexotonic tuple, then* $\mathcal{X}$, *the span of* $\{\Xi_1, \dots, \Xi_g\}$ *is an algebra whose structure matrices are the* Ξ_j; *that is, for all* $1 \le k \le g$ *and words* α,

eq:con
$$\Xi_k \Xi^\alpha = \sum_s (\Xi^\alpha)_{k,s} \Xi_s. \tag{6.5}$$

Moreover, the associated convexotonic rational mappings of equation (1.3)

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \text{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1},$$

are inverses of one another.

Proof. Inducting on the length of α in equation (6.5) and using the relation of equation (1.2) at the third equality, gives

$$\begin{aligned}
\Xi_k \Xi^{\alpha x_\ell} &= \Xi_k \Xi^\alpha \Xi_\ell \\
&= \sum_t (\Xi^\alpha)_{k,t} \Xi_t \Xi_\ell \\
&= \sum_t \sum_s (\Xi^\alpha)_{k,t} (\Xi_\ell)_{t,s} \Xi_s \\
&= \sum_s (\Xi^{\alpha x_\ell})_{k,s} \Xi_s.
\end{aligned}$$

To prove the maps of equation (1.3) are inverses of one another, expand q^t in a series gives to obtain

$$q^t(x) = \sum_j x_j \big(I + \Lambda_\Xi(x)\big)^{-1}_{j.t} = \sum_{j,\alpha\in\langle x\rangle} (-1)^{|\alpha|}\, (\Xi^\alpha)_{j,t} x_j \alpha.$$

Using equation (6.5) at the fourth equality below obtains,

$$\begin{aligned} I - \Lambda_\Xi(q(x)) &= I - \sum_{t=1}^g \Xi_t q^t = I - \sum_t \Xi_t \sum_{j,\alpha} (-1)^{|\alpha|} (\Xi^\alpha)_{j,t} x_j \alpha \\ &= I - \sum_{j,\alpha} (-1)^{|\alpha|} \big(\sum_t (\Xi^\alpha)_{j,t} \Xi_t\big) x_j \alpha = I - \sum_{j,\alpha} (-1)^{|\alpha|} \Xi_j \Xi^\alpha x_j \alpha \\ &= I - \sum_{j,\alpha} (-1)^{|\alpha|} \Xi^{x_j\alpha} x_j \alpha = I + \sum_{|\beta|>0} (-1)^{|\beta|} \Xi^\beta \beta \\ &= \sum_\beta (-1)^{|\beta|} \Xi^\beta \beta = (I + \Lambda_\Xi(x))^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} p\circ q(x) &= q(x)\,[I - \Lambda_\Xi(q(x))]^{-1} = q(x)\big((I+\Lambda_\Xi(x))^{-1}\big)^{-1} \\ &= x\,(I+\Lambda_\Xi(x))^{-1}\,(I+\Lambda_\Xi(x)) = x \end{aligned}$$

and it follows that q is a right inverse for p. By symmetry, it is also a left inverse establishing item (4). ∎

An algebra $\mathscr{A}$ has **order of nilpotence** $N \in \mathbb{N}$ if the product of any N elements of $\mathscr{A}$ is 0 and N is the smallest natural number with this property. Proposition 6.6 below explains how convexotonic maps naturally arise from the algebra-module structure of equation (6.4).

lem:gtg **Proposition 6.3.** *Suppose R and E are g-tuples of matrices of the same size d and let $\mathscr{B}$ denote the span of $\{E_1,\dots,E_g\}$. If the set $\{E_1,\dots,E_g\}$ is linearly independent and $E_jR^\alpha \in \mathscr{B}$ for each $1\le j\le g$ and word α, then the g-tuple Ξ of $g\times g$ matrices uniquely determined by*

eq:defXi
$$E_kR_j = \sum_s (\Xi_j)_{k,s} E_s \tag{6.6}$$

is convexotonic and

eq:EjRa
$$E_kR^\alpha = \sum_{s=1}^{\tilde g} \Xi^\alpha_{k,s} E_s, \tag{6.7}$$

for each $1\le j\le g$ and word α, is convexotonic.

If Ξ is nilpotent of order ν, then $\nu \le g$. Moreover, if $R^\alpha = 0$, then $\Xi^\alpha = 0$ and hence if R is nilpotent of order μ, then $\nu\le\mu$.

Proof. By assumption, for each word α and $1\le t\le g$ the matrix E_tR^α has a unique representation of the form

eq:EtRa
$$E_tR^\alpha = \sum_{k=1}^g (\Xi_\alpha)_{t,k} E_k, \tag{6.8}$$

for some $g\times g$ matrix Ξ_α. We now argue that $\Xi_\alpha = \Xi^\alpha$ by induction on the length of the word α, the case of length 0 holding by the choice of Ξ. Accordingly, suppose $\Xi_\alpha = \Xi^\alpha$. Applying R_u on the right of equation (6.8) gives

$$\sum_{s=1}^{g}(\Xi_{\alpha x_u})_{t,s}E_s = E_tR^{\alpha}R_u = \sum_{k=1}^{g}(\Xi_\alpha)_{t,k}E_kR_u = \sum_{k=1}^{g}(\Xi_\alpha)_{t,k}\sum_{\ell=1}^{g}(\Xi_u)_{k,\ell}E_\ell = \sum_{\ell=1}^{g}(\Xi_\alpha\Xi_u)_{t,\ell}E_\ell.$$

By linear independence of $\{E_1,\dots,E_g\}$ and the induction hypothesis,

$$\Xi_{\alpha x_u} = \Xi_\alpha\Xi_u = \Xi^{\alpha}\Xi_u = \Xi^{\alpha\, x_u}.$$

To prove that the tuple Ξ is convexotonic, fix $1 \le k \le g$ and compute the product $E_kR_jR_\ell$ in two different ways. On the one hand, using equation (6.6) twice,

$$E_kR_jR_\ell = \sum_{t=1}^{g}(\Xi_\ell)_{j,t}E_kR_t = \sum_{t=1}^{g}(\Xi_\ell)_{j,t}\sum_{s=1}^{g}(\Xi_t)_{k,s}E_s = \sum_{s=1}^{g}\sum_{t=1}^{g}(\Xi_\ell)_{j,t}(\Xi_t)_{k,s}E_s.$$

On the other hand, using the already established equation (6.7) with $\alpha = x_j\, x_\ell$,

$$E_kR_jR_\ell = \sum_s(\Xi_j\Xi_\ell)_{k,s}E_s.$$

For a fixed k, the independence of the set $\{E_1,\dots,E_g\}$ implies

$$\sum_t(\Xi_\ell)_{j,t}(\Xi_t)_{k,s} = (\Xi_j\Xi_\ell)_{k,s}$$

for each $1 \le k, s \le g$ and thus,

$$\sum_t(\Xi_\ell)_{j,t}\Xi_t = \Xi_j\Xi_\ell.$$

Hence Ξ is a convexotonic tuple.

To prove the last part of the proposition suppose Ξ is nilpotent of order ν. Thus if $\Xi^\alpha = 0$, then $E_kR^\alpha = 0$ for each k. Let $\mathcal{R}^k$ denote the algebra generated by $\{R^\alpha : |\alpha| = k\}$ and let $\mathcal{B}^k = \mathcal{B}\,\mathcal{R}^k$. Thus the $(\mathcal{B}^k)_k$ is a nested decreasing sequence of subspaces of $\mathcal{B}$ such that $\mathcal{B}^\nu = (0)$. Letting $\mathcal{B}^0 = \mathcal{B}$, it follows, for each $j \ge 1$, that either $\mathcal{B}^j = (0)$ or $\mathcal{B}^{j-1} \supsetneq \mathcal{B}^j$. Thus the dimension of $\mathcal{B}^j$ is at most $\dim(\mathcal{B}) - j$ and hence there is a $\rho \le g$ such that $\mathcal{B}^\rho = (0)$. In particular, for $|\alpha| \ge \rho$ and each k,

$$0 = \sum_{s=1}^{g}(\Xi^\alpha)_{j,s}E_s.$$

From the independence of the set $\{E_1,\dots,E_g\}$, it follows that $\Xi^\alpha = 0$ and hence $\nu \le \rho \le g$. Likewise it follows that if $R^\alpha = 0$, then $\Xi^\alpha = 0$. ■

Corollary 6.4. *Suppose Ξ is a convexotonic g-tuple with associated convexotonic maps p and q as in equation (1.3). If the tuple Ξ is nilpotent, then its order of nilpotency is at most g. Further Ξ is nilpotent if and only if p and q are polynomials. In this case the order of nilpotence of Ξ is the same as the degrees of p and q. In particular, the degrees of p and q are at most g. Finally, there are examples where the degree of p and q are g.*

Proof. As described in Subsection 1.2.1, there exists a tuple $R = (R_1,\dots,R_g)$ such that $\{R_1,\dots,R_g\}$ is linearly independent and spans an algebra with structure matrices Ξ. Hence, choosing $E = R$ in Proposition 6.3, it follows that if Ξ is nilpotent, then its order of nilpotency is at most g. The remainder of the corollary follows immediately from the form of p and q and the bound on the order of nilpotency of Ξ. ■

Example 6.5. Given g, let S denote a (square) matrix nilpotent of order $g+1$ and let $R_j = S^j$. Let R denote the tuple $(R_1, \dots, R_g)$. On $\mathbb{C}^g$ with its standard orthonormal basis $\{e_1, \dots, e_g\}$, define $Se_j = e_{j-1}$ for $j \geq 2$ and $Se_1 = 0$. Thus S is the truncated backward shift. The structure matrices Ξ_j for the algebra generated by R are then $\Xi_j = S^j$. In this case the convexotonic polynomial p associated to Ξ is

$$p = x(I - \Lambda_\Xi(x))^{-1} = (p^1, \dots, p^g),$$

where

$$p^m = \sum \prod_{\sum j_k = m} x_{j_k}.$$

In particular, p^m has degree m and hence p has degree g. $\lozenge$

Proposition 6.6. *Let $A = (A_1, \dots, A_g) \in M_d(\mathbb{C})^g$ be given and assume that $\{A_1, \dots, A_g\}$ is linearly independent. Suppose C is a $d \times d$ matrix such that, for each $1 \leq j \leq g$ there exists a matrix Ξ_j such that for each $1 \leq k \leq g$ equation (6.4) holds. Let $R = (C - I)A$ and let $\Xi = (\Xi_1, \dots, \Xi_g)$. Then:*

(1) *the span $\mathcal{R}$ of $\{R_1, \dots, R_g\}$ is an algebra;*
(2) *the span $\mathcal{M}$ of $\{A_1, \dots, A_g\}$ is a right $\mathcal{R}$-module and*

$$A_k R^\alpha = \sum_t (\Xi^\alpha)_{k,t} A_t; \tag{6.9}$$

(3) *the tuple $(\Xi_1, \dots, \Xi_g)$ is convexotonic;*
(4) *the convexotonic rational mappings p and q associated to Ξ by equation (1.3) are inverses of one another;*
(5) *if $R^\alpha = 0$, then $\Xi^\alpha = 0$ and conversely, if $\Xi^\alpha = 0$, then $R_j R^\alpha = 0$ for all $1 \leq j \leq g$;*
(6) *$\mathcal{R}$ is nilpotent if and only if $\mathcal{X}$, the span of Ξ, is nilpotent. In this case, letting μ and ν denote the orders of nilpotency of $\mathcal{R}$ and $\mathcal{X}$ respectively, $\mu \leq \nu \leq \mu + 1$, and $\mu \leq \min\{\dim(\mathcal{R}) + 1, g\}$.*

Proof. From Proposition 6.3,

$$A_k R^\alpha = \sum_{j=1}^g \Xi^\alpha_{k,j} A_j. \tag{6.10}$$

Multiplying (6.10) on the left by $(C - I)$ gives,

$$R_k R^\alpha = \sum_{j=1}^g \Xi^\alpha_{k,j} R_j.$$

Thus the set $\{R_1, \dots, R_g\}$ spans an algebra $\mathcal{R}$ and equation (6.10) says the span $\mathcal{M}$ of the set $\{A_1, \dots, A_g\}$ is a module over $\mathcal{R}$. At this point items (1) and (2) have been established.

Item (3) follows from Proposition 6.3 by choosing $E = A$. Item (4) is contained in Proposition 6.2. Item (5) more is contained in Proposition 6.3 as is most of item (6). To prove the last part of item (6), multiply equation (6.10) on the left by $(C - I)$ to obtain

$$0 = R_k R^\alpha = \sum_{s=1}^g (\Xi^\alpha)_{k,s} R_s.$$

Thus, if Ξ is nilpotent of order ν, then R is nilpotent of order at most $\mu+1$ and hence $\mu \leq \nu \leq \mu+1$. Finally, if R is nilpotent, then its order of nilpotency is at most the dimension of $\mathcal{R}$. $\blacksquare$

6.2. **The convexotonic map p and its inverse q.** The following theorem is the main result of this section. Its proof relies on Proposition 6.6.

Theorem 6.7. *Suppose A, B are g-tuples of matrices of the same size d, $\{A_1, \dots, A_g\}$ is linearly independent and $p = (p^1, \dots, p^g)$ where each p^j is a formal power series, $p(0) = 0$ and $p'(0) = I$. If there exists a $d \times d$ matrix-valued free formal power series W such that equation (6.1) and the identities of equation (4.2) with $G(x) = \Lambda_B(p(x))$ hold, then*

(1) *there exists a uniquely determined $d \times d$ unitary matrix $\mathscr{W}$ and a unitary $d \times d$ matrix C such that, with $R = (C - I)A$, the functions G and W are given as in equations (1.6) and (1.7) and $B = \mathscr{W}^* C A \mathscr{W}$ (meaning $B_j = \mathscr{W}^* C A_j \mathscr{W}$ for $j = 1, 2, \dots, g$);*
(2) *there is a convexotonic tuple Ξ satisfying (6.4) (equivalently (6.9)). In particular, the set of matrices $\{R_j = (C - I)A_j : j = 1, \dots, g\}$ spans an algebra $\mathcal{R}$;*
(3) *letting p and q denote the convexotonic mappings of equation (1.3) associated to Ξ, we have $p : \mathcal{D}_A \to \mathcal{D}_B$ is bianalytic with inverse q;*
(4) *p is a polynomial if and only if the algebra $\mathcal{X}$ spanned by $\{\Xi_j : 1 \le j \le g\}$ is nilpotent and in this case q is also a polynomial and the degrees of p and q and the order of nilpotence of Ξ are all the same and at most g, and there are examples where this degree is g.*

Conversely, if $A = (A_1, \dots, A_g)$ is a linearly independent tuple of $d \times d$ matrices and C is a $d \times d$ matrix that is unitary on the span of the ranges of the A_j such that for each j, k the matrix $A_k(C - I)A_j$ is in $\mathcal{M}$, the span of $\{A_1, \dots, A_g\}$, then $\mathcal{R}$ equals the span of $\{R_j = (C - I)A_j : 1 \le j \le g\}$ is an algebra and $\mathcal{M}$ is a right module over the algebra $\mathcal{R}$. Let $\Xi = (\Xi_1, \dots, \Xi_g)$ denote the structure matrices for the module $\mathcal{M}$ over the algebra $\mathcal{R}$. Given a unitary $\mathscr{W}$ and letting $B = \mathscr{W}^* C A \mathscr{W}$, the function W given by equation (1.7) and the rational function $p(x) = x(I - \Lambda_\Xi(x))^{-1}$ together satisfies equation (6.1) and hence items (1) through (4).

Before turning the proof of Theorem 6.7, we indicate how to use it to deduce Theorem 1.1.

Proof of Theorem 1.1. Suppose $\mathcal{R}$ is a g-dimensional algebra spanned by the g-tuple of $d \times d$ matrices $(R_1, \dots, R_g)$, the matrix C is a $d \times d$ unitary and $A = (A_1, \dots, A_g)$ are as described in the definition of a spectrahedral pair associated to the algebra $\mathcal{R}$. In particular, letting $\Xi = (\Xi_1, \dots, \Xi_g)$ denote the structure matrices for $\mathcal{R}$ (with the basis $\{R_1, \dots, R_g\}$), equation (1.4) holds. Since the set $\{R_1, \dots, R_g\}$ is linearly independent, so is the set $\{A_1, \dots, A_g\}$. We shall use the converse portion of Theorem 6.7. Choosing $\mathscr{W} = I$ gives rise to the convexotonic mapping p associated to Ξ. Further, by item (3) of Theorem 6.7, this p is a bianalytic mapping between $\mathcal{D}_A$ and $\mathcal{D}_{CA}$. ■

Proof of Theorem 6.7. To prove item (1) apply Proposition 4.3 to equation (6.1) and use the finite dimensionality of $\mathcal{E} = \mathbb{C}^d$ in the present context to obtain a $d \times d$ unitary matrix C such that W and G have the form given in equations (1.7) and (1.6) with $\mathscr{W} = W_\varnothing$. Since $\mathscr{W}$ is an isometric mapping from $\mathbb{C}^d$ to $\mathbb{C}^d$ it is unitary. Thus, by Lemma 6.1, equation (6.4) holds. (See the discussion surrounding equation (6.4).) Hence, by Proposition 6.6, item (2) holds. In fact, since C and $\mathscr{W}$ are unitary, from equation (6.3) of Lemma 6.1,

$$A_k R^\omega = \sum_{j=1}^{g} p^j_{x_k \omega} \, A_j. \tag{6.11}$$

Comparing equations (6.9) (or (6.4)) and (6.11) shows $p^j_{x_k \omega} = (\Xi_\omega)_{k,j}$ and hence

$$\begin{aligned} x(I - \Lambda_\Xi(x))^{-1} &= \begin{pmatrix} \sum_{j,\alpha} (\Xi^\alpha)_{1,j} x_j \alpha & \dots & \sum_{j,\alpha} (\Xi^\alpha)_{g,j} x_j \alpha \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j,\alpha} p^j_{x_1 \alpha} x_j \alpha & \dots & \sum_{j,\alpha} p^j_{x_g \alpha} x_j \alpha \end{pmatrix} = p(x). \end{aligned}$$

Thus item (3) holds.

The converse statements of the theorem are established by verifying that, with the choices of $A, B, \mathscr{W}, C$ and W and finally p, equation (6.1) holds. ■

6.3. **Proper analytic mappings.** In this section we apply Theorem 6.7 to the case of a mapping $p = (p^1, \dots, p^{\tilde{g}})$ in g ($g < \tilde{g}$) variables x where each p^j is a formal power series.

prop:gvtg **Proposition 6.8.** *Suppose*

(a) $A \in M_d(\mathbb{C})^g$ *and* $B \in M_d(\mathbb{C})^{\tilde{g}}$*;*
(b) $p(0) = 0$*;*
(c) $p'(0) = (I \ \ 0)$*; and*
(d) *the set* $\{B_1, \dots, B_{\tilde{g}}\}$ *is linearly independent.*

If there exists a matrix-valued formal power series W with coefficients from $M_d(\mathbb{C})$ such that

$$L_B(p(x)) = W(x)^* L_A(x) W(x),$$

and the identities of equation (4.2) with $G(x) = \Lambda_B(p(x))$ hold, then there exists a $\hat{g}$ and a convexotonic $\hat{g}$-tuple Ξ of $\hat{g} \times \hat{g}$ matrices such that $P(x, 0_\tau) = (p(x), 0_\sigma)$, where $P(x, y)$ is the convexotonic rational function in the variables $(x_1, \dots, x_g, y_1, \dots, y_\tau)$ (and where $\tau = \hat{g} - g$ and $\sigma = \hat{g} - \tilde{g}$) associated to Ξ,

$$P(x, y) = \begin{pmatrix} x & y \end{pmatrix} (I - \Lambda_\Xi(x, y))^{-1}.$$

Proof. The strategy is to reduce to the case $\tilde{g} = g$. From $L_B(p(x)) = W(x)^* L_A(x) W(x)$ and Proposition 4.3, it follows that there exists $d \times d$ unitary matrices C and $\mathscr{W}$ such that, with $R = (C - I)A$, the formal power series W is the rational function $W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W}$. Further, by Lemma 6.1, for $1 \le j \le g$,

$$B_j = \mathscr{W}^* C A_j \mathscr{W},$$

and generally $\mathscr{W}^* C A_j R^\omega \mathscr{W}$ is a linear combination of $\{B_1, \dots, B_{\tilde{g}}\}$. Thus,

$$A R^\omega \in \mathfrak{B},$$

where $\mathfrak{B}$ denotes the span of $\{C^* \mathscr{W} B_1 \mathscr{W}^*, \dots, C^* \mathscr{W} B_{\tilde{g}} \mathscr{W}^*\}$. (In particular, $C^* \mathscr{W} B_j \mathscr{W}^* = A_j$ for $1 \le j \le g$.) Let $\mathcal{E} = \{E_1, \dots, E_{\hat{g}}\}$ be any linearly independent subset of $M_d(\mathbb{C})$ such that $E_j = C^* \mathscr{W} B_j \mathscr{W}^*$ for $1 \le j \le \tilde{g}$ and

$$E_k (C - I) E_j \in \operatorname{span} \mathcal{E}.$$

In particular $\hat{g} \ge \tilde{g}$. Let $F = \mathscr{W}^* C E \mathscr{W}$, set $S = (C - I)E$ and let $Y(x, y) = (I - \Lambda_S(x, y))^{-1} \mathscr{W}$. By the converse portion of Proposition 4.3,

$$Y(x, y)^* L_E(x, y) Y(x, y) = L_F(P(x, y)),$$

for some power series P. Indeed, by Theorem 6.7, P is the convexotonic rational function associated to Ξ and is a bianalytic map between the free spectrahedra determined by E and F. Observe that $Y(x, 0) = W(x)$, $L_E(x, 0) = L_A(x)$. Hence

$$L_F(P(x, 0)) = Y(x, 0)^* L_E(x, 0) Y(x, 0) = W(x)^* L_A(x) W(x) = L_B(p(x)).$$

Since $F_j = B_j$ for $1 \le j \le \tilde{g}$, the linear independence assumption implies $P(x, 0) = p(x)$. ■

7. Bianalytic Maps

Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$, the domains $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded, $p : \mathcal{D}_A \to \mathcal{D}_B$ is an analytic mapping such that $p(0) = 0$, $p'(0) = I$ and p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$. Equivalently, p is proper and thus bianalytic [HKM11]. In this section we will see, up to mild assumptions on A and B, that $d = e$ and the hypothesis of Theorem 6.7 are met and hence p is convexotonic.

7.1. **An irreducibility condition.** In this subsection we introduce irreducibility conditions on tuples A and B that ultimately allow the application of Theorem 6.7.

7.1.1. *Singular vectors.* The following is an elementary linear algebra fact.

Lemma 7.1. *Suppose T is an $M \times N$ matrix of norm one and let $\mathscr{E}$ and $\mathscr{E}_*$ denote the eigenspaces corresponding to the (largest) eigenvalue 1 of T^*T and TT^* respectively. Thus, for instance,*

$$\mathscr{E} = \{x \in \mathbb{C}^N : T^*Tx = x\}.$$

(1) *The dimensions of $\mathscr{E}$ and $\mathscr{E}_*$ are the same.*
(2) *The mapping $x \mapsto Tx$ is a unitary map from $\mathscr{E}$ to $\mathscr{E}_*$ with inverse $y \mapsto T^*y$.*
(3) *Letting*

$$J = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix},$$

the kernel of J is the set $\{-Tu \oplus u : u \in \mathscr{E}\}$.

Proof. Simply note, if $T^*Tx = x$, then $TT^*(Tx) = Tx$ and conversely if $TT^*y = y$, then $T^*T(T^*y) = T^*y$ to prove the first two items. To prove the last item, observe that vectors of the form $-Tu \oplus u$ are in the kernel of J. On the other hand, if $v \oplus w$ is in the kernel of J, then $v + Tw = 0$ and $T^*v + w = 0$. From the first equation $T^*v + T^*Tw = 0$ and from the second $T^*Tw = w$. Thus $w \in \mathscr{E}$ and $v \oplus w = -Tw \oplus w$. ∎

Lemma 7.2. *Suppose $d, e, g \le \tilde{g}$ are positive integers and*
(a) *$A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde{g}}$;*
(b) *H is a Hilbert space;*
(c) *C is a bounded linear operator on $H \otimes \mathbb{C}^d$;*
(d) *$\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ is an isometry;*
(e) *$p = (p^1, \dots, p^{\tilde{g}})$ is a free analytic mapping $\mathcal{D}_A \to \mathcal{D}_B$ with $p(0) = 0$ and linear term ℓ such that*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x), \tag{7.1}$$

where

$$W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W} \tag{7.2}$$

and $R = (C - I)A$; and
(f) *$\alpha \in (\mathbb{C}^{n\times n})^g$ and the largest eigenvalue of $\Lambda_A(\alpha)\Lambda_A(\alpha)^*$ and $\Lambda_A(\alpha)^*\Lambda_A(\alpha)$ is 1; the eigenspaces of $\Lambda_A(\alpha)\Lambda_A(\alpha)^*$ and $\Lambda_A(\alpha)^*\Lambda_A(\alpha)$ corresponding to the eigenvalue 1 are one-dimensional, spanned by the unit vectors u_1, u_2 in $\mathbb{C}^{nd}$ respectively; and*
(g) *$v_1 \in \mathbb{C}^{ne}$ is a unit vector and $\Lambda_B(\ell(\alpha))\Lambda_B(\ell(\alpha))^* v_1 = v_1$.*

Let $v_2 = -\Lambda_B(\ell(\alpha))^* v_1$ (note that v_2 is a unit vector) and write, for $j = 1, 2$ and $\{e_1, \dots, e_n\}$ a basis for $\mathbb{C}^n$,

$$u_j = \sum_{k=1}^n u_{j,k} \otimes e_k \in \mathbb{C}^d \otimes \mathbb{C}^n = \mathbb{C}^{nd}$$

and similarly for v_j. Then there is a unit vector $\lambda \in H$ (depending on α, u_j and v_j) such that,

(1) $\Lambda_A(\alpha)u_2 = -u_1$ *and* $\Lambda_A(\alpha)^* u_1 = -u_2$;
(2) $\Lambda_B(\ell(\alpha))v_2 = -v_1$ *and* $\Lambda_B(\ell(\alpha))^* v_1 = -v_2$;
(3) $\mathscr{W} v_{2,k} = \lambda \otimes u_{2,k}$ *for each* $1 \le k \le n$;
(4) $\mathscr{W} v_{1,k} = C(\lambda \otimes u_{1,k})$ *for each* $1 \le k \le n$; *and*
(5) *if* $A = B$ *and* $\ell(x) = x$, *then, without loss of generality,* $v_1 = u_1$ *and* $v_2 = u_2$.

Note that if $X \in M(\mathbb{C})^g$ is of sufficiently small norm or nilpotent, then we may substitute X for x in equation (7.1) by using the formulas for G and W in Proposition 4.3. Moreover, in this case we can evaluate $W(x)$ from (7.2) at X as

$$\begin{aligned} W(X) &= \big(I \otimes I_n - [(C-I)\otimes I_n]\Lambda_{I_H \otimes A}(X)\big)^{-1}(\mathscr{W}\otimes I_n) \\ &= \big(I \otimes I_n - [(C-I)\otimes I_n][I_H \otimes \Lambda_A(X)]\big)^{-1}(\mathscr{W}\otimes I_n) \end{aligned}$$

rather than appealing to convergence of a series expansion for W.

Proof of Lemma 7.2. Let

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $X = S \otimes \alpha$. Thus X has size $2n$. Conjugating $L_A(X)$ by the permutation matrix that implements the unitary equivalence of $A \otimes S \otimes \alpha$ with $S \otimes A \otimes \alpha$ shows, up to this unitary equivalence,

$$L_A(X) = \begin{pmatrix} I & \Lambda_A(\alpha) \\ \Lambda_A(\alpha)^* & I \end{pmatrix}.$$

Thus the assumptions on $\Lambda_A(\alpha)$ and Lemma 7.1 imply that $L_A(X)$ is positive semidefinite with a nontrivial kernel spanned by

$$u = \sum_{j=1}^{2} e_j \otimes u_j = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2 \otimes (\mathbb{C}^n \otimes \mathbb{C}^d).$$

In particular, if z is in the kernel of $I_H \otimes L_A(X)$, then there is a vector $\lambda \in H$ such that $z = \lambda \otimes u$. Also note, $\|u_1\| = \|u_2\|$ and we assume both are unit vectors.

Since, by assumption $p(x) = \ell(x) + h(x)$, where ℓ is linear and h consists of higher order terms and X is (jointly) nilpotent (of order 2),

$$p(X) = \begin{pmatrix} 0 & \ell(\alpha) \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$L_B(p(X)) = \begin{pmatrix} I & \Lambda_B(\ell(\alpha)) \\ \Lambda_B(\ell(\alpha))^* & I \end{pmatrix}.$$

Since $L_A(X)$ is positive semidefinite, equation (7.1) implies $L_B(p(X))$ is positive semidefinite. Moreover, the hypotheses imply that the vector $v = \sum_{j=1}^2 e_j \otimes v_j$ satisfies $L_B(p(X))v = 0$. Another application of equation (7.1) shows $W(X)v$ is in the kernel of $L_{I_H \otimes A}(X)$; i.e., $W(X)v = \lambda \otimes u$, for some vector $\lambda \in H$ with $\|\lambda\| = \|W(X)v\|$. Hence,

$$\lambda \otimes (\sum_{j=1}^{2} u_j \otimes e_j) = \lambda \otimes u = W(X)v = (I - \sum_{i=1}^{g} R_i \otimes X_i)^{-1}(\mathscr{W} \otimes I_n \otimes I_2)v.$$

Multiplying by $(I-\sum_{i=1}^g R_i\otimes X_i)$ on the left yields

$$\sum_{j=1}^2([\mathscr{W}\otimes I_n]v_j)\otimes e_j = [I-[(C-I)\otimes I_{2n}](I\otimes \Lambda_{I_H\otimes A}(X))](\lambda\otimes u)$$
$$= \big(\lambda\otimes u_1 - [(C-I)\otimes I_n](\lambda\otimes\Lambda_A(\alpha)u_2)\big)\otimes e_1 + \lambda\otimes u_2\otimes e_2.$$

It follows that $(\mathscr{W}\otimes I_n)v_2=\lambda\otimes u_2$ and, since $\mathscr{W}$ is an isometry, $\|\lambda\|=\|v_2\|$. Further,

$$(\mathscr{W}\otimes I_n)v_1 = \lambda\otimes u_1 - [(C-I)\otimes I_n]\big(\lambda\otimes\Lambda_A(\alpha)u_2\big).$$

Using $\Lambda_A(\alpha)u_2=-u_1$ gives $[\mathscr{W}\otimes I_n]v_1=[C\otimes I_n](\lambda\otimes u_1)$.

To complete the proof observe that

$$(\mathscr{W}\otimes I_n)v_2=\sum_{k=1}^n \mathscr{W}v_{2,k}\otimes e_k.$$

Thus, $\mathscr{W}v_{2,k}=\lambda\otimes u_{2,k}$. Similarly,

$$(C\otimes I_n)(\lambda\otimes u_1) = (C\otimes I_n)(\sum_{k=1}^n \lambda\otimes u_{1,k}\otimes e_k) = \sum_{k=1}^n C(\lambda\otimes u_{1,k})\otimes e_k.$$

Thus, $\mathscr{W}v_{1,k}=C(\lambda\otimes u_{1,k})$ for each $1\le k\le n$. ∎

`sssec:eig`

7.1.2. *The Eig-generic condition.* We now introduce some refinements of the notion of sv-generic we saw in the introduction. A subset $\{b_1,\dots,b_{\ell+1}\}$ of a finite-dimensional vector space V is a **hyperbasis** if each subset of ℓ vectors is a basis. In particular, if $\{b_1,\dots,b_\ell\}$ is a basis for V and $b_{\ell+1}=\sum_{j=1}^{\tilde g} c_jb_j$ and $c_j\ne 0$ for each j, then $\{b_1,\dots,b_{\ell+1}\}$ is a hyperbasis and conversely each hyperbasis has this form. Given a tuple $A\in M_d(\mathbb{C})^g$, let

$$\ker(A)=\bigcap_{j=1}^g \ker(A_j).$$

Given a positive integer m, let $\{e_j: 1\le j\le m\}$ denote the standard basis for $\mathbb{C}^m$.

`eneric-weak` **Definition 7.3.** The tuple $A\in M_d(\mathbb{C})^g$ is **weakly eig-generic** if there exists an $\ell\le d+1$ and, for $1\le j\le \ell$, positive integers n_j and tuples $\alpha^j\in(\mathbb{C}^{n_j\times n_j})^g$ such that

`it:oneD` (a) for each $1\le j\le\ell$, the eigenspace corresponding to the largest eigenvalue of $\Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ has dimension one and hence is spanned by a vector $u^j=\sum_{a=1}^{n_j}u_a^j\otimes e_a$; and

`it:span` (b) the set $\mathscr{U}=\{u_a^j: 1\le j\le\ell,\ 1\le a\le n_j\}$ contains a hyperbasis for $\ker(A)^\perp=\operatorname{rg}(A^*)$.

The tuple is **eig-generic** if it is weakly eig-generic and $\ker(A)=(0)$. Equivalently, $\operatorname{rg}(A^*)=\mathbb{C}^d$.

Finally, a tuple A is **$*$-generic** (resp. **weakly $*$-generic**) if there exists an $\ell\le d$ and tuples β^j such that the kernels of $I-\Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ have dimension one and are spanned by vectors $\mu^j=\sum\mu_a^j\otimes e_a$ for which the set $\{\mu_a^j: j,a\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(A)=\ker(A^*)^\perp$).

`pecial-eigs` **Remark 7.4.** It is illustrative to consider two special cases of the weak eig-generic condition. First suppose $n_j=1$ for all $1\le j\le\ell$. The kernel of $I-\Lambda_A(\alpha^j)^*\Lambda_A(\alpha)$ is spanned by a a single (non-zero) vector $u^j\in\mathbb{C}^d$ and the set $\{u^1,\dots,u^\ell\}$ is a hyperbasis for $\ker(A)^\perp$. Hence $\ell-1$ is the dimension of $\ker(A)^\perp$. If we also assume $\ker(A)^\perp=(0)$ and there exists β^j for $j=1,\dots,n$ such that $I-\Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ is positive definite with one-dimensional kernel spanned by v^j and moreover $\{v^1,\dots,v^d\}$ is a basis for $\mathbb{C}^d$, then A is sv-generic as defined in the introduction.

For the second case, suppose, for simplicity, that $\ker(A) = (0)$. If there exists an $\alpha^1 \in (\mathbb{C}^{n\times n})^g$ such that $I - \Lambda_A(\alpha^1)^*\Lambda_A(\alpha^1)$ is positive semidefinite with a one-dimensional kernel spanned by

$$u^1 = \sum_{k=1}^{n} u_k^1 \otimes e_k \in \mathbb{C}^d \otimes \mathbb{C}^n$$

and if the set $\{u_k^1 : 1 \le k \le n\}$ spans $\mathbb{C}^d$, then A is eig-generic. To prove this statement, suppose, without loss of generality, $\{u_k^1 : 1 \le k \le g\}$ is a basis for $\mathbb{C}^d$. Now take a unitary matrix T such that $T_{k,1} \ne 0$ for each $1 \le k \le d$ and $T_{k,1} = 0$ for each $d+1 \le k \le n$. Let $\alpha^2 = T\alpha^1 T^*$. It follows that $I - \Lambda_A(\alpha^2)^*\Lambda_A(\alpha^2)$ is positive semidefinite with a one-dimensional kernel spanned by the vector $u^2 = (I_d \otimes T)u^1$ and further

$$u^2 = \sum_{k=1}^{n} u_k^1 \otimes Te_k = \sum_{j=1}^{n}\sum_{k=1}^{n} u_k^1 \otimes T_{k,j}e_j = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} T_{k,j}u_k^1\right) \otimes e_j.$$

Thus, in view of the assumptions on T,

$$u_1^2 = \sum_{k=1}^{n} T_{k,1}u_k^1 = \sum_{k=1}^{d} T_{k,1}u_k^1.$$

Since $T_{k,1} \ne 0$ for $1 \le k \le d$, the set $\{u_1^1, \dots, u_g^1, u_1^2\}$ is a hyperbasis for $\mathbb{C}^d$ and the tuple A is eig-generic. $\Diamond$

rem:sv=gen **Remark 7.5.** Let us explain why sv-genericity is a generic property in the standard algebraic geometric sense. First notice that for a generic tuple $A \in M_d(\mathbb{C})^g$, the real-valued polynomial

$$p(\alpha) = \det\left(I - \Lambda_A(\alpha)^*\Lambda_A(\alpha)\right) = \det\begin{pmatrix} I & \Lambda_A(\alpha) \\ \Lambda_A(\alpha)^* & I \end{pmatrix}$$

is irreducible and changes sign on $\mathbb{R}^{2g}$; here we consider $p(\alpha)$ as a real polynomial in the real and imaginary parts of the complex variables $\alpha \in \mathbb{C}^g$. This fact is easily established by simply giving a tuple A with this property. As a consequence, [BCR98, Theorem 4.5.1] implies that each polynomial vanishing on the zero set of p must be a multiple of p.

If A is not sv-generic, it fails one of the two properties in its definition. (It suffices to show this while omitting the positive semidefiniteness condition.) Assume A fails the first property. Then for every choice of $d+1$ vectors $\alpha^j \in \mathbb{C}^d$ for which $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ is singular with a one-dimensional kernel spanned by u^j, the set $\{u^1, \dots, u^{d+1}\}$ is not a hyperbasis. Observe that in this case u^j can be chosen to be a column of the adjugate matrix of $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$.

The latter condition can be expressed by saying that one of the $d \times d$ minors of the matrix $\begin{pmatrix} u^1 & \cdots & u^{d+1} \end{pmatrix}$, whose columns u^j are columns of the adjugate of $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$, vanishes. Equivalently, the product q of all these $d \times d$ minors vanishes on the zero set of p.

But, as follows from the first paragraph, on a generic set of As, this means that q is a multiple of p. However, it is easy to find examples of A for which this fails. The argument is similar if A fails the second property of the definition of sv-generic. Hence being sv-generic is a generic property. $\Diamond$

em:generic+ **Remark 7.6.** The one-dimensional kernel assumption is key for the eig-generic property, and has been successfully analyzed in the two papers [HKV, KV17]. Namely, if the tuple $A \in M_d(\mathbb{C})^g$ is minimal w.r.t. the size needed to describe the free spectrahedron $\mathcal{D}_A$, then $\dim\ker L_A(X) = 1$ for all X in an open and dense subset of the boundary $\partial\mathcal{D}_A(n)$ provided n is large enough. $\Diamond$

7.2. **The structure of bianalytic maps.** In this section our main results on bianalytic maps between free spectrahedra appear as Theorem thm:one-sided 7.10 and Corollary cor:main 7.11. We begin by collecting consequences of the eig-generic assumptions.

c in action **Lemma 7.7.** *Suppose*

(a) $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde{g}}$;

(b) *H is a Hilbert space, C is an isometry on $H \otimes \mathbb{C}^d$ and*

$$W(x) = (I - \Lambda_R(x))^{-1}\mathscr{W},$$

where $R = (C - I)[I_H \otimes A]$ and $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ is an isometry;

it:nils (c) *$p = (p^1, \dots, p^{\tilde{g}})$ is a free analytic mapping $\mathcal{D}_A \to \mathcal{D}_B$ such that $p(0) = 0$, and*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x),$$

in the sense that

$$L_B(p(X)) = W(X)^* L_{I_H \otimes A}(X) W(X)$$

for each nilpotent $X \in M(\mathbb{C})^g$;

it:pproper (d) *p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$; and*

it:alphas (e) *there is a positive integer ℓ and, for $1 \le j \le \ell$, tuples α^j in $(\mathbb{C}^{n_j \times n_j})^g$ such that $I - \Lambda(\alpha^j)^*\Lambda(\alpha^j)$ is positive definite with a one-dimensional kernel spanned by*

$$u_2^j = \sum_{k=1}^{n_j} u_{2,k}^j \otimes e_k;$$

action dlee (1) *If A is eig-generic $\big($resp. weakly eig-generic$\big)$, then $d \le \dim(\operatorname{rg}(B^*)) \le e$ $\big($resp. $\dim(\operatorname{rg}(A^*) \le \dim(\operatorname{rg}(B^*))\big)$;*

:action d=e (2) *If $e = d$ $\big($resp. $\dim(\operatorname{rg}(A^*)) = \dim(\operatorname{rg}(B^*))\big)$ and the tuples α^j and unit vectors u_2^j validate the eig-generic $\big($resp. weak eig-generic$\big)$ assumption for A, then there exists a unit vector $\lambda \in H$ and unit vectors*

$$v_2^j = \sum_{k=1}^{n_j} v_{2,k}^j \otimes e_j$$

in the kernel of $I - \Lambda_B(p(\alpha^j))^*\Lambda_B(p(\alpha^j))$ such that if $\mathcal{I} \subseteq \{(j,k) : 1 \le j \le l,\ 1 \le k \le n_j\}$ and $\{u_{2,k}^j : (j,k) \in \mathcal{I}\}$ is a hyperbasis for $\mathbb{C}^d$ $\big($resp. $\operatorname{rg}(A^*)\big)$, then $\{v_{2,k}^j : (j,k) \in \mathcal{I}\}$ is a hyperbasis for $\mathbb{C}^d$ $\big($resp. $\operatorname{rg}(B^*)\big)$, and for all $(j,k) \in \mathcal{I}$,

$$\mathscr{W} v_{2,k}^j = \lambda \otimes u_{2,k}^j;$$

ion d=e alt (3) *If $e = d$ and A is eig-generic $\big($resp. $\dim(\operatorname{rg}(A^*)) = \dim(\operatorname{rg}(B^*)$ and A is weakly eig-generic$\big)$, then there exists a unit vector $\lambda \in H$ and a $d \times d$ unitary M $\big($resp. a unitary map M from $\operatorname{rg}(B^*)$ to $\operatorname{rg}(A^*)$) such that $\mathscr{W} = \lambda \otimes M$ $\big($resp. $\mathscr{W} v = \lambda \otimes M v$ for $v \in \operatorname{rg}(B^*)\big)$; and*

on one-term (4) *If A is eig-generic and $*$-generic and $e = d$ $\big($resp. A is weakly eig-generic and weakly $*$-generic, $\dim(\operatorname{rg}(A^*)) = \dim(\operatorname{rg}(B^*))$ and $\dim(\operatorname{rg}(A)) = \dim(\operatorname{rg}(B))\big)$, then there is a vector $\lambda \in H$ and $d \times d$ unitary matrices M and Z such that $\mathscr{W} = \lambda \otimes M$ and $C(\lambda \otimes I_d) = \lambda \otimes Z$ $\big($resp. a unitary map M and an isometry N from $\operatorname{rg}(B^*)$ to $\operatorname{rg}(A^*)$ and from $\operatorname{rg}(B^*) \cap \operatorname{rg}(B)$ into $\operatorname{rg}(A)$ respectively such that $\mathscr{W} v = \lambda \otimes M v$ for $v \in \operatorname{rg}(B^*)$ and $C(\lambda \otimes N v) = \lambda \otimes M v$ for $v \in \operatorname{rg}(B^*) \cap \operatorname{rg}(B)\big)$.*

Remark 7.8. Note that the hypotheses on α^j and u_2^j imply that each $u_{2,k}^j \in \operatorname{rg}(A^*)$. Likewise $v_{2,k}^j \in \operatorname{rg}(B^*)$. The eig-generic hypothesis in item (it:action dlee 1) can be relaxed to $\{u_{2,k}^j : j,k\}$ *spans* $\mathbb{C}^d$ *(respectively* $\operatorname{rg}(A^*)$*)*, rather than that *it contains a hyperbasis.* ♢

Proof. We begin with some calculations preliminary to proving all items claimed in the lemma. Let α^j and u_2^j be as described in item (e) (but do not necessarily assume that $\{u_2^j : j\}$ contains a hyperbasis yet). Let $\mathscr{J} = \{(j,k) : 1 \le j \le \ell,\ 1 \le k \le n_j\}$ and, as in the proof of Lemma 7.2, let

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $1 \le j \le \ell$, let $X^j = S \otimes \alpha^j$. The hypotheses imply X^j is in the boundary of $\mathcal{D}_A$. By item (d), $p(X)$ is in the boundary of $\mathcal{D}_B$. Observe that $p(X) = \ell(X)$, where ℓ is the linear part of p. Thus (up to unitary equivalence)

$$L_B(p(X)) = \begin{pmatrix} I & \Lambda_B(\ell(\alpha^j)) \\ \Lambda_B(\ell(\alpha^j))^* & I \end{pmatrix}$$

is positive semidefinite and there exist unit vectors $v_i^j \in \mathbb{C}^{ne}$ such that $v^j = e_1 \otimes v_1^j + e_2 \otimes v_2^j$ lies in the kernel of $L_B(p(X))$. Hence, by Lemma 7.1, $\Lambda_B(\ell(\alpha^j))\Lambda_B(\ell(\alpha^j))^* v_1^j = v_1^j$ and $v_2^j = -\Lambda_B(\alpha^j)^* v_1^j$. Consequently, Lemma 7.2 applies. In particular, writing

$$v_i^j = \sum_{k=1}^{n_j} v_{i,k}^j \otimes e_k,$$

there exist unit vectors $\lambda_j \in H$ such that

$$\mathscr{W} v_{2,k}^j = \lambda_j \otimes u_{2,k}^j$$

for $(j,k) \in \mathscr{J}$.

Fix $\mathcal{I} \subseteq \mathscr{J}$ and let $\mathscr{U}_\mathcal{I} = \{u_{2,k}^j : (j,k) \in \mathcal{I}\} \subseteq \mathrm{rg}(A^*) \subseteq \mathbb{C}^d$. Suppose

$$0 = \sum_{(j,k)\in\mathcal{I}} c_k^j v_{2,k}^j. \tag{7.3}$$

Applying $\mathscr{W}$ to equation (7.3)

$$0 = \sum_{(j,k)\in\mathcal{I}} c_k^j \lambda_j \otimes u_{2,k}^j.$$

Given a vector $\eta \in H$, applying the operator $\eta^* \otimes I$ yields

$$0 = \sum_{(j,k)\in\mathcal{I}} c_k^j \eta^* \lambda_j u_{2,k}^j = \sum_{(j,k)\in\mathcal{I}} a_k^j u_{2,k}^j, \tag{7.4}$$

where $a_k^j = c_k^j \eta^* \lambda_j$.

Suppose $\mathscr{U}_\mathcal{I}$ is linearly independent and $(p,m) \in \mathcal{I}$. Choosing $\eta = \lambda_p$ in equation (7.4) gives $a_m^p = c_m^p \lambda_p^* \lambda_p = 0$. Thus $c_m^p = 0$ for each $(p,m) \in \mathcal{I}$. Hence $\mathscr{V}_\mathcal{I} := \{v_{2,k}^j : (j,k) \in \mathcal{I}\} \subseteq \mathrm{rg}(B^*) \subseteq \mathbb{C}^e$ is linearly independent and in particular the cardinality of $\mathcal{I}$ is at most $\dim(\mathrm{rg}(B^*))$. If $\mathscr{U}_\mathscr{J} = \{u_{j,k}^2 : j,k\}$ spans $\mathrm{rg}(A^*)$, then choosing $\mathscr{U}_\mathcal{I}$ a basis for $\mathrm{rg}(A^*)$ shows $\dim(\mathrm{rg}(A^*) \le \dim(\mathrm{rg}(B^*)$. Further if $\dim(\mathrm{rg}(A^*)) = d$, then $d \le \mathrm{rg}(B^*) \le e$.

To prove item (1), simply observe if α^j and u_j^2 validate the eig-generic (resp. weak eig-generic) hypothesis, then $\mathscr{U}_\mathscr{J}$ does span $\mathbb{C}^d$ (resp. $\mathrm{rg}(A^*)$) and hence $d \le e$ (resp. $\dim(\mathrm{rg}(A^*)) \le \dim(\mathrm{rg}(B^*))$).

To prove item (2), suppose the tuples α^j and the vectors u^j validate the (resp. weakly) eig-generic assumption. Thus, there is an $\mathcal{I} \subseteq \mathscr{J}$ such that $\mathscr{U}_\mathcal{I}$ is a hyperbasis for $\mathbb{C}^d$ (resp. $\mathrm{rg}(A^*)$). Since a hyperbasis for $\mathbb{C}^d$ contains $d+1$ (resp. $\dim(\mathrm{rg}(A^*))+1$) elements, the cardinality of $\mathcal{I}$ is $d+1$ (resp. $\dim(\mathrm{rg}(A^*))+1$). Assuming $d = e$ (resp. $\dim(\mathrm{rg}(A^*)) = \dim(\mathrm{rg}(B^*))$), the set $\{v_k^j : (j,k) \in \mathcal{I}\}$ is then a set of $e+1$ (resp. $\dim(\mathrm{rg}(B^*))+1$) elements in $\mathbb{C}^e$ (resp. $\mathrm{rg}(B^*)$), so is linearly dependent. On the other hand, if $\mathcal{I}'$ is a subset of $\mathcal{I}$ of cardinality d, then $\mathcal{U}_{\mathcal{I}'}$ is a basis for $\mathbb{C}^d$ (resp. $\mathrm{rg}(A^*)$) and hence $\mathscr{V}_{\mathcal{I}'} = \{v_{j,k}^2 : (j,k) \in \mathcal{I}'\}$ is a basis for $\mathbb{C}^d$ (resp. $\mathrm{rg}(B^*)$). Hence $\mathscr{V}_\mathcal{I}$ is a hyperbasis for $\mathbb{C}^d$ (resp. $\operatorname{rg}(B^*)$). Hence, there exists, for $(j,k)\in\mathcal{I}$, scalars c_k^j none of which are 0 such that equation (7.3) holds. Given $(p,m)\in\mathcal{I}$, an application of equation (7.4) with a (nonzero) vector η orthogonal to λ_p gives $a_m^p=0$ and hence, again by the hyperbasis property, $a_k^j=0$ for all $(j,k)\in\mathcal{I}$. Since $c_k^j\neq 0$, it follows that η is orthogonal to each λ_j and consequently the unit vectors λ_j are all colinear. By multiplying v^j by a unimodular constant as needed, it may be assumed that there is a unit vector $\lambda\in H$ such that $\lambda_j=\lambda$ for all j. With this re-normalization, for $(j,k)\in\mathcal{I}$,

$$\mathscr{W}v_k^j=\lambda\otimes u_{2,k}^j, \tag{7.5}$$

completing the proof of item (2).

Turning to the proof of item (3), it follows immediately from equation (7.5) that that for each $v\in\mathbb{C}^d$ (resp. $v\in\operatorname{rg}(B^*)$) there is a $u\in\mathbb{C}^d$ (resp. $u\in\operatorname{rg}(A^*)$) such that

$$\mathscr{W}v=\lambda\otimes u,$$

since $\{v_{2,k}^j:(j,k)\in\mathcal{I}\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(B^*)$). Hence, by linearity and since $\mathscr{W}$ is an isometry, there is a unitary mapping $M:\mathbb{C}^d\to\mathbb{C}^d$ (resp. $M:\operatorname{rg}(B^*)\to\operatorname{rg}(A^*)$) such that $\mathscr{W}=\lambda\otimes M$ (resp. $\mathscr{W}|_{\operatorname{rg}(B^*)}=\lambda\otimes M$).

For the proof of item (4), assuming A is $*$-generic (resp. weakly $*$-generic), for $1\le j\le\ell$, there exists tuples β^j of sizes n_j and vectors

$$\mathrm{u}_1^j=\sum_{k=1}^{n_j}\mathrm{u}_{1k}^j\otimes e_k$$

satisfying the $*$-generic (resp. weak $*$-generic) condition for A. That is, $I-\Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ is positive semidefinite with one-dimensional kernel spanned by u_1^j and the set of vectors $\{\mathrm{u}_{1k}^j:1\le j\le\ell,\ 1\le k\le n_j\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(A)$). By Lemma 7.2, there exist vectors

$$\mathrm{u}_2^j=\sum_{k=1}^{n_j}\mathrm{u}_{2k}^j\otimes e_k$$

such that $I-\Lambda_A(\beta^j)^*\Lambda_A(\beta^j)$ has a one-dimensional kernel spanned by u_2^j. On the other hand, the tuples

$$X^j=\begin{pmatrix}0&\beta^j\\0&0\end{pmatrix}$$

lie in the boundary of $\mathcal{D}_A$. Hence, as before $p(X^j)$ lies in the boundary of $\mathcal{D}_B$. Thus

$$L_B(p(X^j))=\begin{pmatrix}I&\Lambda_B(\ell(\beta^j))\\\Lambda_B(\ell(\beta^j))^*&I\end{pmatrix}$$

is positive semidefinite and has a kernel. Hence, there exists vectors $\mathrm{v}^j=\mathrm{v}_1^j\oplus\mathrm{v}_2^j$ such that $L_B(p(X^j))\mathrm{v}^j=0$. By Lemma 7.1 these vectors are related by

$$\begin{aligned}\Lambda_A(\beta^j)^*\mathrm{u}_1^j&=-\mathrm{u}_2^j, &\qquad \Lambda_B(\ell(\beta^j))^*\mathrm{v}_1^j&=-\mathrm{v}_2^j,\\ \Lambda_A(\beta^j)\mathrm{u}_2^j&=-\mathrm{u}_1^j, &\qquad \Lambda_B(\ell(\beta^j))\mathrm{v}_2^j&=-\mathrm{v}_1^j.\end{aligned}$$

Write

$$\mathrm{v}_i^j=\sum_{k=1}^{n_j}\mathrm{v}_{i,k}^j\otimes e_k.$$

By Lemma 7.2, for each j there exists a vector $\tau_j\in H$ such that

$$\mathscr{W}\mathrm{v}_{2,k}^j=\tau_j\otimes\mathrm{u}_{2,k}^j\qquad\text{and}\qquad\mathscr{W}\mathrm{v}_{1,k}^j=C\tau_j\otimes\mathrm{u}_{1,k}^j$$

for each $1 \le k \le n_j$. Now suppose further that A is weakly eig-generic and $\dim(\operatorname{rg}(B^*)) = \dim(\operatorname{rg}(A^*))$. In this case, by the already proved item (3), there is a unit vector λ and unitary mapping $M : \operatorname{rg}(B^*) \to \operatorname{rg}(A^*)$ such that $\mathscr{W}\mathrm{v} = \lambda \otimes M\mathrm{v}$ on $\operatorname{rg}(B^*)$. Since $\mathrm{v}^j_{2k} \in \operatorname{rg}(B^*)$ and $\mathscr{W}$ and C are isometries, it follows that $\tau_j = \rho_j \lambda$ for some scalar $\rho_j \neq 0$. Hence,

eq:C*sW

$$\mathscr{W}\mathrm{v}^j_{1,k} = C\lambda \otimes \rho_j \mathrm{u}^j_{1,k} \tag{7.6}$$

for each $1 \le j \le \ell$ and $1 \le k \le n_j$. Since $\{\mathrm{u}^j_{1,k} : j,k\}$ spans $\operatorname{rg}(A)$ and both C and $\mathscr{W}$ are unitary, equation (7.6) implies there is an isometry $Z : \operatorname{rg}(A) \to \operatorname{rg}(B)$ such that

$$C(\lambda \otimes \mathrm{u}) = \mathscr{W} Z\mathrm{u}$$

for $\mathrm{u} \in \operatorname{rg}(A)$. In particular, $\dim(\operatorname{rg}(A)) \le \dim(\operatorname{rg}(B))$. Hence, if $\operatorname{rg}(A) = \mathbb{C}^d$ (as is the case if A is $*$-generic) and $e = d$, then $\operatorname{rg}(B) = \mathbb{C}^d$ and Z is onto. In the case that A is only weakly $*$-generic, we have assumed the dimensions of $\operatorname{rg}(A)$ and $\operatorname{rg}(B)$ are the same and so again Z is onto. So in either case, Z is unitary. In particular, given $\mathrm{v} \in \operatorname{rg}(B) \cap \operatorname{rg}(B^*)$, there is a $\mathrm{u} \in \operatorname{rg}(A)$ such that $Z\mathrm{u} = \mathrm{v}$ and

$$C(\lambda \otimes Z^*\mathrm{v}) = \mathscr{W}\mathrm{v}.$$

On the other hand, as $\mathrm{v} \in \operatorname{rg}(B^*)$, we have $\mathscr{W}\mathrm{v} = \lambda \otimes M\mathrm{v}$. Hence,

eq:Clambda

$$C(\lambda \otimes Z^*\mathrm{v}) = \lambda \otimes M\mathrm{v}. \tag{7.7}$$

Hence, letting N denote the restriction of Z^* to $\operatorname{rg}(B) \cap \operatorname{rg}(B^*)$ the desired conclusion follows.

We now take up the case A is eig-generic and $*$-generic and $e = d$. In this case, M is a $d \times d$ unitary matrix by item (3). Moreover, as noted above $d = \dim(\operatorname{rg}(A)) \le \dim(\operatorname{rg}(B)) \le d$. On the other hand, from item (1), $d \le \dim(\operatorname{rg}(B^*)) \le d$ and hence $\operatorname{rg}(B^*) = \mathbb{C}^d$. It follows that $Z : \mathbb{C}^d \to \mathbb{C}^d$ is unitary and letting $\mathrm{u} = Z^*\mathrm{v}$ in equation (7.7) gives,

$$C(\lambda \otimes \mathrm{u}) = \lambda \otimes MN\mathrm{u}$$

and the proof of item (4) is complete. ■

rem:weakC **Remark 7.9.** In the context of item (4), the dimension of $\operatorname{rg}(B) \cap \operatorname{rg}(B^*)$ is at most the dimension of $\operatorname{rg}(A) \cap \operatorname{rg}(A^*)$. In the case that these dimensions coincide, the identity $C^*(\lambda \otimes Mv) = \lambda \otimes Nv$ for $v \in \operatorname{rg}(B) \cap \operatorname{rg}(B^*)$ implies there is a unitary mapping Z of $\operatorname{rg}(A) \cap \operatorname{rg}(A^*)$ such that $C^*(\lambda \otimes z) = \lambda \otimes Zz$ for $z \in \operatorname{rg}(A) \cap \operatorname{rg}(A^*)$; i.e., $C^* = I \otimes Z$ on $\mathbb{C}\lambda \otimes \operatorname{rg}(A) \cap \operatorname{rg}(A^*)$. ◊

m:one-sided **Theorem 7.10.** *Suppose*

(a) $A, B \in M_d(\mathbb{C})^g$;
(b) $\mathcal{D}_A$ *is bounded;*
(c) p *is a mapping from* $\mathcal{D}_A$ *into* $\mathcal{D}_B$ *that is analytic and bounded on a free pseudoconvex set* $\mathcal{G}_Q$ *containing* $\mathcal{D}_A$*;*
(d) p *maps the boundary of* $\mathcal{D}_A$ *into the boundary of* $\mathcal{D}_B$.

If A is eig-generic and $*$-generic and $p(x) = x + f(x)$, where f consists of terms of degree two and higher, then there exists a $d \times d$ matrix-valued analytic function $\mathtt{W}$ *such that*

$$L_B(p(x)) = \mathtt{W}(x)^* L_A(x) \mathtt{W}(x)$$

and thus the conclusions of Theorem 6.7 hold. In particular, there exist $d \times d$ unitary matrices $C, \mathscr{W}$ such that, $B = \mathscr{W}^* C A \mathscr{W}$ and $(\mathcal{D}_A, \mathcal{D}_B)$ is a spectrahedral pair with associated convexotonic map p.

cor:main **Corollary 7.11.** *Suppose*

(a) $A \in M_d(\mathbb{C})^g$ *and* $B \in M_e(\mathbb{C})^g$;
(b) $\mathcal{D}_A$ *is bounded;*
(c) A *is eig-generic and* $*$*-generic;*

(d) *p is an analytic mapping $\mathcal{D}_A \to \mathcal{D}_B$ analytic and bounded on a free pseudoconvex set $\mathcal{G}_P$ containing $\mathcal{D}_A$ with $p(x) = x + f(x)$ where f consists of terms of degree two and higher;*
(e) *r is an analytic mapping $\mathcal{D}_B \to \mathcal{D}_A$ analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_B$, with $r(x) = x + h(x)$ where h consists of terms of degree two and higher;*
(f) *p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$ and r maps the boundary of $\mathcal{D}_B$ to the boundary of $\mathcal{D}_A$.*

If B is eig-generic, then $d = e$. In any case, if $d = e$, then the maps p and r are bianalytic (convexotonic) between $\mathcal{D}_A$ and $\mathcal{D}_B$ and between $\mathcal{D}_B$ and $\mathcal{D}_A$ respectively. Moreover, for each the conclusions of Theorem 6.7 hold. In particular, there exist $d \times d$ unitary matrices $C, \mathscr{W}$ such that, $B = \mathscr{W}^* C A \mathscr{W}$ and $(\mathcal{D}_A, \mathcal{D}_B)$ is a spectrahedral pair with associated convexotonic map p.

Proof. By Lemma 7.7 (1) the assumption A is eig-generic implies $d \le e$. If B is assumed eig-generic, then reversing the roles of A and B and using r in place of p, implies $e \le d$. Thus in any case $d = e$ and Theorem 7.10 applies to complete the proof. ■

Proof of Theorem 7.10. We begin by considering the case, as in Remark 7.12 below, that $\tilde{g} \ge g$ and $B \in M_d(\mathbb{C})^{\tilde{g}}$, leaving the special case $\tilde{g} = g$ for later. Since p maps $\mathcal{D}_A$ into $\mathcal{D}_B$ and $p(0) = 0$, by the Analytic Positivstellensatz (here is where the hypothesis $\mathcal{D}_A$ is bounded is used), Corollary 1.10, there exists a Hilbert space H, a unitary mapping C on $H \otimes \mathbb{C}^d$ and an isometry $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that

$$L_B(p(x)) = W(x)^* \big(I_H \otimes L_A(x)\big) W(x),$$

where

$$W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W}$$

and $R = (C - I)(I_H \otimes A)$. Moreover,

$$L_B(p(X)) = W(X)^* L_{I_H \otimes A}(X) W(X)$$

holds for nilpotent $X \in M(\mathbb{C})^g$ and the identities of equation (4.2) hold with $G(x) = \Lambda_B(p(x))$.

Lemma 7.7 (4) implies there is a vector $\lambda \in H$ and $d \times d$ unitary matrices M and N such that $\mathscr{W} = \lambda \otimes M$ and $C(\lambda \otimes I) = \lambda \otimes N$. To complete the proof, let

$$\mathtt{W}(x) = [\lambda^* \otimes I] W(x).$$

Importantly, $\mathtt{W}$ is a square $(d \times d)$ matrix-valued analytic function. Further,

$$\begin{aligned} L_B(p(x)) = W(x)^*(I_H \otimes L_A(x))W(x) &= \mathtt{W}(x)^*[\lambda^* \otimes I](I_H \otimes L_A(x))[\lambda \otimes I]\mathtt{W}(x) \\ &= \mathtt{W}(x)^* L_A(x) \mathtt{W}(x). \end{aligned}$$

If $\tilde{g} = g$ then Theorem 6.7 applies and concludes the proof. ■

Remark 7.12. If, in the setting of Theorem 7.10 or Corollary 7.11, the assumptions are relaxed to $B \in M_d(\mathbb{C})^{\tilde{g}}$ with $\tilde{g} \ge g$, then we can conclude that p satisfies the conclusions of Proposition 6.8. ◊

This section concludes with a proof of Theorem 1.5.

Proof of Theorem 1.5. The assumption that A and B are both sv-generic immediately imply both are eig-generic and $*$-generic. By assumption, p has inverse r and r is a bianalytic map from $\mathcal{D}_B$ to $\mathcal{D}_A$. Thus the hypotheses of Corollary 7.11 are validated by those of Theorem 1.5 and the result thus follows from Corollary 7.11. ■

8. Affine Linear Change of Variables

This section describes the effects of change of variables by way of pre and post composition with an affine linear map on an analytic mapping between free spectrahedra.

Suppose $A = (A_1, \dots, A_g) \in M_d(\mathbb{C})^g$ determines a bounded LMI domain $\mathcal{D}_A$, $B = (B_1, \dots, B_{\tilde{g}}) \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is analytic with $p(0) = b$.

We first, in Subsection 8.1, turn our attention to conditions on A and B that guarantee $p'(0)$ is one to one. Next, in Subsection 8.2, assuming $p'(0)$ is one to one, we apply a linear transforms on the range of p placing p into the canonical form $p(x) = (x, 0) + h(x)$, where $h(x)$ consists of higher order terms ($h(0) = 0$ and $h'(0) = 0$). In Subsection 8.3 we consider an affine linear change of variables on the domain of p.

8.1. Conditions guaranteeing $p'(0)$ is one to one.

Natural hypotheses on a mapping $\mathcal{D}_A$ to $\mathcal{D}_B$ via p lead to the conclusion that $p'(0)$ is one-one.

Lemma 8.1. *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is analytic and $p(0)$ is in the interior of $\mathcal{D}_B$. If*

(a) *p is proper; and*
(b) *$\mathcal{D}_A$ is bounded,*

then $p'(0)$ is one-one.

Proof. Let b denote the constant term of p. Thus b is a row vector of length $\tilde{g}$ with entries from $\mathbb{C}$ and $q(x) = p(x) - b$ satisfies $q(0) = 0$. Let $\mathfrak{B} = L_B(b)$. In particular, $\mathfrak{B}$ is an $e \times e$ matrix (since $B \in M_e(\mathbb{C})^{\tilde{g}}$). It is also positive definite, since $\mathfrak{B} = L_B(p(0))$ and $p(0)$ is in the interior of $\mathcal{D}_B$. Let $\mathfrak{H}$ denote the positive square root of $\mathfrak{B}$ and define $F = \mathfrak{H}^{-1} B \mathfrak{H}^{-1}$. Thus, $F \in M_e(\mathbb{C})^{\tilde{g}}$ is a $\tilde{g}$ tuple of $e \times e$ matrices and

$$\mathfrak{H}^{-1} L_B(p(x)) \mathfrak{H}^{-1} = L_F(q(x)).$$

In particular, for a given n and tuple $X \in M_n(\mathbb{C})^g$, we have $L_B(p(X)) \succeq 0$ if and only if $L_F(q(X)) \succeq 0$ and q is proper since p is assumed to be. If $p'(0) = q'(0)$ is not one-one, then there exists a non-zero $a \in \mathbb{C}^g$ such that $q'(0)a = 0$. Given S, a non-zero matrix nilpotent of order two, let $X = a \otimes S = \big(a_1 S, \dots, a_g S\big)$. It follows that

$$q(tX) = t(q'(0)a) \otimes S = 0.$$

Since $\mathcal{D}_A$ is bounded and contains 0 in its interior and $X \neq 0$, there exists a t such that tX is in the boundary of $\mathcal{D}_A$. On the other hand, since $q(tX) = 0$, the tuple tX is not in the boundary of $\mathcal{D}_F$, contradicting the fact that q is proper. Hence $q'(0)$ is one-one. ■

8.2. Affine linear change of variables for the range of p.

In this section we compute explicitly the effect of an affine linear change of variables in the range space of p. This change of variable can be used to produce a new map $\tilde{p}$ with $\tilde{p}'(0) = I$, used later in the proof of Theorem 10.2. Given a $g \times g$ matrix M and an analytic mapping $q = \begin{pmatrix} q^1 & \cdots & q^g \end{pmatrix}$, let qM denote the analytic mapping,

$$q(x)M = \begin{pmatrix} q^1(x) & \dots q^g(x) \end{pmatrix} M = \left(\sum q^j(x) M_{j,1}, \dots, \sum q^j(x) M_{j,g} \right).$$

On the other hand, for $B \in M_n(\mathbb{C})^g$, we often write MB for $(M \otimes I)B$ where B is treated as a column vector. Thus,

$$MB = \begin{pmatrix} \sum M_{1,j} B_j \\ \vdots \\ \sum M_{g,j} B_j \end{pmatrix}.$$

Since we are viewing x and $p(x)$ as row vectors, in the case p has $\tilde{g}$ entries, $p'(0)$ is a $g \times \tilde{g}$ matrix.

prop:dF **Proposition 8.2.** *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is an analytic map with $p(0) = b \in \mathbb{C}^{\tilde{g}}$. Let $\mathfrak{H}$ denote the positive square root of*

$$\mathfrak{B} = L_B(b) = I + \sum_{j=1}^{g} b_j B_j + \sum_{j=1}^{g} (b_j B_j)^*,$$

and let $F = p'(0)\,\mathfrak{H}^{-1} B \mathfrak{H}^{-1} = \mathfrak{H}^{-1}\, p'(0) B\, \mathfrak{H}^{-1}$.

Suppose $p'(0)$ is one-one and choose any invertible $\tilde{g}\times\tilde{g}$ matrix M whose first g rows are those of $p'(0)$ and let ℓ denote the affine linear polynomial $\ell(x) = (-b + x)M^{-1}$. The analytic map $\tilde{p} = \ell \circ p$ maps $\mathcal{D}_A$ into $\mathcal{D}_F$ and satisfies $\tilde{p}(0) = 0$ and $\tilde{p}'(0) = \begin{pmatrix} I_d & 0 \end{pmatrix}$. Thus $\tilde{p}(x) = \begin{pmatrix} x & 0 \end{pmatrix} + h(x)$ where $h(0) = 0$ and $h'(0) = 0$. In particular, if p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$, then $\tilde{p}$ maps the boundary of $\mathcal{D}_A$ to the boundary of $\mathcal{D}_F$; and if p is bianalytic, then so is $\tilde{p}$.

Written in more expansive notation, for each $1 \le i \le g$,

$$F_i = \mathfrak{H}^{-1}(MB)_i \mathfrak{H}^{-1} = \sum_{j=1}^{g} p'(0)_{i,j} \mathfrak{H}^{-1} B_j \mathfrak{H}^{-1},$$

$$B_i = \mathfrak{H}(M^{-1}F)_i \mathfrak{H} = \sum_{j=1}^{g} \left(M^{-1}\right)_{i,j} \mathfrak{H} F_j \mathfrak{H}.$$

Proof. Consider

$$\begin{aligned}
\sum_{k=1}^{\tilde{g}} (MB)_k \otimes \tilde{p}(x)_k &= \sum_{k=1}^{\tilde{g}} (MB)_k \otimes \big((-b + p(x))M^{-1}\big)_k \\
&= \sum_{k=1}^{\tilde{g}} \left[\left(\sum_{j=1}^{\tilde{g}} M_{k,j} B_j \right) \otimes \left(\sum_{i=1}^{\tilde{g}} (-b_i + p_i(x)) \left(M^{-1}\right)_{i,k} \right) \right] \\
&= \sum_{j=1}^{\tilde{g}} \sum_{i=1}^{\tilde{g}} \left(\sum_{k=1}^{\tilde{g}} (M^{-1})_{i,k} M_{k,j} \right) \Big[B_j \otimes \big(- b_i + p_i(x) \big) \Big] \\
&= \sum_{j=1}^{\tilde{g}} \sum_{i=1}^{\tilde{g}} (I_{i,j}) \Big[B_j \otimes \big(- b_i + p_i(x) \big) \Big] = \sum_{j=1}^{\tilde{g}} B_j \otimes p_j(x) - \sum_{j=1}^{\tilde{g}} B_j b_j.
\end{aligned}$$

Given a tuple X, it follows that

$$\begin{aligned}
L_F(\tilde{p}(X)) &= I + \sum_{j=1}^{\tilde{g}} F_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} F_j{}^* \otimes \tilde{p}_j(X)^* \\
&= I + \sum_{j=1}^{\tilde{g}} \left(\mathfrak{H}^{-1}(MB)\mathfrak{H}^{-1}\right)_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} \left(\mathfrak{H}^{-1}(MB)\mathfrak{H}^{-1}\right)_j^* \otimes \tilde{p}_j(X)^* \\
&= (\mathfrak{H}^{-1} \otimes I) \left(\mathfrak{B} + \sum_{j=1}^{\tilde{g}} (MB)_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} (MB)_j^* \otimes \tilde{p}_j(X)^* \right) (\mathfrak{H}^{-1} \otimes I) \\
&= (\mathfrak{H}^{-1} \otimes I) \left(I + \sum_{j=1}^{\tilde{g}} B_j \otimes p_j(X) + \sum_{j=1}^{\tilde{g}} B_j^* \otimes p(X)_j^* \right) (\mathfrak{H}^{-1} \otimes I) \\
&= (\mathfrak{H}^{-1} \otimes I) L_B(p(X)) (\mathfrak{H}^{-1} \otimes I).
\end{aligned}$$

Since $\mathfrak{H}^{-1} \otimes I$ is invertible, $L_F(\tilde{p}(X)) \succeq 0$ if and only if $L_B(p(X)) \succeq 0$. Assuming $p : \mathcal{D}_A \to \mathcal{D}_B$ and $X \in \mathcal{D}_A$, it follows that $\tilde{p}(X) \in \mathcal{D}_F$.

Next we note that $\tilde{p}(0) = (-b + p(0))M^{-1} = 0$ and that $\ell'(x) = M^{-1}$. Hence, with $P = \begin{pmatrix} I & 0 \end{pmatrix}$,

$$\tilde{p}'(0) = \ell'\big(p(0)\big)p'(0) = M^{-1}p'(0)$$

and thus, $\tilde{p}'(0) = p(0)M^{-1} = PMM^{-1} = P$. ∎

Remark 8.3. In Theorem 1.5, without the assumption $p'(0) = I$, the remaining hypotheses imply $p'(0)$ is invertible [HKM11, Theorem 3.4]. Applying Proposition 8.2 with $b = 0$ and $M = p'(0)$, gives $\mathfrak{H} = I$ and $F = p'(0)B$. Moreover, since B is sv-generic, so is F. The resulting $\tilde{p}$ thus does satisfy the hypotheses of Theorem 1.5. It is now just a matter of undoing the linear change of variables that sent B to F. ◊

8.3. **Change of basis in the $\mathcal{R}$ module generated by $\mathcal{A}$.** In the context of the results of Theorem 1.5, the formula for the convexotonic mapping p depends (only) upon the structure matrices Ξ for the module generated by the tuple A over the algebra generated by the tuple $R = (C - I)A$ with respect to the basis implicitly given by $A = (A_1, \dots, A_g)$. We now see that a linear change of the A variables produces a simple linear "similarity" transform $\tilde{p}$ of the mapping p.

Starting with the identity of equation (6.4), consider a linear change of variables determined by an invertible matrix $M \in \mathbb{C}^{g \times g}$. That is, $\tilde{A} = MA$ where A is regarded as the column of matrices $\begin{pmatrix} A_1 \\ \vdots \\ A_g \end{pmatrix}$, so $\tilde{A}_i = \sum_{j=1}^{g} M_{ij} A_j$ for $i = 1, \dots, g$. The matrix M implements a change of basis on the span of $\{A_1, \dots, A_g\}$. We emphasize that the vectors of variables and maps are row vectors. Observe, in view of equation (6.4),

$$\begin{aligned} \tilde{A}(C - I)\tilde{A}_j &= MA(C - I)(\sum_{k=1}^{g} M_{jk}A_k) = M(\sum_{k=1}^{g} M_{jk}(A(C - I)A_k)) \\ &= M(\sum_{k=1}^{g} M_{jk}(\Xi_k A)) = (M(\sum_{k=1}^{g} M_{jk}\Xi_k)M^{-1})\tilde{A}. \end{aligned}$$

Thus, $(\tilde{A}, C)$ satisfy the hypotheses of Proposition 6.6 with structure matrices

$$\widetilde{\Xi}_j = M(\sum_{k=1}^{g} M_{jk}\Xi_k)M^{-1}.$$

Concretely,

$$(\widetilde{\Xi}_j)_{s,q} = \sum_{t,k,p} M_{s,t}M_{j,k}(\Xi_k)_{t,p}(M^{-1})_{p,q}. \tag{8.1}$$

8.3.1. *Computation of the mappings after linear change of coordinates.* Recall the convexotonic mapping $p(y) = y(1 - \sum_{i=1}^{g} y_i\, \Xi_i)^{-1}$ associated to the convexotonic tuple Ξ. We now look at the effect of the linear change of variable implemented by M on p. The rational function $\tilde{p}$ determined by $\widetilde{\Xi}_k$ of equation (8.1), is

$$\begin{aligned} \tilde{p}(y) &= y(1 - \sum_{i=1}^{g} y_i \widetilde{\Xi}_i)^{-1} = y(1 - \sum_{i=1}^{g} y_i M(\sum_{k=1}^{g} M_{ik}\Xi_k)M^{-1})^{-1} \\ &= yM(1 - \sum_{k=1}^{g}(\sum_{i=1}^{g} y_i M_{ik})\Xi_k)^{-1}M^{-1}. \end{aligned}$$

Note that $\sum_{i=1}^{g} y_i M_{ik}$ is the k-th column of yM, thus $\tilde{p}(y) = p(yM)M^{-1}$. Similarly for $\tilde{p}$ inverse, denoted $\tilde{q}$, so we can summarize this as

$$\tilde{p}(y) = p(yM)M^{-1}, \qquad \tilde{q}(y) = q(yM)M^{-1}.$$

(For an example see (8.2) below.)

The following proposition summarizes the mapping implications of this change of variable.

Proposition 8.4. *If $p : \mathcal{D}_A \to \mathcal{D}_B$, then $\tilde{p} : \mathcal{D}_{\tilde{A}} \to \mathcal{D}_{\tilde{B}}$ where*

$$\mathcal{D}_{\tilde{A}} := \{y : yM \in \mathcal{D}_A\}, \qquad \mathcal{D}_{\tilde{B}} := \{z : zM \in \mathcal{D}_B\}$$

Proof. Given $y \in \mathcal{D}_{\tilde{A}}$, set $yM = x$, which by definition is in $\mathcal{D}_A$. By the formula above $\tilde{p}(y) = p(x)M^{-1} =: z$. Thus $zM = p(x) \in \mathcal{D}_B$, hence by definition of $\mathcal{D}_{\tilde{B}}$, we have $\tilde{p}(y) = z \in \mathcal{D}_{\tilde{B}}$. ■

sec:compose

8.4. **Composition of convexotonic maps is not necessarily convexotonic.** Suppose $p : \mathcal{D}_A \to \mathcal{D}_B$ and $q : \mathcal{D}_B \to \mathcal{D}_E$ are convexotonic maps between the spectrahedral pairs $(\mathcal{D}_A, \mathcal{D}_B)$ and $(\mathcal{D}_B, \mathcal{D}_E)$. In particular, the pairs (A, B) and (B, E) must satisfy rather stringent algebraic conditions. In this case, generically $q \circ p$ is again convexotonic by Theorem 7.10. On the other hand, in general, given convexotonic maps p and q (without specifying domains and codomains), there is no reason to expect that the composition $q \circ p$ is convexotonic. Indeed the following example shows it need not be the case.

Let $g = 2$ and let p be the indecomposable convexotonic map of Type I from Section 9. Let $\tilde{p}$ denote the convexotonic map

$$\tilde{p} = (x_1 + x_2^2, x_2).$$

It can be obtained by reversing the roles of x_1 and x_2 in p or observing it belongs to the convexotonic tuple

$$\Xi_1 = 0, \quad \Xi_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In terms of the formalism in Subsection 8.3, consider the change of basis matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and note

eq:chCT

$$\tilde{p}(x, y) = p(y, x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x + y^2 & y \end{pmatrix}. \tag{8.2}$$

Now

$$p(\tilde{p}(x, y)) = \begin{pmatrix} x + y^2 & y + x^2 + xy^2 + y^2x + y^4 \end{pmatrix}$$

is not convexotonic by Proposition 6.3, since it is a polynomial of degree exceeding two.

sec:examples

9. Constructing all Convexotonic Maps

To construct all convexotonic maps in g variables first one lists the indecomposable ones, i.e., those associated with an indecomposable algebra. Then build general convexotonic maps as direct sums of these. We illustrate this by giving all convexotonic maps in dimension 2 in Subsections 9.1 and 9.2. Finally, in Subsection 9.3 we show how the automorphisms of the complex wild ball $\sum X_j^* X_j \preceq I$ are, after affine linear changes of variables, convexotonic.

sec:two dim

9.1. Convexotonic maps for $g = 2$. In very small dimensions $g \leq 5$ indecomposable algebras are classified [Maz79]. We work out the corresponding convexotonic maps for $g = 2$. The following is the list of indecomposable two-dimensional algebras over $\mathbb{C}$ (with basis R_1, R_2).

notation	nonzero products			properties	
I	$R_1^2 = R_2$			commutative	nilpotent
II	$R_1^2 = R_1$	$R_1R_2 = R_2$			
III	$R_1^2 = R_1$	$R_2R_1 = R_2$			
IV	$R_1^2 = R_1$	$R_1R_2 = R_2$	$R_2R_1 = R_2$	commutative	with identity

Accordingly we refer to these as algebras of type I – IV.

9.1.1. *Type I.* If R_1 is nilpotent of order 3, then $\Xi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \Xi_2 = 0$. These structure matrices produce the convexotonic maps

$$p(x_1, x_2) = \begin{pmatrix} x_1 & x_2 + x_1^2 \end{pmatrix} \qquad q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 - x_1^2 \end{pmatrix}.$$

Note $p(0) = 0$, $p'(0) = I$ and likewise for q.

9.1.2. *Type II.* Let $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The corresponding structure matrices Ξ are $\Xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So

$$p(x) = \begin{pmatrix} (1 - x_1)^{-1}x_1 & (1 - x_1)^{-1}x_2 \end{pmatrix} \qquad q(x) = \begin{pmatrix} (1 + x_1)^{-1}x_1 & (1 + x_1)^{-1}x_2 \end{pmatrix}.$$

9.1.3. *Type III.* Let $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The structure matrices are $\Xi_1 = I_2$, $\Xi_2 = 0$. So

$$p(x) = \begin{pmatrix} x_1(1 - x_1)^{-1} & x_2(1 - x_1)^{-1} \end{pmatrix} \qquad q(x) = \begin{pmatrix} x_1(1 + x_1)^{-1} & x_2(1 + x_1)^{-1} \end{pmatrix}.$$

9.1.4. *Type IV.* Take $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The corresponding structure matrices are $\Xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So

$$\begin{aligned} p(x) &= \begin{pmatrix} x_1(1 - x_1)^{-1} & (1 - x_1)^{-1}x_2(1 - x_1)^{-1} \end{pmatrix} \\ q(x) &= \begin{pmatrix} x_1(1 + x_1)^{-1} & (1 + x_1)^{-1}x_2(1 + x_1)^{-1} \end{pmatrix}. \end{aligned}$$

The other indecomposable convexotonic maps correspond to these after a linear change of basis, cf. Section 8.3. If the change of basis corresponds to an invertible 2×2 matrix M, then the corresponding convexotonic map is

$$\tilde{p}(x) = p(xM)M^{-1}.$$

c:decompose

9.2. Convexotonic maps associated to decomposable algebras. Here we explain which convexotonic maps arise from decomposable algebras. Suppose $\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}''$ and $\mathcal{R}', \mathcal{R}''$ are indecomposable finite-dimensional algebras. Let $\{R_1, \ldots, R_g\}$ be a basis for $\mathcal{R}'$ and let $\{R_{g+1}, \ldots, R_h\}$ be a basis for $\mathcal{R}''$. Then $\{R_1 \oplus 0, \ldots, R_g \oplus 0, 0 \oplus R_{g+1}, \ldots, 0 \oplus R_h\}$ is a basis for $\mathcal{R}$ with the corresponding structure matrices

$$\Xi_j = \begin{cases} \Xi_j' \oplus 0 & : j \leq g \\ 0 \oplus \Xi_j'' & : j > g, \end{cases}$$

where Ξ'_j and Ξ''_j denote the structure matrices for $\mathcal{R}'$ and $\mathcal{R}''$, respectively. The convexotonic map corresponding to $\mathcal{R}$ is

$$\begin{aligned} p_{\mathcal{R}}(x) &= \begin{pmatrix} x_1 & \cdots & x_g & x_{g+1} & \cdots & x_h \end{pmatrix} \Big(I - \sum_{j=1}^{h} \Xi_j x_j \Big)^{-1} \\ &= \begin{pmatrix} x_1 & \cdots & x_g & x_{g+1} & \cdots & x_h \end{pmatrix} \begin{pmatrix} I - \sum_{j=1}^{g} \Xi'_j x_j & 0 \\ 0 & I - \sum_{j=g+1}^{h} \Xi''_j x_j \end{pmatrix}^{-1} \\ &= \begin{pmatrix} p_{\mathcal{R}'}(x_1, \dots, x_g) & p_{\mathcal{R}''}(x_{g+1}, \dots, x_h) \end{pmatrix}. \end{aligned}$$

9.3. **Biholomorphisms of balls.** In this subsection we show how (linear fractional) biholomorphisms of balls in $\mathbb{C}^g$ can be presented using convexotonic maps. Let $\{\hat{e}_1, \dots, \hat{e}_{g+1}\}$ denote the standard basis of row vectors for $\mathbb{C}^{g+1}$ and let $A_j = \hat{e}_1^* \hat{e}_{j+1}$ for $j = 1, \dots, g$. Since $\mathcal{D}_A(1) = \{z \in \mathbb{C}^g : \sum_j |z_j|^2 \leq 1\}$ is the unit ball in $\mathbb{C}^g$, the free spectrahedron $\mathcal{D}_A$ is a free version of the ball. That is, $\mathcal{D}_A = \{X : \sum X_j^* X_j \preceq I\}$ consisting of all row contractions. Fix a (row) vector $v \in \mathbb{C}^g$ with $\|v\| < 1$ and let $xv^* = \sum \overline{v}_j x_j$, where $x = (x_1, \dots, x_g)$ is a row vector of free variables. By [Pop10], up to rotation, automorphisms of $\mathcal{D}_A$ have the form,

$$\mathcal{F}_v(x) = v - \big(1 - vv^*\big)^{\frac{1}{2}} \big(1 - xv^*\big)^{-1} x \big(I - v^* v\big)^{\frac{1}{2}}.$$

Modulo affine linear transformations, $\mathcal{F}_v$ is of the form

$$\big(1 - xv^*\big)^{-1} x = \begin{pmatrix} (1 - xv^*)^{-1} x_1 & \cdots & (1 - xv^*)^{-1} x_g \end{pmatrix}$$

since $\big(1 - vv^*\big)^{\frac{1}{2}}$ is a number and $\big(I - v^* v\big)^{\frac{1}{2}}$ is a matrix independent of x. Further,

$$\big(1 - xv^*\big)^{-1} x = x\big(I - v^* x\big)^{-1}.$$

Now let Ξ denote the g-tuple of $g \times g$ matrices $\Xi_j = e_j \otimes v^*$, where $\{e_1, \dots, e_g\}$ is the standard basis of row vectors for $\mathbb{C}^g$. Then $\Xi_j \Xi_k = \overline{v}_j \Xi_k = \sum_s (\Xi_k)_{j,s} \Xi_s$, so the tuple Ξ satisfies (1.2); i.e., it is convexotonic. Moreover,

$$x(I - \lambda_\Xi(x))^{-1} = x(I - v^* x)^{-1} = (1 - xv^*)^{-1} x.$$

Thus $\mathcal{F}_v$ is a convexotonic map.

10. Bianalytic Spectrahedra that are not Affine Linearly Equivalent

In this section we present bounded free spectrahedra that are polynomially equivalent, but not affine linearly equivalent (over $\mathbb{C}$).

Suppose A and B are eig-generic tuples, $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded and there is a polynomial bianalytic map $p : \mathcal{D}_A \to \mathcal{D}_B$ with $p(x) = x + h(x)$ (h for higher order terms). In particular, by Theorem 6.7, A and B have the same size and $B = W^* V A W$ for unitaries V and W. Further, there is a representation for p in terms of the g-tuple Ξ of $g \times g$ matrices determined by

$$A_k (V - I) A_j = \sum_{s=1}^{g} (\Xi_j)_{ks} A_s. \qquad (10.1)$$

10.1. **A class of examples.** Let Q be an invertible 2×2 matrix, so that

$$\mathcal{D}_Q(1) = \big\{c \in \mathbb{C} : I_2 + \big((cQ)^* + cQ\big) \succeq 0\big\}$$

is bounded. Choose P_{12}, P_{21}, P_{22} invertible, same size as Q with $P_{21}P_{12} = -2Q$. Now let

eq:thisisA
$$A_1 = \begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}. \tag{10.2}$$

Given γ unimodular, let

eq:thisisVg
$$V_\gamma = \begin{pmatrix} \gamma I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{10.3}$$

prop:VA **Proposition 10.1.** *With notation as above,*

eq:VA
$$A_k(V_\gamma - I)A_j = \sum_{s=1}^{2} (\Xi_j)_{ks} A_s, \tag{10.4}$$

where $\Xi = (\Xi_1, \Xi_2)$ *is the tuple defined by*

$$\Xi_1 = \begin{pmatrix} 0 & -2(\gamma-1) \\ 0 & 0 \end{pmatrix}$$

and $\Xi_2 = 0$*. Thus the polynomial mapping*

$$p_\gamma(x_1,x_2) = x(I - \Lambda_\Xi(x))^{-1} = (x_1, x_2 + 2(1-\gamma)x_1^2)$$

is a bianalytic $p : \mathcal{D}_A \to \mathcal{D}_B$ *with* $B = V_\gamma A$.

Moreover, if $P_{22} = \alpha_1 Q + \alpha_3(P_{12}^* P_{12} + P_{21}P_{21}^*)$*, then for each unimodular* φ*,*

$$\begin{aligned} s_\varphi = {}& (\varphi x_1,\ -(1-\varphi)\left(4\overline{\alpha_3}\varphi - \alpha_1\right)x_1 + x_2 + 2(1-\varphi^2)x_1^2) \\ & + (\overline{\alpha_3}(1-\varphi),\ -\overline{\alpha_3}(1-\varphi)\left(2\overline{\alpha_3}(1-\varphi) + \alpha_1\right)). \end{aligned}$$

is a polynomial automorphism of $\mathcal{D}_A$.

Proof. Equation (eq:VA 10.4) follows from the computations, $(V_\gamma - I)A_2 = 0 = A_2(V_\gamma - I)$ and

$$A_1(V_\gamma - I)A_1 = -2(\gamma-1)A_2.$$

The converse portion of Theorem thm:shotinthedark 6.7 now implies that

$$\begin{aligned} p = x(I - \Lambda_\Xi(x))^{-1} &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2(\gamma-1)x_1 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2(1-\gamma)x_1 \\ 0 & 1 \end{pmatrix} = (x_1, 2(1-\gamma)x_1^2 + x_2) \end{aligned}$$

is bianalytic between $\mathcal{D}_A$ and $\mathcal{D}_{V_\gamma A}$ as claimed.

To prove the second part of the proposition, suppose φ is unimodular. Let $\delta = \overline{\alpha_3}(1-\varphi)$ and $\eta = -4\varphi\delta + (1-\varphi)\alpha_1$ and let ρ denote the affine linear polynomial,

$$\rho(x_1,x_2) = (\varphi x_1,\ x_2 + \eta x_1) + \delta(1,\ 2\delta - \alpha_1).$$

With these notations,

$$L_A(\rho(x_1,x_2)) = L_A(\rho(0,0)) + (\varphi A_1 + \eta A_2)x_1 + A_2 x_2 + (\overline{\varphi}A_1^* + \overline{\eta}A_2^*)x_1^* + A_2^* x_2^*$$

and, using $P_{22}-\alpha_1 Q=\alpha_3(P_{12}^*P_{12}+P_{21}P_{21}^*)$,

$$\begin{aligned}L_A(\rho(0,0)) &= I+\delta A_1+\delta^* A_1^*+(2\delta^2-\delta\alpha_1)A_2+\overline{(2\delta^2-\delta\alpha_1)}A_2^*\\ &=\begin{pmatrix} I & \delta P_{12}+\overline{\delta}P_{21}^* \\ \overline{\delta}P_{12}^*+\delta P_{21} & \delta\alpha_3(P_{12}^*P_{12}+P_{21}P_{21}^*)+\overline{\delta\alpha_3}(P_{12}^*P_{12}+P_{21}P_{21}^*)-2\delta^2 Q-2(\overline{\delta})^2Q^*\end{pmatrix}\\ &=\mathscr{Y}^*\mathscr{Y},\end{aligned}$$

where

$$\mathscr{Y}=\begin{pmatrix}\varphi I & \varphi\big(\delta P_{12}+\overline{\delta}P_{21}^*\big)\\ 0 & I\end{pmatrix}.$$

Indeed, the only entry of this equality that is not immediate occurs in the $(2,2)$ entry. Since φ is unimodular, $|\delta|^2=\alpha_3\delta+\overline{\delta\alpha_3}$ and thus the $(2,2)$ entry of $\mathscr{Y}^*\mathscr{Y}$ is

$$\begin{aligned}I+\big(\delta P_{12}+\overline{\delta}P_{21}^*\big)^*\big(\delta P_{12}+\overline{\delta}P_{21}^*\big) &= I+|\delta|^2\big(P_{12}^*P_{12}+P_{21}P_{21}^*\big)+\delta^2P_{21}P_{12}+\overline{\delta}^2P_{12}^*P_{21}^*\\ &= I+\delta\alpha_3(P_{12}^*P_{12}+P_{21}P_{21}^*)+\overline{\delta\alpha_3}(P_{12}^*P_{12}+P_{21}P_{21}^*)-2\delta^2Q-2(\overline{\delta})^2Q^*,\end{aligned}$$

where $P_{21}P_{12}=-2Q$ was also used.

Next, let $B=V_\gamma A$, where $\gamma=\varphi^2$. For notational ease let $Y=\delta P_{12}+\overline{\delta}P_{21}^*$ and verify

$$\begin{aligned}\varphi[P_{21}Y+Y^*P_{12}]+P_{22} &= \varphi[\delta(P_{21}P_{12}+P_{21}P_{12})+\delta^*(P_{12}P_{12}^*+P_{21}P_{21}^*)]+P_{22}\\ &= \varphi[-4\delta Q+(1-\overline{\varphi})\alpha_3(P_{12}P_{12}^*+P_{21}P_{21}^*)]+P_{22}\\ &= -4\varphi\delta Q+(\varphi-1)(P_{22}-\alpha_1 Q)+P_{22}\\ &= ((1-\varphi)\alpha_1-4\varphi\delta)Q+\varphi P_{22}\\ &= \eta Q+\varphi P_{22}.\end{aligned}$$

Hence,

$$\mathscr{Y}^*B_1\mathscr{Y}=\begin{pmatrix}0 & \varphi P_{12}\\ P_{21} & \varphi[P_{21}Y+Y^*P_{12}]+P_{22}\end{pmatrix}=\varphi A_1+\eta A_2.$$

Likewise,

$$\mathscr{Y}^*B_2\mathscr{Y}=\mathscr{Y}^*A_2\mathscr{Y}=A_2.$$

It follows that

$$L_A(\rho(x))=\mathscr{Y}^*L_B(x)\mathscr{Y}$$

and thus, as $\mathscr{Y}$ is invertible, $\rho=\rho_\varphi$ is a bianalytic affine linear map from $\mathcal{D}_B$ to $\mathcal{D}_A$. Thus, $\rho_\varphi\circ p_\gamma$ is a polynomial automorphism of $\mathcal{D}_A$. Finally, since

$$\rho_\varphi\circ p_\gamma(x)=s_\varphi(x),$$

the proof of the proposition is complete. ■

The next objective is to establish a converse of Proposition prop:VA 10.1 under some mild additional assumptions on P_{ij} and Q. As a corollary, we produce examples of tuples A and B such that $\mathcal{D}_A$ and $\mathcal{D}_B$ are polynomially, but not linearly, bianalytic.

thm:PQ **Theorem 10.2.** *Suppose $\{P_{12}^*P_{12},P_{21}P_{21}^*\}$ is linearly independent and $\mathcal{D}_Q$ is bounded. In this case $\mathcal{D}_A$ is bounded.*

Suppose further that A is eig-generic and $*$-generic and either B is eig-generic or has size 4 (the same size as A).

it:bddpq (1) *If $p:\mathcal{D}_A\to\mathcal{D}_B$ is a polynomial bianalytic map with $p(x)=x+h(x)$, then there is a unimodular γ such that, up to unitary equivalence, $B=V_\gamma A$ and*

$$p=(x_1,x_2+2(1-\gamma)x_1^2).$$

Now suppose further that $\{Q, Q^*, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ *is linearly independent, there is a* $c \neq 0$ *so that* $P_{21}^* + cP_{12}$ *is not invertible but* $P_{21} - cP_{12}$ *is invertible.*

(2) *If* $\{Q, P_{22}, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ *is linearly independent, then* $\mathcal{D}_A$ *has no non-trivial polynomial automorphisms: if* $q : \mathcal{D}_A \to \mathcal{D}_A$ *is a bianalytic polynomial, then* $q(x) = x$.

(3) *If* $P_{22} = \alpha_1 Q + \alpha_2 Q^* + \alpha_3 P_{12}^*P_{12} + \alpha_4 P_{21}P_{21}^*$, *then either*

   (a) $\alpha_2 \neq 0$ *and conclusion of item* (2) *holds; or*

   (b) $\alpha_2 = 0$ *in which case* $\alpha_3 = \alpha_4$ *and a polynomial automorphism* s *of* $\mathcal{D}_A$ *must have the form*

$$s = s_\varphi = \left(\varphi x_1,\ -(1-\varphi)\,(4\overline{\alpha_3}\varphi - \alpha_1)\,x_1 + x_2 + 2(1-\varphi^2)x_1^2\right) \\ + \left(\overline{\alpha_3}(1-\varphi),\ -\overline{\alpha_3}(1-\varphi)\,(2\overline{\alpha_3}(1-\varphi) + \alpha_1)\right)$$

   for some unimodular φ.

Remark 10.3. Of course the polynomial automorphisms of $\mathcal{D}_A$ form a group under composition. In fact, as is straightforward to verify, $s_\varphi \circ s_\psi = s_{\varphi\psi}$. Further, combining items (3) and (1) of Theorem 10.2, produces a parameterization of all bianalytic polynomials $\mathcal{D}_A$ to $\mathcal{D}_B$ (under the prevailing assumptions on A and B). ♢

Example 10.4. As a concrete example, choose

$$Q = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

We note that $xQ + \overline{x}Q$ has both positive and negative eigenvalues for $x \neq 0$, so $\mathcal{D}_Q$ is bounded. Let

$$P_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{21} = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, \quad P_{22} = I_2,$$

and writing A as was done above we claim A, as described in equation (10.2) is eig-generic. Furthermore $\{Q, Q^*, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ and $\{Q, P_{22}, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ are linearly independent, so Theorem 10.2 applies, thus $p(x) = (x_1, x_2 + 4x_1^2)$ is the unique bianalytic map between $\mathcal{D}_A$ and $\mathcal{D}_B$, where $B = V_{-1}A$ and V_{-1} is defined by equation (10.3). In particular, $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded and polynomially equivalent, but they are not affine linearly equivalent.

Alternatively, let

$$P_{22} = \begin{pmatrix} 10 & -1 \\ -1 & 2 \end{pmatrix} = P_{21}P_{21}^* + P_{12}^*P_{12},$$

then we have a form for $q(x)$ and a class of affine linear automorphisms of $\mathcal{D}_A$.

Finally, letting $P_{22} = 0$, we get our family of automorphisms of $\mathcal{D}_A$ parameterized by the unimodular complex numbers. ♢

10.2. **The proof of Theorem 10.2.** Before turning to the proof of Theorem 10.2 proper, we record a few preliminary results.

Proposition 10.5. *Let* L *be a linear pencil. If* $\mathcal{D}_L$ *is bounded, then* $\mathcal{D}_L(1)$ *is bounded. Conversely, if* $\mathcal{D}_L$ *is not bounded, then there exists* $\alpha \in \mathbb{C}^g$ *such that* $t\alpha \in \mathcal{D}_L(1)$ *for all* $t \in \mathbb{R}_{>0}$.

Proof. This result is the complex version of the full strength of [HKM13, Proposition 2.4]. Unfortunately, the statement of the result there is weaker than what is actually proved. Simply note that over the complex numbers, if T is a matrix and $\langle T\gamma, \gamma\rangle = 0$ for all vectors γ, then, by polarization, $\langle T\gamma, \delta\rangle = 0$ for all vectors γ, δ and hence $T = 0$. (By comparison, over the real numbers the same conclusion holds provided T is self-adjoint.) ■

Corollary 10.6. *Let* L *be a monic linear pencil with truly linear part* Λ. *Thus,* $L(x) = I + \Lambda(x) + \Lambda(x)^*$. *The domain* $\mathcal{D}_L$ *is bounded if and only if* $\Lambda(\alpha)$ *has both positive and negative eigenvalues for each* $\alpha \in \mathbb{C}^g \setminus \{0\}$.

Proof of Theorem 10.2. First observe that independence of $\{P_{12}^*P_{12}, P_{21}P_{21}^*\}$ implies independence of $\{P_{12}, P_{21}^*\}$ since $P_{12} = tP_{21}^*$ implies $P_{12}^*P_{12} = |t|^2 P_{21}P_{21}^*$. Let

$$Z = xA_1 + \overline{x}A_1^* + yA_2 + \overline{y}A_2^* = \begin{pmatrix} 0 & M \\ M^* & N \end{pmatrix}.$$

We claim Z has both positive and negative eigenvalues, provided not both x and y are 0.

In the case $M \neq 0$, the matrix Z has both positive and negative eigenvalues. Note $M = xP_{12} + \overline{x}P_{21}^*$ and by independence $M = 0$ if and only if $x = 0$. In the case $x = 0$, (and thus $y \neq 0$), $N = yQ + (yQ)^*$. Since, by hypothesis, $\mathcal{D}_Q$ is bounded, Corollary 10.6 implies N has both positive and negative eigenvalues. Therefore, once again by Corollary 10.6, $\mathcal{D}_A$ is a bounded domain.

To prove item (1), observe the hypotheses (and they imply $\mathcal{D}_A$ is bounded) allow the application of Corollary 7.11. In particular, there exists a unitary V satisfying equation (10.1) and, in terms of the tuple Ξ of structure matrices,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1}.$$

Write $V = (V_{jk})$ as a 2×2 matrix to match the 2×2 block structure of A. Straightforward computation gives,

$$(V - I)A_2 = \begin{pmatrix} 0 & V_{12}Q \\ 0 & (V_{22} - I)Q \end{pmatrix}.$$

Hence

$$A_1(V - I)A_2 = \begin{pmatrix} 0 & P_{12}(V_{22} - I)Q \\ 0 & P_{21}V_{21}Q \end{pmatrix}.$$

Since P_{12} and Q are invertible and, by equation (10.1), $A_1(V - I)A_2$ lies in the span of $\{A_1, A_2\}$, it follows that $V_{22} - I = 0$. Since V is unitary, $VV^* = I$. Thus

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & I \end{pmatrix} \begin{pmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & I \end{pmatrix} = \begin{pmatrix} V_{11}V_{11}^* + V_{12}V_{12}^* & V_{11}V_{21}^* + V_{12} \\ V_{21}V_{11}^* + V_{12}^* & V_{21}V_{21}^* + I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

It follows that $V_{21} = 0$ and thus $V_{12} = 0$ as well. Finally,

$$A_1(V - I)A_1 = \begin{pmatrix} 0 & 0 \\ 0 & P_{21}(V_{11} - I)P_{12} \end{pmatrix}.$$

Hence equation (10.1) holds in this case ($j = 1 = k$) if and only if there is a λ such that $P_{21}(V_{11} - I)P_{12} = \lambda Q$ (note that $A_1(V - I)A_1 = \delta A_1 + \lambda A_2$, but $\delta = 0$). Since $P_{21}P_{12} = -2Q$, it follows that

$$V_{11} - I = \lambda P_{21}^{-1}QP_{12}^{-1} = -\frac{1}{2}\lambda I.$$

Thus $V_{11} = (1 - \frac{1}{2}\lambda)I$ and $|1 - \frac{1}{2}\lambda| = 1$. Hence,

$$V = V_\gamma = \begin{pmatrix} \gamma I & 0 \\ 0 & I \end{pmatrix}$$

for some unimodular γ. The tuple Ξ of structure matrices and polynomial p are thus described in Proposition 10.1.

Turning to the second part of the theorem, suppose $q : \mathcal{D}_A \to \mathcal{D}_A$ is a polynomial automorphism. Let $b = q(0)$ and let $\mathcal{H}$ denote the positive square root of $L_A(b)$. By Proposition 8.2, there exist F and a bianalytic polynomial $\tilde{q} : \mathcal{D}_A \to \mathcal{D}_F$ with $\tilde{q}(x) = (-b + q(x))q'(0)^{-1}$, such that $\tilde{q}(0) = 0$, $\tilde{q}'(0) = I$,

$$A_j = (q'(0)^{-1})_{j,1}\mathcal{H}F_1\mathcal{H} + \cdots + (q'(0)^{-1})_{j,g}\mathcal{H}F_g\mathcal{H}$$

and

$$F_j = q'(0)_{j,1}\mathcal{H}^{-1}A_1\mathcal{H}^{-1} + \cdots + q'(0)_{j,g}\mathcal{H}^{-1}A_g\mathcal{H}^{-1}.$$

Now F is the same size as A and A is eig-generic and A^* is $*$-generic, hence we can apply item (1) to the bianalytic polynomial $\tilde{q} : \mathcal{D}_A \to \mathcal{D}_F$. In particular, there is a unimodular γ such that $F = V_\gamma A$ and $\tilde{q} = (x_1, x_2 + 2(1-\gamma)x_1^2)$. By Proposition 8.2,

$$A_i = \sum_{j=1}^{g} \left(q'(0)^{-1}\right)_{i,j} \mathcal{H} F_j \mathcal{H}.$$

Since $F_j = \mathcal{V}^* V A_j \mathcal{V}$,

$$\mathscr{H}^{-*} A_i \mathscr{H}^{-1} = \sum_{j=1}^{g} \left(q'(0)^{-1}\right)_{i,j} V A_j,$$

where $\mathscr{H} = \mathcal{V}\mathcal{H}$. Setting

$$q'(0)^{-1} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix},$$

gives

$$\mathscr{H}^{-*} A_1 \mathscr{H}^{-1} = \mathscr{H}^{-*} \begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \mathscr{H}^{-1} = \begin{pmatrix} 0 & \lambda_1 \gamma P_{12} \\ \lambda_1 P_{21} & \mu_1 Q + \lambda_1 P_{22} \end{pmatrix} \tag{10.5}$$

and likewise

$$\mathscr{H}^{-*} A_2 \mathscr{H}^{-1} = \mathscr{H}^{-*} \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \mathscr{H}^{-1} = \begin{pmatrix} 0 & \lambda_2 \gamma P_{12} \\ \lambda_2 P_{21} & \mu_2 Q + \lambda_2 P_{22} \end{pmatrix}. \tag{10.6}$$

By equation (10.5), $\lambda_1 \neq 0$ since A_1 is invertible.

Let $Y = \mathscr{H}^{-1}$ and write $Y = (Y_{j,k})$ in the obvious way as a 2×2 matrix with 2×2 block entries. From equation (10.6),

$$\begin{pmatrix} Y_{21}^* Q Y_{21} & Y_{21}^* Q Y_{12} \\ * & Y_{22}^* Q Y_{22} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_2 \gamma P_{12} \\ \lambda_2 P_{21} & \mu_2 Q + \lambda_2 P_{22} \end{pmatrix}.$$

Thus, as Q is invertible, $Y_{21} = 0$ and therefore $\lambda_2 = 0$. We also record $Y_{22}^* Q Y_{22} = \mu_2 Q$ or equivalently $Y_{22}^* P_{21} P_{12} Y_{22} = \mu_2 P_{21} P_{12}$. Turning to equation (10.5),

$$\begin{pmatrix} 0 & Y_{11}^* P_{12} Y_{22} \\ Y_{22}^* P_{21} Y_{11} & Y_{22}^* P_{21} Y_{12} + Y_{12}^* P_{12} Y_{22} + Y_{22}^* P_{22} Y_{22} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \gamma P \\ \lambda_1 P & \mu_1 Q + \lambda_1 P_{22} \end{pmatrix}.$$

Hence,

$$\begin{aligned} \lambda_2 &= 0 \\ Y_{22}^* P_{21} P_{12} Y_{22} &= \mu_2 P_{21} P_{12} \\ Y_{11}^* P_{12} Y_{22} &= \lambda_1 \gamma P_{12} \\ Y_{22}^* P_{21} Y_{11} &= \lambda_1 P_{21} \\ Y_{22}^* P_{21} Y_{12} + Y_{12}^* P_{12} Y_{22} + Y_{22}^* P_{22} Y_{22} &= \mu_1 Q + \lambda_1 P_{22}. \end{aligned} \tag{10.7}$$

Taking determinants in the second of these equations gives $|\det(Y_{22})|^2 = \mu_2^2$ and therefore μ_2 is real. Taking determinants in the third and fourth equations gives $\overline{\lambda_1^2} = \lambda_1^2 \gamma^2$. Thus,

$$\overline{\lambda_1} = \pm \lambda_1 \gamma.$$

In particular,

$$|\overline{\lambda_1}|^2 = \pm \lambda_1^2 \gamma. \tag{10.8}$$

Multiplying the third equation on the left by the fourth equation gives

$$Y_{22}^* P_{21} Y_{11} Y_{11}^* P_{12} Y_{22} = \lambda_1^2 \gamma P_{21} P_{12}.$$

Using the second equation

$$Y_{22}^* P_{21} Y_{11} Y_{11}^* P_{12} Y_{22} = \frac{\lambda_1^2 \gamma}{\mu_2} Y_{22}^* P_{21} P_{12} Y_{22}.$$

Since $Y_{22}^* P_{21}$ and $P_{12} Y_{22}$ are invertible,

$$Y_{11} Y_{11}^* = \frac{\lambda_1^2 \gamma}{\mu_2} > 0.$$

In particular, Y_{11} is a multiple of a unitary.

Next multiply the fourth equation by its adjoint on the right to obtain

$$|\lambda_1|^2 P_{21} P_{21}^* = Y_{22}^* P_{21} Y_{11} (Y_{22}^* P_{21} Y_{11})^* = \frac{\lambda_1^2 \gamma}{\mu_2} Y_{22}^* P_{21} P_{21}^* Y_{22}.$$

Multiplying the third equation by its adjoint on the left gives (as $|\gamma| = 1$)

$$|\lambda_1|^2 P_{12}^* P_{12} = (Y_{11}^* P_{12} Y_{22})^* Y_{11}^* P_{12} Y_{22} = \frac{\lambda_1^2 \gamma}{\mu_2} Y_{22}^* P_{12}^* P_{12} Y_{22}.$$

In view of equation (10.8), if $\mu_2 > 0$, then $\lambda_1^2 \gamma = |\lambda_1|^2$ and if $\mu_2 < 0$, then $-\lambda_1^2 \gamma = |\lambda_1|^2$. Hence, with $|\kappa|^2 = |\mu_2|$ and $Z = \kappa Y_{22}$ either $\mu_2 > 0$ and

$$\begin{aligned} Z^* P_{21} P_{12} Z &= P_{21} P_{12} \qquad & Z^* P_{21} P_{21}^* Z &= P_{21} P_{21}^* \\ Z^* P_{12}^* P_{21}^* Z &= P_{12}^* P_{21}^* \qquad & Z^* P_{12}^* P_{12} Z &= P_{12}^* P_{12} \end{aligned}$$

or $\mu_2 < 0$ and

$$\begin{aligned} Z^* P_{21} P_{12} Z &= -P_{21} P_{12} \qquad & Z^* P_{21} P_{21}^* Z &= P_{21} P_{21}^* \\ Z^* P_{12}^* P_{21}^* Z &= -P_{12}^* P_{21}^* \qquad & Z^* P_{12}^* P_{12} Z &= P_{12}^* P_{12}. \end{aligned}$$

We will argue that this second case does not occur. Recall we are assuming P_{21} and P_{12} are both invertible. This implies Z is invertible. Observe that, assuming this second set of equations, for complex numbers c,

eq:pmc
$$Z^*((P_{12} + cP_{21}^*)^* (P_{12} + cP_{21}^*)) Z = (P_{12} - cP_{21}^*)^* (P_{12} - cP_{21}^*). \tag{10.9}$$

By assumption there is a $c \neq 0$ such that $P_{21}^* + cP_{12}$ is not invertible but $P_{21}^* - cP_{12}$ is invertible, leading to the contradiction that the left hand side of equation (10.9) is invertible, but the right hand side is not. It follows that $\mu_2 > 0$ and $\lambda_1^2 \gamma = |\lambda_1|^2$.

Assuming $\{Q, Q^*, P_{21} P_{21}^*, P_{12}^* P_{12}\}$ is linearly independent, this set spans the 2×2 matrices. Hence (using the fact that $A^* X A = X$ for all 2×2 matrices X implies A is a multiple of the identity) $Y_{22} = \kappa I$ for some κ with $|\kappa|^2 = \mu_2$. Hence many of the identities in equation (10.7) now imply that Y_{11} is also a multiple of the identity. For instance, using the third equality,

$$\lambda_1 P_{21} = Y_{22}^* P_{21} Y_{11} = \overline{\kappa} P_{21} Y_{11}$$

and hence $Y_{11} = \frac{\lambda_1}{\overline{\kappa}} I$.

Thus,

$$\mathscr{H}^{-1} = Y = \begin{pmatrix} \frac{\lambda_1}{\overline{\kappa}} I & Y_{12} \\ 0 & \kappa I \end{pmatrix}$$

and consequently

$$\mathcal{V}\mathcal{H} = \mathscr{H} = \begin{pmatrix} \frac{\overline{\kappa}}{\lambda_1} & -\frac{\overline{\kappa}}{\kappa \lambda_1} Y_{12} \\ 0 & \frac{1}{\kappa} \end{pmatrix}.$$

It follows that

$$\mathcal{H}^2 = \mathscr{H}^* \mathscr{H} = \begin{pmatrix} \frac{|\kappa|^2}{|\lambda_1|^2} I & -\frac{\bar{\kappa}}{|\lambda_1|^2} Y_{12} \\ -\frac{\kappa}{|\lambda_1|^2} Y_{12}^* & \frac{1}{|\kappa|^2} I + \frac{1}{|\lambda_1|^2} Y_{12}^* Y_{12} \end{pmatrix}$$

On the other hand,

$$\mathcal{H}^2 = L_A(b) = (I + \sum b_j A_j + \sum (b_j A_j)^*)$$
$$= \begin{pmatrix} I & b_1 P_{12} + \overline{b_1} P_{21}^* \\ b_1 P_{21} + \overline{b_1} P_{12}^* & I + b_2 Q + \overline{b_2} Q^* + b_1 P_{22} + \overline{b_1} P_{22}^* \end{pmatrix}.$$

It follows that,

eq:YP

$$\begin{aligned} \frac{|\kappa|^2}{|\lambda_1|^2} &= 1 \\ -\frac{\bar{\kappa}}{|\lambda_1|^2} Y_{12} &= b_1 P_{12} + \overline{b_1} P_{21}^* \\ \frac{1}{|\kappa|^2} I + \frac{1}{|\lambda_1|^2} Y_{12}^* Y_{12} &= I + b_2 Q + \overline{b_2} Q^* + b_1 P_{22} + \overline{b_1} P_{22}^*. \end{aligned} \tag{10.10}$$

Note that combining the first two of these equations gives,

eq:YP+

$$Y_{12} = -\kappa(b_1 P_{12} + \overline{b_1} P_{21}^*). \tag{10.11}$$

Since $Y_{22} = \kappa I$, the last equality in equation (10.7) gives

$$\overline{\kappa} P_{21} Y_{12} + \kappa Y_{12}^* P_{12} + |\kappa|^2 P_{22} = \mu_1 Q + \lambda_1 P_{22}.$$

It follows, using the second equality in equation (10.10),

$$\begin{aligned} \mu_1 Q + (\lambda_1 - |\kappa|^2) P_{22} &= -|\lambda_1|^2 P_{21}(b_1 P_{12} + \overline{b_1} P_{21}^*) - |\lambda_1|^2 (\overline{b_1} P_{12}^* + b_1 P_{21}) P_{12} \\ &= -|\lambda_1|^2 \left(b_1 P_{21} P_{12} + \overline{b_1} P_{21} P_{21}^* + \overline{b_1} P_{12}^* P_{12} + b_1 P_{21} P_{12} \right). \end{aligned}$$

Simplifying with $P_{21} P_{12} = -2Q$ and bringing to one side gives

eq:P lin com

$$0 = (\mu_1 + 4 b_1 |\lambda_1|^2) Q + \overline{b_1} |\lambda_1|^2 (P_{21} P_{21}^* + P_{12}^* P_{12}) + (\lambda_1 - |\kappa|^2) P_{22}. \tag{10.12}$$

We now proceed to prove item (2). Assuming $\{Q, P_{12}^* P_{12}, P_{21} P_{21}^*, P_{22}\}$ is linearly independent, equation (10.12) and the fact that $\lambda_1 \neq 0$ yields $b_1 = 0$. So $\mu_1 = 0$ and $\lambda_1 = |\kappa|^2$. Furthermore, $|\kappa| = |\lambda_1|$, implies $\lambda_1 = |\lambda_1|$. Hence $\lambda_1 = 1$, $|\kappa| = 1$ and $\gamma = 1$. It also follows that $Y_{12} = 0$ by the third equation in (10.10). Furthermore,

$$I = I + b_2 Q + \overline{b_2} Q^*,$$

so $b_2 = 0$, as $\{Q, Q^*\}$ is linearly independent. Hence $L_A(b) = I = \mathcal{H}$ and

$$\mathcal{V} = \overline{\kappa} I_4.$$

Finally, $Y_{12} = 0$ also implies $\mu_2 = 1$. Thus, $F = A$ and $V = V_\gamma = I$, $q(0) = 0$ and finally, $q'(0) = I$ too. Hence, $q(x) = x$ and the proof of item (2) is complete.

Moving on to item (3), assume now

$$P_{22} = \alpha_1 Q + \alpha_2 Q^* + \alpha_3 P_{12}^* P_{12} + \alpha_4 P_{21} P_{21}^*.$$

If $\alpha_2 \neq 0$ then $\{Q, P_{12}^* P_{12}, P_{21} P_{21}^*, P_{22}\}$ must also be linearly independent, and hence the conclusions of item (2) hold.

To complete the proof of the theorem, suppose $\alpha_2 = 0$ and recall equation (10.12),

$$\begin{aligned} 0 = &(\mu_1 + 4 b_1 |\lambda_1|^2 + \alpha_1(\lambda_1 - |\kappa|^2)) Q \\ &+ (\overline{b_1} |\lambda_1|^2 + \alpha_3(\lambda_1 - |\kappa|^2)) P_{12}^* P_{12} + (\overline{b_1} |\lambda_1|^2 + \alpha_4(\lambda_1 - |\kappa|^2)) P_{21} P_{21}^*. \end{aligned}$$

Since $\{Q, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent,

$$\begin{aligned}\mu_1 + 4b_1|\lambda_1|^2 + \alpha_1(\lambda_1 - |\lambda_1|^2) &= 0\\ \overline{b_1}|\lambda_1|^2 + \alpha_3(\lambda_1 - |\lambda_1|^2) &= 0\\ \overline{b_1}|\lambda_1|^2 + \alpha_4(\lambda_1 - |\lambda_1|^2) &= 0.\end{aligned}$$

It follows that $\alpha_3 = \alpha_4$ and

eq:b1

$$b_1 = \frac{\overline{\alpha_3}(\lambda_1 - 1)}{\lambda_1}. \tag{10.13}$$

Now, using equation (10.11) and looking at the third equation in equation (10.10),

$$\begin{aligned}\frac{1}{|\kappa|^2}I + \frac{|\kappa|^2}{|\lambda_1|^2}(|b_1|^2P_{12}^*P_{12} + 2b_1^2Q + 2\overline{b_1}^2Q^* + |b_1|^2P_{21}P_{21}^*)\\ = I + b_2Q + \overline{b_2}Q^* + b_1P_{22} + \overline{b_1}P_{22}^*.\end{aligned}$$

Using $P_{22} = \alpha_1 Q + \alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*)$,

eq:YComb

$$\begin{aligned}0 = \left(1 - \frac{1}{|\kappa|^2}\right) I + (b_2 - 2b_1^2 + \alpha_1 b_1)Q + (\overline{b_2} - 2\overline{b_1}^2 + \overline{\alpha_1}\overline{b_1})Q^*\\ + (b_1\alpha_3 + \overline{b_1}\overline{\alpha_3} - |b_1|^2)(P_{12}^*P_{12} + P_{21}P_{21}^*).\end{aligned} \tag{10.14}$$

Using equation (10.13),

$$\begin{aligned}b_1\alpha_3 + \overline{b_1}\overline{\alpha_3} - |b_1|^2 &= |\alpha_3|^2\left(\frac{\lambda_1 - 1}{\lambda_1} + \frac{\overline{\lambda_1} - 1}{\overline{\lambda_1}} - \frac{(\lambda_1 - 1)(\overline{\lambda_1} - 1)}{\lambda_1\overline{\lambda_1}}\right)\\ &= |\alpha_3|^2\left(1 - \frac{1}{|\lambda_1|^2}\right) = |\alpha_3|^2\left(1 - \frac{1}{|\kappa|^2}\right).\end{aligned}$$

Let $z = (b_2 - 2b_1^2 + \alpha_1 b_1)$, solving for the Q and Q^* terms, equation (10.14) becomes

$$zQ + \overline{z}Q^* = \left(1 - \frac{1}{|\kappa|^2}\right)\left(I + |\alpha_3|^2(P_{12}^*P_{12} + P_{21}P_{21}^*)\right)$$

Write $C = (1 - |\kappa|^{-2})$, let $t \in \mathbb{R}$ with $tC > 0$ and consider

$$L_Q(tz) = I + tzQ + t\overline{z}Q^* = (1 + tC)I + |\alpha_3|^2 tC(P_{12}^*P_{12} + P_{21}P_{21}^*).$$

But $P_{12}^*P_{12}, P_{21}P_{21}^* \succeq 0$, so $L_Q(tz) \succeq 0$ for all t with $tC > 0$, contradicting the boundedness of $\mathcal{D}_Q$. Hence both $z = 0$ and $C(I + |\alpha_3|^2(P_{12}^*P_{12} + P_{21}P_{21}^*)) = 0$. So either $C = 0$ or $I = -|\alpha_3|^2(P_{12}^*P_{12} + P_{21}P_{21}^*)$. However $I \succeq 0$ while $-|\alpha_3|^2(P_{12}^*P_{12} + P_{21}P_{21}^*) \preceq 0$, hence this second equality never holds. Thus $C = 0$.

It follows that $|\kappa|^2 = |\lambda_1|^2 = \mu_2 = 1$, so

$$\begin{aligned}b_1 &= \overline{\alpha_3}(1 - \overline{\lambda_1}) & \mu_1 &= -\lambda_1(1 - \overline{\lambda_1})\left(4\overline{\alpha_3}\overline{\lambda_1} + \alpha_1\right)\\ b_2 &= \overline{\alpha_3}(1 - \overline{\lambda_1})\left(2\overline{\alpha_3}(1 - \overline{\lambda_1}) - \alpha_1\right) & Y_{12} &= \kappa(\overline{\alpha_3}(1 - \overline{\lambda_1})P_{12} - \alpha_3(1 - \lambda_1)P_{21}^*),\end{aligned}$$

and

$$F_1 = \mathcal{H}^{-1}\left(\overline{\lambda_1}B_1 + (1 - \overline{\lambda_1})(4\overline{\alpha_3}\overline{\lambda_1} + \alpha_1)B_2\right)\mathcal{H}^{-1}, \qquad F_2 = \mathcal{H}^{-1}B_2\mathcal{H}^{-1}.$$

Recall,

$$q(0) = (b_1, b_2), \quad q'(0) = \begin{pmatrix}\lambda_1 & \mu_1\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\overline{\lambda_1} & -\overline{\lambda_1}\mu_1\\ 0 & 1\end{pmatrix},$$

so plugging in;

$$q(0) = \left(\overline{\alpha_3}\left(1 - \overline{\lambda_1}\right), \quad \overline{\alpha_3}\left(1 - \overline{\lambda_1}\right)\left(2\overline{\alpha_3}(1 - \overline{\lambda_1}) - \alpha_1\right)\right),$$

and

$$q'(0) = \begin{pmatrix} \overline{\lambda_1} & (1-\overline{\lambda_1})(4\overline{\alpha_3}\overline{\lambda_1}+\alpha_1) \\ 0 & 1 \end{pmatrix}.$$

Next, we know that $\ell(x) = (-b+x)q'(0)^{-1}$ and $\ell^{-1}(x) = xq'(0)+b$, , so yet again plugging in;

$$\ell(x) = (-\lambda_1 q(0)_1 + \lambda_1 x_1, \lambda_1 q'(0)_{1,2} - \lambda_1 q'(0)_{1,2}x_1 + x_2),$$
$$\ell^{-1}(x) = \left(q(0)_1 + \overline{\lambda_1}x_1, q(0)_2 + q'(0)_{1,2}x_1 + x_2\right).$$

Using the fact that $q = \ell^{-1}\circ\tilde{q}$,

$$q(x) = \ell^{-1}\left(\tilde{q}(x)\right) = \left(q(0)_1 + \overline{\lambda_1}x_1, \quad q(0)_2 + q'(0)_{1,2}x_1 + x_2 + 2(1-\overline{\lambda_1}^2)x_1^2\right).$$

Observe $q = q_{\overline{\lambda_1}}$ i.e. q depends upon the choice of the unimodular $\overline{\lambda_1}$. Thus, taking a unimodular ϕ and setting $s_\phi = q_\phi$,

$$s^1_\phi(x) = \overline{\alpha_3}(1-\phi) + \phi x_1$$
$$s^2_\phi(x) = -\overline{\alpha_3}(1-\phi)\left(2\overline{\alpha_3}(1-\phi)+\alpha_1\right) - (1-\phi)\left(4\overline{\alpha_3}\phi - \alpha_1\right)x_1 + x_2 + 2(1-\phi^2)x_1^2,$$

which by construction is an automorphism of $\mathcal{D}_A$. Moreover, if ψ is another unimodular, then

$$s_\phi \circ s_\psi = s_{\phi\psi}$$

These automorphisms must be the only automorphisms of $\mathcal{D}_A$, since if there were some other form of automorphism then by composing with q we would get a different form for a bianalytic polynomial from $\mathcal{D}_A$ to $\mathcal{D}_A$ which cannot happen. ■

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