Convexity and Semidefinite Programming in dimension-free matrix unknowns

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Summary. One of the main applications of semidefinite programming lies in linear systems and control theory. Many problems in this subject, certainly the textbook classics, have matrices as variables, and the formulas naturally contain non-commutative polynomials in matrices. These polynomials depend only on the system layout and do not change with the size of the matrices involved, hence such problems are called "dimension-free". Analyzing dimension-free problems has led to the development recently of a non-commutative (nc) real algebraic geometry (RAG) which, when combined with convexity, produces dimension-free Semidefinite Programming. This article surveys what is known about convexity in the non-commutative setting and nc SDP and includes a brief survey of nc RAG. Typically, the qualitative properties of the non-commutative case are much cleaner than those of their scalar counterparts - variables in $\mathbb{R}^g$. Indeed we describe how relaxation of scalar variables by matrix variables in several natural situations results in a beautiful structure.

1 Introduction

Given symmetric $\ell \times \ell$ symmetric matrices with real entries A_j, the expression

$$L(x) = I_\ell + \sum_{j=1}^{g} A_j x_j \succ 0 \tag{1}$$

is a **linear matrix inequality** (LMI). Here $\succ 0$ means positive definite, $x = (x_1, \ldots, x_g) \in \mathbb{R}^g$ and of interest is the set of solutions x. Taking advantage of the Kronecker (tensor) product $A \otimes B$ of matrices, it is natural to consider, for tuples of symmetric $n \times n$ matrices $X = (X_1, \ldots, X_g) \in (\mathbb{SR}^{n\times n})^g$, the inequality

$$L(X) = I_\ell \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j \succ 0. \tag{2}$$

For reasons which will become apparent soon, we call expression (2) a **non-commutative LMI** (nc LMI). Letting $\mathcal{D}_L(n)$ denote the solutions X of size $n \times n$, note that $\mathcal{D}_L(1)$ is the solution set of equation (1). In many areas of mathematics and its applications, the inequality (2) is called the **quantized** version of inequality (1).

Quantizing a polynomial inequality requires the notion of a non-commutative (free) polynomial which can loosely be thought of as a polynomial in matrix unknowns. Section 1.2 below gives the details on these polynomails. For now we limit the discussion to the example,

$$p(x, y) = 4 - x - y - (2x^2 + xy + yx + 2y^2). \tag{3}$$

Of course, for symmetric $n \times n$ matrices X, Y,

$$p(X, Y) = 4I_n - X - Y - (2X^2 + XY + YX + 2Y^2). \tag{4}$$

The set $\{(x, y) \in \mathbb{R}^2 : p(x, y) > 0\}$ is a semi-algebraic set. By analogy, the set $\{(X, Y) : p(X, Y) \succ 0\}$ is a **non-commutative semi-algebraic set**.

nc LMIs, and more generally non-commutative semi-algebraic sets, arise naturally in semidefinite programming (SDP) and in linear systems theory problems determined by a signal-flow diagram. They are of course basic objects in the study of operator spaces and thus are related to problems like Connes' embedding conjecture [Con76, KS08a] and the Bessis-Moussa-Villani (BMV) conjecture [BMV75] from quantum statistical mechanics [KS08b]. As is seen in Theorem 4 below, they even have something to say about their scalar (commutative) counterparts. For some these non-commutative considerations have their own intrinsic interest as a free analog to classical semi-algebraic geometry.

Non-commutative will often be shortened to nc.

1.1 The roadmap

The paper treats four areas of research concerning nc LMIs and nc polynomials.

In the remainder of the introduction we first, in Subsection 1.2, give additional background on a core object of our study, polynomials in non-commuting variables. The initiated reader may wish to skip this subsection. Subsections 1.3, 1.4, 1.5, and 1.6 give overviews of the four main topics of the survey.

The body of the paper consists of six sections. The first four give further detail on the main topics. Except for Section 3 which has its own motivation subsection, motivation for our investigations is weaved into the discussion. Convexity is a recurring theme. Section 6 offers a list of computer algebra packages for work in a free $*$-algebra revolving around convexity and positivity.

1.2 Non-commutative polynomials

Let $\mathbb{R}\langle x\rangle$ denote the real algebra of polynomials in the non-commuting indeterminates $x = (x_1, \ldots, x_g)$. Elements of $\mathbb{R}\langle x\rangle$ are **non-commutative polynomials**, abbreviated to **nc polynomials** or often just **polynomials**. Thus, a non-commutative polynomial p is a finite sum,

$$p = \sum p_w w, \tag{5}$$

where each w is a word in $(x_1, \ldots, x_g)$ and the coefficients $p_w \in \mathbb{R}$. The polynomial p of equation (3) is a non-commutative polynomial of degree two in two variables. The polynomial

$$q = x_1 x_2^3 + x_2^3 x_1 + x_3 x_1 x_2 + x_2 x_1 x_3 \tag{6}$$

is an non-commutative polynomial of degree four in three variables.

Involution

There is a natural **involution** * on $\mathbb{R}\langle x\rangle$ given by

$$p^* = \sum p_w w^*, \tag{7}$$

where, for a word w,

$$w = x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto w^* = x_{j_n} \cdots x_{j_2} x_{j_1}. \tag{8}$$

A polynomial p is **symmetric** if $p^* = p$. For example, the polynomials of equation (3) is symmetric, whereas the q of equation (6) is not. In particular, $x_j^* = x_j$ and for this reason the variables are sometimes referred to as symmetric non-commuting variables.

Denote, by $\mathbb{R}\langle x\rangle_d$, the polynomials in $\mathbb{R}\langle x\rangle$ of (total) degree d or less.

Substituting Matrices for Indeterminates

Let $(\mathbb{SR}^{n\times n})^g$ denote the set of g-tuples $X = (X_1, \dots, X_g)$ of real symmetric $n \times n$ matrices. A polynomial $p(x) = p(x_1, \dots, x_g) \in \mathbb{R}\langle x\rangle$ can naturally be evaluated at a tuple $X \in (\mathbb{SR}^{n\times n})^g$ resulting in an $n \times n$ matrix. Equations (3) and (4) are illustrative. In particular, the constant term $p_\varnothing$ of $p(x)$ becomes $p_\varnothing I_n$; i.e., the empty word evaluates to I_n. Often we write $p(0)$ for $p_\varnothing$ interpreting the 0 as $0 \in \mathbb{R}^g$. As a further example, for the polynomial q from equation (6),

$$q(X) = X_1X_2^3 + X_2^3X_1 + X_3X_1X_2 + X_2X_1X_3.$$

The involution on $\mathbb{R}\langle x\rangle$ that was introduced earlier is compatible with evaluation at X and matrix transposition, i.e.,

$$p^*(X) = p(X)^*,$$

where $p(X)^*$ denotes the transpose of the matrix $p(X)$. Note, if p is symmetric, then so is $p(X)$.

Matrix-Valued Polynomials

Let $\mathbb{R}\langle x\rangle^{\delta\times\delta'}$ denote the $\delta \times \delta'$ matrices with entries from $\mathbb{R}\langle x\rangle$. In particular, if $p \in \mathbb{R}\langle x\rangle^{\delta\times\delta'}$, then

$$p = \sum p_w w, \tag{9}$$

where the sum is finite and each p_w is a real $\delta\times\delta'$ matrix. Denote, by $\mathbb{R}\langle x\rangle_d^{\delta\times\delta'}$, the subset of $\mathbb{R}\langle x\rangle^{\delta\times\delta'}$ whose polynomial entries have degree d or less.

Evaluation at $X \in (\mathbb{SR}^{n\times n})^g$ naturally extends to $p \in \mathbb{R}\langle x\rangle^{\delta\times\delta'}$ via the Kronecker tensor product, with the result, $p(X)$, a $\delta \times \delta'$ block matrix with $n \times n$ entries. The involution * naturally extends to $\mathbb{R}\langle x\rangle^{\delta\times\delta}$ by

$$p = \sum p_w^* w^*, \tag{10}$$

for p given by equation (9). A polynomial $p \in \mathbb{R}\langle x\rangle^{\delta\times\delta}$ is symmetric if $p^* = p$ and in this case $p(X) = p(X)^*$.

A simple method of constructing new matrix valued polynomials from old ones is by direct sum. For instance, if $p_j \in \mathbb{R}\langle x\rangle^{\delta_j\times\delta_j}$ for $j = 1, 2$, then

$$p_1 \oplus p_2 = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \in \mathbb{R}\langle x\rangle^{(\delta_1+\delta_2)\times(\delta_1+\delta_2)}.$$

Linear Matrix Inequalities (LMIs)

Given symmetric $\ell \times \ell$ matrices $A_0, A_1, \dots, A_g$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j \tag{11}$$

is an affine linear nc matrix polynomial, better known as a **linear** (or affine linear) **pencil**. In the case that $A_0 = 0$, L is a **truly linear pencil**; and when $A_0 = I$, we say L is a **monic linear pencil**.

The inequality $L(x) \succ 0$ for $x \in \mathbb{R}^g$ is a **linear matrix inequality (LMI)**. LMIs are ubiquitous in science and engineering. Evaluation of L at $X \in (\mathbb{SR}^{n\times n})^g$ is most easily described using tensor products as in equation (2) and the expression $L(X) \succ 0$ is a **non-commutative LMI**, or nc LMI for short.

1.3 LMI Domination and Complete Positivity

This section discusses the nc LMI versions of two natural LMI domination questions. To fix notation, let

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j,$$

be a given linear pencil (thus A_j are symmetric $\ell \times \ell$ matrices). For a fixed n the solution set of all $X \in (\mathbb{SR}^{n\times n})^g$ satisfying $L(X) \succ 0$ is denoted $\mathcal{D}_L(n)$ and the sequence (graded set) $(\mathcal{D}_L(n))_{n\in\mathbb{N}}$ is written $\mathcal{D}_L$. Note that $\mathcal{D}_L(1)$ is the solution set of the classical (commutative) LMI, $L(x) \succ 0$.

Given linear matrix inequalities (LMIs) L_1 and L_2 it is natural to ask:

(Q_1) when does one dominate the other, that is, when is $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$?
(Q_2) when are they mutually dominant, that is, $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(2)$?

While such problems can be NP-hard, their nc relaxations have elegant answers. Indeed, they reduce to constructible semidefinite programs. We chose to begin with this topic because it offers the most gentle introduction to our matrix subject.

To describe a sample result, assume there is an $x \in \mathbb{R}^g$ such that both $L_1(x)$ and $L_2(x)$ are both positive definite, and suppose $\mathcal{D}_{L_1}(1)$ is bounded. If $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$ for every n, then there exist matrices V_j such that

$$L_2(x) = V_1^* L_1(x) V_1 + \cdots + V_\mu^* L_1(x) V_\mu. \tag{A$_1$}$$

The converse is of course immediate. As for (Q_2) we show that L_1 and L_2 are mutually dominant ($\mathcal{D}_{L_1}(n) = \mathcal{D}_{L_2}(n)$ for all n) if and only if, up to certain redundancies described in detail in Section 2, L_1 and L_2 are unitarily equivalent.

It turns out that our matrix variable LMI domination problem is equivalent to the classical problem of determining if a linear map τ from one subspace of matrices to another is "completely positive". Complete positivity is one of the main techniques of modern operator theory and the theory of operator algebras. On one hand it provides tools for studying LMIs and on the other hand, since completely positive maps are not so far from representations and generally are more tractable than their merely positive counterparts, the theory of completely positive maps provides perspective on the difficulties in solving LMI domination problems. nc LMI domination is the topic of Section 2.

1.4 Non-commutative Convex Sets and LMI Representations

Section 1.3 dealt with the (matricial) solution set of a Linear Matrix Inequality

$$\mathcal{D}_L = \{X : L(X) \succ 0\}.$$

The set $\mathcal{D}_L$ is convex in the sense that each $\mathcal{D}_L(n)$ is convex. It is also a non-commutative basic open semi-algebraic set (in a sense we soon define). The main theorem of this section is the converse, a result which has implications for both semidefinite programming and systems engineering.

Let $p \in \mathbb{R}\langle x\rangle^{\delta\times\delta}$ be a given symmetric non-commutative $\delta \times \delta$-valued matrix polynomial. Assuming that $p(0) \succ 0$, the positivity set $\mathcal{D}_p(n)$ of a non-commutative symmetric polynomial p in dimension n is the component of 0 of the set

$$\{X \in (\mathbb{SR}^{n\times n})^g :\ p(X) \succ 0\}.$$

The **positivity set**, $\mathcal{D}_p$, is the sequence of sets $(\mathcal{D}_p(n))$, which is the type of set we call a **non-commutative basic open** semi-algebraic set. The non-commutative set $\mathcal{D}_p$ is called **convex** if, for each n, $\mathcal{D}_p(n)$ is convex. A set is said to have a **Linear Matrix Inequality Representation** if it is the set of all solutions to some LMI, that is, it has the form $\mathcal{D}_L$ for some $L(x) = I + \sum_j A_j x_j$.

The main theorem of Section 3 says: if $p(0) \succ 0$ and $\mathcal{D}_p$ is bounded, then $\mathcal{D}_p$ has an LMI representation if and only if $\mathcal{D}_p$ is convex.

1.5 Non-commutative Convex Polynomials have Degree Two

We turn now from non-commutative convex sets to non-commutative convex polynomials. The previous section exposed the rigid the structure of sets which are both convex and the sublevel set of a non-commutative polynomial. Of course if p is concave ($-p$ is convex), then its sublevel sets are convex. But more is true.

A symmetric polynomial p is **matrix convex**, if for each positive integer n, each pair of tuples of symmetric matrices $X \in (\mathbb{SR}^{n\times n})^g$ and $Y \in (\mathbb{SR}^{n\times n})^g$, and each $0 \le t \le 1$,

$$p\big(tX + (1-t)Y\big) \preceq tp(X) + (1-t)p(Y).$$

The main result on convex polynomials, given in Section 4, is that every symmetric non-commutative polynomial which is matrix convex has degree two or less.

1.6 Algebraic certificates of non-commutative positivity: Positivstellensätze

An algebraic certificate for positivity of a polynomial p on a semi-algebraic set S is a Positivstellensatz. The familiar fact that a polynomial p in one-variable which is positive on $S = \mathbb{R}$ is a sum of squares is an example.

The theory of Positivstellensätze - a pillar of the field of semi-algebraic geometry - underlies the main approach currently used for global optimization of polynomials. See [Par00, Las01] for a beautiful treatment of this, and other, applications of commutative semi-algebraic geometry. Further, because convexity of a polynomial p on a set S is equivalent to positivity of the Hessian of p on S, this theory also provides a link between convexity and semi-algebraic geometry. Indeed, this link in the non-commutative setting ultimately leads to the conclusion the a matrix convex non-commutative polynomial has degree at most two.

Polynomial optimization problems involving non-commuting variables also arise naturally in many areas of quantum physics, see [PNA10, NPA].

Positivstellensätze in various incarnations appear throughout this survey as they arise naturally in connection with the previous topics. Section 5 contains a brief list of algebraic certificates for positivity like conditions for non-commutative polynomials in both symmetric and non-symmetric nc variables, Thus, this section provides an overview of non-commutative semi-algebraic geometry with the theme being that nc Positivstellensätze are cleaner and more rigid than there commutative counterparts.

2 LMI Domination and Complete Positivity

In this section we expand upon the discussion of nc LMI domination of Subsection 1.3. Recall, a monic linear pencil is an expression of the form

$$L(x) = I + \sum_{j=1}^{g} A_j x_j,$$

where, for some ℓ, the A_j are symmetric $\ell \times \ell$ matrices with real entries and I is the $\ell \times \ell$ identity. For a given positive integer n,

$$\mathcal{D}_L(n) = \{X \in (\mathbb{SR}^{n\times n})^g : L(X) \succ 0\}$$

and let $\mathcal{D}_L$ denote the sequence of sets $(\mathcal{D}_L(n))_{n\in\mathbb{N}}$. Thus $\mathcal{D}_L$ is the solution set of the nc LMI $L(X) \succ 0$ and $\mathcal{D}_L(1)$ is the solution set of the traditional LMI $L(x) \succ 0$ ($x \in \mathbb{R}^g$). We call $\mathcal{D}_L$ an **nc LMI**.

2.1 Certificates for LMI Domination

This subsection contains precise algebraic characterizations of nc LMI domination. Algorithms, the connection to complete positivity, examples, and the application to a new commutative Positivstellensatz follow in succeeding subsections.

Theorem 1 (Linear Positivstellensatz [HKMb]). *Let $L_j \in \mathbb{SR}^{d_j\times d_j}\langle x\rangle$, $j = 1, 2$, be monic linear pencils and assume $\mathcal{D}_{L_1}(1)$ is bounded. Then $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if there is a $\mu \in \mathbb{N}$ and an isometry $V \in \mathbb{R}^{\mu d_1\times d_2}$ such that*

$$L_2(x) = V^*\big(I_\mu \otimes L_1(x)\big)V = \sum_{j=1}^{\mu} V_j^* L_1(x) V_j. \tag{1}$$

Suppose $L \in \mathbb{SR}^{d\times d}\langle x\rangle$,

$$L = I + \sum_{j=1}^{g} A_j x_j$$

is a monic linear pencil. A subspace $\mathcal{H} \subseteq \mathbb{R}^d$ is **reducing for** L if $\mathcal{H}$ reduces each A_j; i.e., if $A_j\mathcal{H} \subseteq \mathcal{H}$. Since each A_j is symmetric, it also follows that $A_j\mathcal{H}^\perp \subseteq \mathcal{H}^\perp$. Hence, with respect to the decomposition $\mathbb{R}^d = \mathcal{H} \oplus \mathcal{H}^\perp$, L can be written as the direct sum,

$$L = \tilde{L} \oplus \tilde{L}^\perp = \begin{bmatrix} \tilde{L} & 0 \\ 0 & \tilde{L}^\perp \end{bmatrix} \qquad \text{where} \qquad \tilde{L} = I + \sum_{j=1}^{g} \tilde{A}_j x_j,$$

and $\tilde{A}_j$ is the restriction of A_j to $\mathcal{H}$. (The pencil $\tilde{L}^\perp$ is defined similarly.) If $\mathcal{H}$ has dimension ℓ, then by identifying $\mathcal{H}$ with $\mathbb{R}^\ell$, the pencil $\tilde{L}$ is a monic linear pencil of size ℓ. We say that $\tilde{L}$ is a **subpencil** of L. If moreover, $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$, then $\tilde{L}$ is a **defining subpencil** and if no proper subpencil of $\tilde{L}$ is a defining subpencil for $\mathcal{D}_L$, then $\tilde{L}$ is a **minimal defining** (**sub**)**pencil**.

Theorem 2 (Linear Gleichstellensatz [HKMb]). *Suppose L_1, L_2 are monic linear pencils with $\mathcal{D}_{L_1}(1)$ bounded. Then $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$ if and only if minimal defining pencils $\tilde{L}_1$ and $\tilde{L}_2$ for $\mathcal{D}_{L_1}$ and $\mathcal{D}_{L_2}$ respectively, are unitarily equivalent. That is, there is a unitary matrix U such that*

$$\tilde{L}_2(x) = U^* \tilde{L}_1(x) U. \tag{2}$$

2.2 Algorithms for LMIs

Of widespread interest is determining if

$$\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1), \tag{3}$$

or if $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(1)$. For example, the paper of Ben-Tal and Nemirovski [BTN02] exhibits simple cases where determining this is NP-hard. While we do not give details here we guide the reader to [HKMb, Section 4] where we prove that $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ is equivalent to the feasibility of a certain semidefinite program which we construct explicitly in [HKMb, Section 4.1]. Of course, if $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$, then $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$. Thus our algorithm is a type of relaxation of the problem (3).

Also in [HKMb] is an algorithm (Section 4.2) easily adapted from the first to determine if $\mathcal{D}_L$ is bounded, and what its "radius" is. By [HKMb, Proposition 2.4], $\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded. Our algorithm thus yields an upper bound of the radius of $\mathcal{D}_L(1)$. In [HKMb, Section 4.3] we solve a matricial relaxation of the classical matrix cube problem, finding the biggest matrix cube contained in $\mathcal{D}_L$. Finally, given a matricial LMI set $\mathcal{D}_L$, [HKMb, Section 4.4] gives an algorithm to compute the linear pencil $\tilde{L} \in \mathbb{SR}^{d\times d}\langle x\rangle$ with smallest possible d satisfying $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

2.3 Complete Positivity and LMI Inclusion

To monic linear pencils L_1 and L_2,

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j \times d_j}\langle x\rangle, \quad j = 1, 2 \tag{4}$$

are the naturally associated subspaces of $d_j \times d_j$ $(j = 1, 2)$ matrices

$$\mathcal{S}_j = \operatorname{span}\{I, A_{j,\ell} \colon \ell = 1, \dots, g\} = \operatorname{span}\{L_j(X) \colon X \in \mathbb{R}^g\} \subseteq \mathbb{SR}^{d_j \times d_j}. \tag{5}$$

We shall soon see that the condition L_2 dominates L_1, equivalently $\mathcal{D}_{L_1} \subset \mathcal{D}_{L_2}$, is equivalent to a property called complete positivity, defined below, of the unital linear mapping $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ determined by

$$\tau(A_{1,\ell}) = A_{2,\ell}. \tag{6}$$

A recurring theme in the non-commutative setting, such as that of a subspace of C*-algebra [Arv69, Arv72, Arv08] or in free probability [Voi04, Voi05] to give two of many examples, is the need to consider the **complete matrix structure** afforded by tensoring with $n \times n$ matrices (over positive integers n). The resulting theory of operator algebras, systems, spaces and matrix convex sets has matured to the point that there are now several excellent books on the subject including [BLM04, Pau02, Pis03].

Let $\mathcal{T}_j \subseteq \mathbb{R}^{d_j \times d_j}$ be unital linear subspaces closed under the transpose, and $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ a unital linear $*$-map. For $n \in \mathbb{N}$, ϕ induces the map

$$\phi_n = I_n \otimes \phi : \mathbb{R}^{n \times n} \otimes \mathcal{T}_1 = \mathcal{T}_1^{n \times n} \to \mathcal{T}_2^{n \times n}, \quad M \otimes A \mapsto M \otimes \phi(A),$$

called an **ampliation** of ϕ. Equivalently,

$$\phi_n\left(\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}\right) = \begin{bmatrix} \phi(T_{11}) & \cdots & \phi(T_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(T_{n1}) & \cdots & \phi(T_{nn}) \end{bmatrix}$$

for $T_{ij} \in \mathcal{T}_1$. We say that ϕ is **k-positive** if ϕ_k is a positive map. If ϕ is k-positive for every $k \in \mathbb{N}$, then ϕ is **completely positive**.

2.4 The Map τ is Completely Positive

A basic observation is that n-positivity of τ is equivalent to the inclusion $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$. Hence $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ is equivalent to complete positivity of τ, an observation whcih ultimately leads to the algebraic characterization of Theorem 1.

Theorem 3. *Consider the monic linear pencils of equation* (4) *and assume that $\mathcal{D}_{L_1}(1)$ is bounded. Let $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ be the unital linear map of equation* (6).

(1) *τ is n-positive if and only if $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$;*
(2) *τ is completely positive if and only if $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$.*

Conversely, suppose $\mathcal{D}$ is a unital $*$-subspace of $\mathbb{SR}^{d \times d}$ and $\tau : \mathcal{D} \to \mathbb{SR}^{d' \times d'}$ is completely positive. Given a basis $\{I, A_1, \dots, A_g\}$ for $\mathcal{D}$, let $B_j = \tau(A_j)$. Let

$$L_1(x) = I + \sum A_j x_j, \quad L_2(x) = I + \sum B_j x_j.$$

The complete positivity of τ implies, if $L_1(X) \succ 0$, then $L_2(X) \succ 0$ and hence $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$. Hence the completely positive map τ (together with a choice of basis) gives rise to an LMI domination.

2.5 An Example

The following example illustrates the constructs of the previous two subsections. Let

$$L_1(x_1,x_2) = I + \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix} x_1 + \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix} x_2 = \begin{bmatrix} 1&x_1&x_2\\x_1&1&0\\x_2&0&1 \end{bmatrix} \in \mathbb{SR}^{3\times 3}\langle x\rangle$$

and

$$L_2(x_1,x_2) = I + \begin{bmatrix} 1&0\\0&-1 \end{bmatrix} x_1 + \begin{bmatrix} 0&1\\1&0 \end{bmatrix} y_2 = \begin{bmatrix} 1+x_1&x_2\\x_2&1-x_1 \end{bmatrix} \in \mathbb{SR}^{2\times 2}\langle x\rangle.$$

Then

$$\begin{aligned} \mathcal{D}_{L_1} &= \{(X_1,X_2)\colon 1 - X_1^2 - X_2^2 \succ 0\},\\ \mathcal{D}_{L_1}(1) &= \{(X_1,X_2)\in\mathbb{R}^2\colon X_1^2 + X_2^2 < 1\},\\ \mathcal{D}_{L_2}(1) &= \{(X_1,X_2)\in\mathbb{R}^2\colon X_1^2 + X_2^2 < 1\}. \end{aligned}$$

Thus $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(1)$. On one hand,

$$\left(\begin{bmatrix} \frac{1}{2}&0\\0&0 \end{bmatrix}, \begin{bmatrix} 0&\frac{3}{4}\\ \frac{3}{4}&0 \end{bmatrix} \right) \in \mathcal{D}_{L_1} \smallsetminus \mathcal{D}_{L_2},$$

so $L_1(X_1,X_2)\succ 0$ does not imply $L_2(X_1,X_2)\succ 0$.

On the other hand, $L_2(X_1,X_2)\succ 0$ does imply $L_1(X_1,X_2)\succ 0$. The map $\tau : \mathcal{S}_2 \to \mathcal{S}_1$ in our example is given by

$$\begin{bmatrix} 1&0\\0&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0\\0&-1 \end{bmatrix} \mapsto \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \quad \begin{bmatrix} 0&1\\1&0 \end{bmatrix} \mapsto \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}.$$

Consider the extension of τ to a unital linear $*$-map $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 3}$, defined by

$$E_{11} \mapsto \frac{1}{2}\begin{bmatrix} 1&1&0\\1&1&0\\0&0&1 \end{bmatrix}, \quad E_{12} \mapsto \frac{1}{2}\begin{bmatrix} 0&0&1\\0&0&1\\1&-1&0 \end{bmatrix},$$

$$E_{21} \mapsto \frac{1}{2}\begin{bmatrix} 0&0&1\\0&0&-1\\1&1&0 \end{bmatrix}, \quad E_{22} \mapsto \frac{1}{2}\begin{bmatrix} 1&-1&0\\-1&1&0\\0&0&1 \end{bmatrix}.$$

(Here E_{ij} are the 2×2 matrix units.) To show that ψ is completely positive compute its Choi matrix defined as

$$C = \begin{bmatrix} \psi(E_{11}) & \psi(E_{12}) \\ \psi(E_{21}) & \psi(E_{22}) \end{bmatrix}. \tag{7}$$

[Pau02, Theorem 3.14] says ψ is completely positive if and only if $C \succeq 0$. The Choi matrix is the key to computational algorithms in [HKMb, Section 4]. In the present case, to see that C is positive semidefinite, note

$$C = \frac{1}{2} W^* W \quad \text{for} \quad W = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

Now ψ has a very nice representation:

$$\psi(S) = \frac{1}{2} V_1^* S V_1 + \frac{1}{2} V_2^* S V_2 = \frac{1}{2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^* \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{8}$$

for all $S \in \mathbb{R}^{2\times 2}$. (Here $V_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, thus $W = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$.) In particular,

$$2L_1(x,y) = V_1^* L_2(x,y) V_1 + V_2^* L_2(x,y) V_2. \tag{9}$$

Hence $L_2(X_1, X_2) \succ 0$ implies $L_1(X_1, X_2) \succ 0$, i.e., $\mathcal{D}_{L_2} \subseteq \mathcal{D}_{L_1}$.

The computations leading up to equation (9) illustrate the proof of our linear Positivstellensatz, Theorem 1. For the details see [HKMb, Section 3.1].

2.6 Positivstellensatz on a Spectrahedron

Our non-commutative techniques lead to a cleaner and more powerful commutative Putinar-type Positivstellensatz [Put93] for p strictly positive on a bounded spectrahedron $\overline{\mathcal{D}}_L(1) = \{x \in \mathbb{R}^g : L(x) \succeq 0\}$. In the theorem which follows, $\mathbb{SR}^{d\times d}[y]$ is the set of symmetric $d \times d$ matrices with entries from $\mathbb{R}[y]$, the algebra of (commutative) polynomials with coefficients from $\mathbb{R}$. Note that an element of $\mathbb{SR}^{d\times d}[y]$ may be identified with a polynomial (in commuting variables) with coefficients from $\mathbb{SR}^{d\times d}$.

Theorem 4. *Suppose $L \in \mathbb{SR}^{d\times d}[y]$ is a monic linear pencil and $\overline{\mathcal{D}}_L(1)$ is bounded. Then for every symmetric matrix polynomial $p \in \mathbb{R}^{\ell\times\ell}[y]$ with $p|_{\overline{\mathcal{D}}_L(1)} \succ 0$, there are $A_j \in \mathbb{R}^{\ell\times\ell}[y]$, and $B_k \in \mathbb{R}^{d\times\ell}[y]$ satisfying*

$$p = \sum_j A_j^* A_j + \sum_k B_k^* L B_k. \tag{10}$$

The Positivstellensatz, Theorem 4, has a non-commutative version for $\delta \times \delta$ matrix valued symmetric polynomials p in non-commuting variables positive on a nc LMI set $\overline{\mathcal{D}}_L$, see [HKMb]. In the case this matrix valued polynomial p is linear, this Positivstellensatz reduces to Theorem 1, which can thus be regarded as a "Linear Positivstellensatz". For perspective we mention that the proofs of our Positivstellensätze actually rely on the linear Positivstellensatz. For experts we point out that the key reason LMI sets behave better is that the quadratic module associated to a monic linear pencil L with *bounded* $\overline{\mathcal{D}}_L$ is *archimedean*.

We shall return to the topic of Positivstellensätze in Section 5.

3 Non-commutative Convex semi-algebraic Sets are LMI Representable

The main result of this section is that a bounded convex non-commutative basic open semi-algebraic set has a monic Linear Matrix Inequality representation. Applications and connections to semidefinite programming and linear systems engineering are discussed in Section 3.4. The work is also of interest in understanding a non-commutative (free) analog of convex semi-algebraic sets [BCR98].

For perspective, in the commutative case of a basic open semi-algebraic subset $\mathcal{C}$ of $\mathbb{R}^g$, there is a stringent condition, called the "*line test*", which, in addition to convexity, is necessary for $\mathcal{C}$ to have an LMI representation. In two dimensions the line test is necessary and sufficient, [HV07], a result used by Lewis-Parrilo-Ramana [LPR05] to settle a 1958 conjecture of Peter Lax on hyperbolic polynomials. Indeed LMI representations are closely tied to properties of hyperbolic polynomials; see this volume, the survey of Helton and Nie.

In summary, if a (commutative) bounded basic open semi-algebraic convex set has an LMI representation, then it must pass the highly restrictive line test; whereas a non-commutative bounded basic open semi-algebraic set has an LMI representation if and only if it is convex.

A subset $\mathcal{S}$ of $(\mathbb{SR}^{n\times n})^g$ is closed under unitary conjugation if for every $X = (X_1, \dots, X_g) \in \mathcal{S}$ and U a $n \times n$ unitary, we have $U^*XU = (U^*X_1U, \dots, U^*X_gU) \in \mathcal{S}$. The sequence $\mathcal{C} = (\mathcal{C}(n))_{n\in\mathbb{N}}$, where $\mathcal{C}(n) \subseteq (\mathbb{SR}^{n\times n})^g$, is a **non-commutative set** if it is closed under unitary conjugation and direct sums; i.e., if $X = (X_1, \dots, X_g) \in \mathcal{C}(n)$ and $Y = (Y_1, \dots, Y_g) \in \mathcal{C}(m)$, then $X \oplus Y = (X_1 \oplus Y_1, \dots, X_g \oplus Y_g) \in \mathcal{C}(n+m)$. Such set $\mathcal{C}$ has an **LMI representation** if there is a monic linear pencil L such that

$$\mathcal{C} = \mathcal{D}_L.$$

Of course, if $\mathcal{C} = \mathcal{D}_L$, then the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ has the representation $\{X\colon L(X) \succeq 0\}$ and so we could also refer to $\overline{\mathcal{C}}$ as having an LMI representation.

Clearly, if $\mathcal{C}$ has an LMI representation, then $\mathcal{C}$ is a convex non-commutative basic open semi-algebraic set. The main result of this section is the converse, under the additional assumption that $\mathcal{C}$ is bounded.

Since we are dealing with matrix convex sets, it is not surprising that the starting point for our analysis is the matricial version of the Hahn-Banach Separation Theorem of Effros and Winkler [EW97] which says that given a point x not inside a matrix convex set there is a (finite) LMI which separates x from the set. For a general matrix convex set $\mathcal{C}$, the conclusion is then that there is a collection, likely infinite, of finite LMIs which cut out $\mathcal{C}$.

In the case $\mathcal{C}$ is matrix convex and also semi-algebraic, the challenge is to prove that there is actually a finite collection of (finite) LMIs which define $\mathcal{C}$. The techniques used to meet this challenge have little relation to previous work on convex non-commutative basic semi-algebraic sets. In particular, they do not involve non-commutative calculus and positivity. See [HM] for the details.

3.1 Non-commutative Basic Open Semi-Algebraic Sets

Suppose $p \in \mathbb{R}\langle x\rangle^{\delta\times\delta}$ is symmetric. In particular, $p(0)$ is a $\delta \times \delta$ symmetric matrix. Assume that $p(0) \succ 0$. For each positive integer n, let

$$\mathfrak{I}_p(n) = \{X \in (\mathbb{SR}^{n\times n})^g \colon p(X) \succ 0\},$$

and define $\mathfrak{I}_p$ to be the sequence (graded set) $(\mathfrak{I}_p(n))_{n=1}^{\infty}$. Let $\mathcal{D}_p(n)$ denote the connected component of 0 of $\mathfrak{I}_p(n)$ and $\mathcal{D}_p$ the sequence (graded set) $(\mathcal{D}_p(n))_{n=1}^{\infty}$. We call $\mathcal{D}_p$ **the positivity set** of p. In analogy with classical real algebraic geometry we call sets of the form $\mathcal{D}_p$ **non-commutative basic open semi-algebraic sets**. (Note that it is not necessary to explicitly consider intersections of non-commutative basic open semi-algebraic sets since the intersection $\mathcal{D}_p \cap \mathcal{D}_q$ equals $\mathcal{D}_{p\oplus q}$.)

Remark 1. By a simple affine linear change of variable the point 0 can be replaced by $\lambda \in \mathbb{R}^g$. Replacing 0 by a fixed $\Lambda \in (\mathbb{SR}^{n\times n})^g$ would require an extension of the theory. ■

3.2 Convex Semi-Algebraic Sets

To say that $\mathcal{D}_p$ is **convex** means that each $\mathcal{D}_p(n)$ is convex (in the usual sense) and in this case we say $\mathcal{D}_p$ is a **convex non-commutative basic open**

semi-algebraic set. In addition, we generally assume that $\mathcal{D}_p$ is bounded; i.e., there is a constant K such for each n and each $X \in \mathcal{D}_p(n)$, we have $\|X\| = \sum \|X_j\| \leq K$. Thus the following list of conditions summarizes our usual assumptions on p.

Assumption 1 *Fix p a $\delta \times \delta$ symmetric matrix of polynomials in g non-commuting variables of degree d. Our standard assumptions are:*

(1) $p(0)$ *is positive definite;*
(2) $\mathcal{D}_p$ *is bounded; and*
(3) $\mathcal{D}_p$ *is convex.*

3.3 The Result

Our main theorem of this section is

Theorem 5 ([HM]). *Every convex non-commutative bounded basic open semi-algebraic set* (*as in Assumption* 1) *has an LMI representation.*

The proof of Theorem 5 yields estimates on the size of the representing LMI.

Theorem 6. *Suppose p satisfies the conditions of Assumption* 1. *Thus p is a symmetric $\delta \times \delta$-matrix polynomial of degree d in g variables. Let $\nu = \delta \sum_{j=0}^{d} g^j$.*

(1) *There is a $\mu \leq \frac{\nu(\nu+1)}{2}$ and a monic linear pencil $L \in \mathbb{SR}^{\mu\times\mu}\langle x\rangle$ such that $\mathcal{D}_p = \mathcal{D}_L$.*
(2) *In the case that $p(0) = I_\delta$, the estimate on the size of the matrices in L reduces to $\frac{\breve{\nu}(\breve{\nu}+1)}{2}$, where $\breve{\nu} = \delta \sum_{j=0}^{\lceil \frac{d}{2} \rceil} g^j$.*

As usual, $\lceil \frac{d}{2} \rceil$ stands for the smallest integer $\geq \frac{d}{2}$. Of course

$$\left\lceil \frac{d}{2} \right\rceil = \frac{d}{2} \text{ when } d \text{ is even} \quad \text{and} \quad \left\lceil \frac{d}{2} \right\rceil = \frac{d+1}{2} \text{ when } d \text{ is odd.}$$

The results above hold even if sets more general than $\mathcal{D}_p$ are used. Suppose $p(0)$ is invertible and define $\mathcal{I}_p$ to be the component of $\{X : p(X) \text{ is invertible}\}$ containing 0. Then if $\mathcal{I}_p$ is bounded and convex, the theorems above still hold for $\mathcal{I}_p$; it has an LMI representation.

An unexpected consequence of Theorem 5 is that projections of non-commutative semi-algebraic sets may not be semi-algebraic. For details and proofs see [HM].

3.4 Motivation

One of the main advances in systems engineering in the 1990's was the conversion of a set of problems to LMIs, since LMIs, up to modest size, can be solved numerically by semidefinite programs [SIG98]. A large class of linear systems problems are described in terms of a signal-flow diagram Σ plus L^2 constraints (such as energy dissipation). Routine methods convert such problems into a non-commutative polynomial inequalities of the form $p(X) \succeq 0$ or $p(X) \succ 0$.

Instantiating specific systems of linear differential equations for the "boxes" in the system flow diagram amounts to substituting their coefficient matrices for variables in the polynomial p. Any property asserted to be true must hold when matrices of any size are substituted into p. Such problems are referred to as dimension free. We emphasize, the polynomial p itself is determined by the signal-flow diagram Σ.

Engineers vigorously seek convexity, since optima are global and convexity lends itself to numerics. Indeed, there are over a thousand papers trying to convert linear systems problems to convex ones and the only known technique is the rather blunt trial and error instrument of trying to guess an LMI. Since having an LMI is seemingly more restrictive than convexity, there has been the hope, indeed expectation, that some practical class of convex situations has been missed. The problem solved here (though not operating at full engineering generality, see [HHLM08]) is a paradigm for the type of algebra occurring in systems problems governed by signal-flow diagrams; such physical problems directly present non-commutative semi-algebraic sets. Theorem 5 gives compelling evidence that all such convex situations are associated to some LMI. Thus we think the implications of our results here are negative for linear systems engineering; for dimension free problems there is no convexity beyond LMIs.

A basic question regarding the range of applicability of SDP is: which sets have an LMI representation? Theorem 5 settles, to a reasonable extent, the case where the variables are non-commutative (effectively dimension free matrices).

4 Convex Polynomials

We turn now from non-commutative convex sets to non-commutative convex polynomials. If p is concave ($-p$ is convex) and monic, then the set $S = \{X : p(X) \succ 0\}$ is a convex non-commutative basic open . If it is also bounded, then, by the results of the previous section, it has an LMI representation.

However, much more is true and the analysis turns on a nc version of the Hessian and connects with nc semi-algebraic geometry.

A symmetric polynomial p is **matrix convex**, or simply **convex** for short, if for each positive integer n, each pair of tuples $X \in (\mathbb{SR}^{n\times n})^g$ and $Y \in (\mathbb{SR}^{n\times n})^g$, and each $0 \le t \le 1$,

$$p(tX + (1-t)Y) \preceq tp(X) + (1-t)p(Y). \tag{1}$$

Even in one-variable, convexity in the non-commutative setting differs from convexity in the commuting case because here Y need not commute with X. For example, to see that the polynomial $p = x^4$ is not matrix convex, let

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and compute

$$\frac{1}{2}X^4 + \frac{1}{2}Y^4 - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

which is not positive semidefinite. On the other hand, to verify that x^2 is a matrix convex polynomial, observe that

$$tX^2 + (1-t)Y^2 - (tX + (1-t)Y)^2 = t(1-t)(X-Y)^2 \succeq 0.$$

It is possible to automate checking for convexity, rather than depending upon lucky choices of X and Y as was done above. The theory described in [CHSY03], leads to and validates a symbolic algorithm for determining regions of convexity of non-commutative polynomials and even of non-commutative rational functions (for non-commutative rationals see [KVV09, HMV06]) which is implemented in NCAlgebra.

Let us illustrate it on the example $p(x) = x^4$. The NCAlgebra command is

NCConvexityRegion[Function F, {Variables x}].

```
In[1]:=  SetNonCommutative[x];
In[2]:=  NCConvexityRegion[ x**x**x**x, {x} ]
Out[2]:= { {2, 0, 0},  {0, 2}, {0, -2} }
```

which we interpret as saying that $p(x) = x^4$ is convex on the set of matrices X for which the the 3×3 block matrix valued non-commutative function

$$\rho(X) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \tag{2}$$

is positive semidefinite. Since $\rho(X)$ is constant and never positive semidefinite, we conclude that p is *nowhere* convex.

This example is a simple special case of the following theorem.

Theorem 7 ([HM03]). *Every convex symmetric polynomial in the free algebra $\mathbb{R}\langle x\rangle$ has degree two or less.*

4.1 The Proof of Theorem 7 and its Ingredients

Just as in the commutative case, convexity of a symmetric $p \in \mathbb{R}\langle x\rangle$ is equivalent to positivity of its Hessian $q(x)[h]$ which is a polynomial in the $2g$ variables $x = (x_1, \dots, x_g)$ and $h = (h_1, \dots, h_g)$. Unlike the commutative case, a positive non-commutative polynomial is a sum of squares. Thus, if p is convex, then its Hessian $q(x)[h]$ is a sum of squares. Combinatorial considerations say that a Hessian which is also a sum of squares must come from a polynomial of degree two. In the remainder of this section we flesh out this argument, introducing the needed definitions, techniques, and results.

Non-commutative Derivatives

For practical purposes, the k^{th}**-directional derivative** of a nc polynomial p is given by

$$p^{(k)}(x)[h] = \frac{d^k}{dt^k} p(x+th)\Big|_{t=0}.$$

Note that $p^{(k)}(x)[h]$ is homogeneous of degree k in h and moreover, if p is symmetric so is $p^k(x)[h]$. For $X, H \in (\mathbb{S}\mathbb{R}^{n\times n})^g$ observe that

$$p'(X)[H] = \lim_{t\to 0} \frac{p(X+tH) - p(X)}{t}.$$

Example 1. The one variable $p(x) = x^4$ has first derivative

$$p'(x)[h] = hxxx + xhxx + xxhx + xxxh.$$

Note each term is linear in h and h replaces each occurrence of x once and only once. The Hessian, or second derivative, of p is

$$p''(x)[h] = 2hhxx + 2hxhx + 2hxxh + 2xhhx + 2xhxh + 2xxhh.$$

Note each term is degree two in h and h replaces each pair of x's exactly once.

Theorem 8 ([HP07]). *Every symmetric polynomial $p \in \mathbb{R}\langle x\rangle$ whose k^{th} derivative is a matrix positive polynomial has degree k or less.*

Proof. See [HP07] for the full proof or [HM03] for case of $k = 2$. The very intuitive proof based upon a little non-commutative semi-algebraic geometry is sketched in the next subsection. ∎

A Little Non-Commutative Semi-Algebraic Geometry

The proof of Theorem 7 employs the most fundamental of all non-commutative Positivstellensätze.

A symmetric non-commutative polynomial p is **matrix positive** or simply **positive** provided $p(X_1, \ldots, X_g)$ is positive semidefinite for every $X \in (\mathbb{SR}^{n\times n})^g$ (and every n). An example of a matrix positive polynomial is a **Sum of Squares** of polynomials, meaning an expression of the form

$$p(x) = \sum_{j=1}^{c} h_j(x)^* h_j(x).$$

Substituting $X \in (\mathbb{SR}^{n\times n})^g$ gives $p(X) = \sum_{j=1}^{c} h_j(X)^* h_j(X) \succeq 0$. Thus p is positive. Remarkably these are the only positive non-commutative polynomials.

Theorem 9 ([Hel02]). *Every matrix positive polynomial is a sum of squares.*

This theorem is just a sample of the structure of non-commutative semi-algebraic geometry, the topic of Section 5.

Suppose $p \in \mathbb{R}\langle x\rangle$ is (symmetric and) convex and $Z, H \in (\mathbb{SR}^{n\times n})^g$ and $t \in \mathbb{R}$ are given. In the definition of convex, choosing $X = Z + tH$ and $Y = Z - tH$, it follows that

$$0 \preceq p(Z + tH) + p(Z - tH) - 2p(Z),$$

and therefore

$$0 \preceq \lim_{t\to 0} \frac{p(X + tH) + p(X - tH) - 2p(X)}{t^2} = p''(X)[H].$$

Thus the Hessian of p is matrix positive and since, in the non-commutative setting, positive polynomials are sums of squares we obtain the following theorem.

Proposition 1. *If p is matrix convex, then its Hessian $p''(x)[h]$ is a sum of squares.*

Proof of Theorem 7 by example

Here we illustrate the proof of Theorem 7 based upon Proposition 1 by showing that $p(x) = x^4$ is not matrix convex. Indeed, if $p(x)$ is matrix convex, then $p''(x)[h]$ is matrix positive and therefore, by Proposition 1, there exists a ℓ and polynomials $f_1(x, h), \ldots, f_\ell(x, h)$ such that

$$\frac{1}{2}p''(x)[h] = hhxx + hxhx + hxxh + xhhx + xhxh + xxhh$$

span

$$= f_1(x,h)^* f_1(x,h) + \cdots + f_\ell(x,h)^* f_\ell(x,h).$$

One can show that each $f_j(x,h)$ is linear in h. On the other hand, some term $f_i^* f_i$ contains $hhxx$ and thus f_i contains hx^2. Let m denote the largest ℓ such that some f_j contains the term hx^ℓ. Then $m \geq 1$ and for such j, the product $f_j^* f_j$ contains the term $hx^{2m}h$ which cannot be cancelled out, a contradiction. ∎

The proof of the more general, order k derivative, is similar, see [HP07].

4.2 Non-commutative Rational and Analytic Functions

A class of functions bigger than nc polynomials is given by nc analytic functions, see e.g. Voiculescu [Voi04, Voi] or the forthcoming paper of Kaliuzhnyi-Verbovetskyi and Vinnikov for an introduction. The rigidity of nc bianalytic maps is investigated by Popescu [Pop10]; see also [HKMS09, HKMa, HKMc]. For other properties of nc analytic functions, a very interesting body of work, e.g. by Popescu [Pop09] can be used as a gateway.

The articles [BGM06, KVV09, HMV06] deal with non-commutative rational functions. For instance, [HMV06] shows that if a non-commutative rational function is convex in an open set, then it is the Schur Complement of some monic linear pencil.

5 Algebraic Certificates of Positivity

In this section we give a brief overview of various free $*$-algebra analogs to the classical Positivstellensätze, i.e., theorems characterizing polynomial inequalities in a purely algebraic way. Here it is of benefit to consider free *non-symmetric* variables. That is, let $x = (x_1, \ldots, x_g)$ be non-commuting variables and $x^* = (x_1^*, \ldots, x_g^*)$ another set of non-commuting variables. Then $\mathbb{R}\langle x, x^*\rangle$ is the free $*$-algebra of polynomials in the non-commuting indeterminates x, x^*.

There is a natural involution $*$ on $\mathbb{R}\langle x, x^*\rangle$ induced by $x_i \mapsto x_i^*$ and $x_j^* \mapsto x_j$. As before, $p \in \mathbb{R}\langle x, x^*\rangle$ is symmetric if $p = p^*$. An element of the form p^*p is a square, and Σ^2 denotes the convex cone of all sums of squares. Given a matrix polynomial $p = \sum_w p_w w \in \mathbb{R}\langle x, x^*\rangle^{\delta\times\delta'}$ and $X \in (\mathbb{R}^{n\times n})^g$, we define the evaluation $p(X, X^*)$ by analogy with evaluation in the symmetric variable case.

5.1 Positivstellensätze

This subsection gives an indication of various free $*$-algebra analogs to the classical theorems characterizing polynomial inequalities in a purely algebraic way. We will start by sketching a proof of the following refinement of Theorem 9.

Theorem 10 ([Hel02]). *Let $p \in \mathbb{R}\langle x, x^* \rangle_d$ be a non-commutative polynomial. If $p(M, M^*) \succeq 0$ for all g-tuples of linear operators M acting on a Hilbert space of dimension at most $N(k) := \dim \mathbb{R}\langle x, x^* \rangle_k$ with $2k \geq d+2$, then $p \in \Sigma^2$.*

Proof. Note that a polynomial p satisfying the hypothesis automatically satisfies $p = p^*$. The only necessary technical result we need is the closedness of the cone Σ_k^2 in the Euclidean topology of the finite dimensional space $\mathbb{R}\langle x, x^* \rangle_k$. This is done as in the commutative case, using Carathéodory's convex hull theorem, more exactly, every polynomial of Σ_k^2 is a convex combination of at most $\dim \mathbb{R}\langle x, x^* \rangle_k + 1$ squares (of polynomials). On the other hand the positive functionals on Σ_k^2 separate the points of $\mathbb{R}\langle x, x^* \rangle_k$. See for details [HMP04].

Assume that $p \notin \Sigma^2$ and let $k \geq (d+2)/2$, so that $p \in \mathbb{R}\langle x, x^* \rangle_{2k-2}$. Once we know that Σ_{2k}^2 is a closed cone, we can invoke Minkowski separation theorem and find a symmetric functional $L \in \mathbb{R}\langle x, x^* \rangle_{2k}'$ providing the strict separation:

$$L(p) < 0 \leq L(f), \quad f \in \Sigma_{2k}^2.$$

Applying the Gelfand-Naimark-Segal construction to L yields a tuple M of operators acting on a Hilbert space H of dimension $N(k)$ and a vector $\xi \in H$, such that

$$0 \leq \langle p(M, M^*)\xi, \xi \rangle = L(p) < 0,$$

a contradiction. ∎

When compared to the commutative framework, this theorem is stronger in the sense that it does not assume a strict positivity of p on a well chosen "spectrum". Variants with supports (for instance for spherical tuples M : $M_1^* M_1 + ... + M_g^* M_g \preceq I$) of the above result are discussed in [HMP04].

To draw a very general conclusion from the above computations: when dealing with positivity in a free $*$-algebra, the standard point evaluations (or more precisely prime or real spectrum evaluations) of the commutative case are replaced by matrix evaluations of the free variables. The positivity can be tailored to "evaluations in a supporting set". The results pertaining to the resulting algebraic decompositions are called Positivstellensätze, see

[PD01] for details in the commutative setting. We state below an illustrative and generic result, from [HM04], for sums of squares decompositions in a free $*$-algebra.

Theorem 11 ([HM04]). *Let $p = p^* \in \mathbb{R}\langle x, x^*\rangle$ and let $q = \{q_1, ..., q_k\} \subseteq \mathbb{R}\langle x, x^*\rangle$ be a set of symmetric polynomials, so that*

$$\mathrm{QM}(q) = \mathrm{co}\{f^* q_i f;\ f \in \mathbb{R}\langle x, x^*\rangle,\ 0 \le i \le k\},\ q_0 = 1,$$

contains $1 - x_1^* x_1 - ... - x_g^* x_g$. If for all tuples of linear bounded Hilbert space operators $X = (X_1, ..., X_g)$, we have

$$q_i(X, X^*) \succeq 0,\ 1 \le i \le k \quad \Rightarrow \quad p(X, X^*) \succ 0, \tag{1}$$

then $p \in \mathrm{QM}(q)$.

Henceforth, call $\mathrm{QM}(q)$ the **quadratic module** generated by the set of polynomials q.

We omit the proof of Theorem 11, as it is very similar to the previous proof. The only difference is in the separation theorem applied. For details, see [HM04].

Some interpretation is needed in degenerate cases, such as those where no bounded operators satisfy the relations $q_i(X, X^*) \succeq 0$. Suppose for example, if ϕ denotes the defining relations for the Weyl algebra and the q_i include $-\phi^*\phi$. In this case, we would say $p(X, X^*) \succ 0$, since there are no X satisfying $q(X, X^*)$, and voila $p \in \mathrm{QM}(q)$ as the theorem says. A non-archimedean Positivstellensatz for the Weyl algebra, which treats unbounded representations and eigenvalues of polynomial partial differential operators, is given in [Sch05].

A paradigm practical question with matrix inequalities is:

Given a non-commutative symmetric polynomial $p(a, x)$ and a $n \times n$ matrix tuple A, find $X \succeq 0$ if possible which makes $p(A, X) \succeq 0$.

As a refinement of this problem, let $q(a, x)$ be a given nc symmetric polynomial. For a given A, find X if possible, such that both $q(A, X)$ and $p(A, X)$ are positive semidefinite. The infeasibility of this latter problem is equivalent to the statement, if $q(A, X) \succeq 0$, then $p(A, X) \not\succeq 0$. There is keen interest in numerical solutios of such problems. The next theorem informs us that the main issue is the matrix coefficients A, as it gives a “certificate of infeasibility” for the problem in the absense of A.

Theorem 12 (Nirgendsnegativsemidefinitheitsstellensatz [KS07]). *Let $p = p^* \in \mathbb{R}\langle x, x^*\rangle$ and let $q = \{q_1, ..., q_k\} \subset \mathbb{R}\langle x, x^*\rangle$ be a set of symmetric polynomials, so that* $\mathrm{QM}(q)$ *contains* $1 - x_1^* x_1 - ... - x_g^* x_g$ *. If for all tuples of linear bounded Hilbert space operators* $X = (X_1, ..., X_g)$*, we have*

$$q_i(X, X^*) \succeq 0,\ 1 \le i \le k \quad \Rightarrow \quad p(X, X^*) \not\preceq 0, \tag{2}$$

then there exists an integer r and $h_1, \dots, h_r \in \mathbb{R}\langle x, x^*\rangle$ with $\sum_{i=1}^r h_i^* p h_i \in 1 + \mathrm{QM}(q)$.

Proof. By (2),

$$\{X \mid q_i(X, X^*) \succeq 0,\ 1 \le i \le k,\ -p(X, X^*) \succeq 0\} = \varnothing.$$

Hence $-1 \in \mathrm{QM}(q, -p)$ by Theorem 11. ∎

5.2 Quotient Algebras

The results from Section 5.1 allow a variety of specializations to quotient algebras. In this subsection we consider a two sided ideal $\mathcal{I}$ of $\mathbb{R}\langle x, x^*\rangle$ which need not be invariant under $*$. Then one can replace the quadratic module QM in the statement of a Positivstellensatz with $\mathrm{QM}(q) + \mathcal{I}$, and apply similar arguments as above. For instance, the next simple observation can be deduced.

Corollary 1. *Assume, in the hypotheses of Theorem* 11*, that the relations* (1) *include some relations of the form $r(X, X^*) = 0$, even with r not symmetric, then*

$$p \in \mathrm{QM}(q) + \mathcal{I}_r \tag{3}$$

where $\mathcal{I}_r$ denotes the two sided ideal generated by r.

Proof. This follows immediately from $p \in \mathrm{QM}(q,\ -r^* r)$ which is a consequence of Theorem 11 and the fact

$$\mathrm{QM}(q,\ -r^* r) \subset \mathrm{QM}(q) + \mathcal{I}_r.$$

∎

For instance, we can look at the situation where r is the commutator $[x_i, x_j]$ as insisting on positivity of $q(X)$ only on commuting tuples of operators, in which case the ideal $\mathcal{I}$ generated by $[x_j^*, x_i^*],\ [x_i, x_j]$ is added to $\mathrm{QM}(q)$. The classical commuting case is captured by the corollary applied to the "commutator ideal": $\mathcal{I}_{[x_j^*, x_i^*],\ [x_i, x_j],\ [x_i, x_j^*]}$ for $i, j = 1, \dots, g$ which requires testing only on commuting tuples of operators drawn from a commuting C^*-algebra. The classical Spectral Theorem, then converts this to testing only on $\mathbb{C}^g$, cf. [HP07].

The situation where one tests for constrained positivity in the absence of an archimedean property is thoroughly analyzed in [SS].

5.3 A Nullstellensatz

With similar techniques (well chosen, separating, $*$-representations of the free algebra) and a rather different "dilation type" of argument, one can prove a series of Nullstellensätze.

We state for information one of them. For an early version see [HMP05].

Theorem 13. *Let $q_1(x), ..., q_m(x) \in \mathbb{R}\langle x\rangle$ be polynomials not depending on the x_j^* variables and let $p(x, x^*) \in \mathbb{R}\langle x, x^*\rangle$. Assume that for every g tuple X of linear operators acting on a finite dimensional Hilbert space H, and every vector $v \in H$, we have:*

$$(q_j(X)v = 0,\ 1 \leq j \leq m) \ \Rightarrow \ (p(X, X^*)v = 0).$$

Then p belongs to the left ideal $\mathbb{R}\langle x, x^*\rangle q_1 + ... + \mathbb{R}\langle x, x^*\rangle q_m$.

Again, this proposition is stronger than its commutative counterpart. For instance there is no need of taking higher powers of p, or of adding a sum of squares to p. Note that here $\mathbb{R}\langle x\rangle$ has a different meaning than earlier, since, unlike previously, the variables are nonsymmetric.

We refer the reader to [HMP07] for the proof of Theorem 13. An earlier, transpose-free Nullstellensatz due to Bergman was given in [HM04].

Here is a theorem which could be regarded as a very different type of non-commutative Nullstellensatz.

Theorem 14 ([KS08a]). *Let $p = p^* \in \mathbb{R}\langle x, x^*\rangle_d$ be a non-commutative polynomial satisfying $\operatorname{tr} p(M, M^*) = 0$ for all g-tuples of linear operators M acting on a Hilbert space of dimension at most d. Then p is a sum of commutators of non-commutative polynomials.*

We end this subsection with an example which goes against any intuition we would carry from the commutative case, see [HM04].

Example 2. Let $q = (x^*x + xx^*)^2$ and $p = x + x^*$ where x is a single variable. Then, for every matrix X and vector v (belonging to the space where X acts), $q(X)v = 0$ implies $p(X)v = 0$; however, there does not exist a positive integer m and $r, r_j \in \mathbb{R}\langle x, x^*\rangle$, so that

$$p^{2m} + \sum r_j^* r_j = qr + r^* q. \tag{4}$$

Moreover, we can modify the example to add the condition $q(X)$ is positive semidefinite implies $p(X)$ is positive semidefinite and still not obtain this representation.

5.4 Tracial Positivstellensatz

Another type of non-commutative positivity is given by the trace. A polynomial $p \in \mathbb{R}\langle x, x^*\rangle$ is called **trace-positive** if $\operatorname{tr} p(X, X^*) \geq 0$ for all $X \in (\mathbb{R}^{n\times n})^g$. The main motivation for studying these comes from two outstanding open problems: Connes' embedding conjecture [Con76] from operator algebras [KS08a] and the Bessis-Moussa-Villani (BMV) conjecture [BMV75] from quantum statistical mechanics [KS08b].

Clearly, a sum of a matrix positive (i.e., sum of hermitian squares by Theorem 10) and a trace-zero (i.e., sum of commutators by Theorem 14) polynomial is trace-positive. However, unlike in the matrix positive case, not every trace-positive polynomial is of this form [KS08a, KS08b].

Example 3. Let x denote a single non-symmetric variable and

$$M_0 := 3x^4 - 3(xx^*)^2 - 4x^5x^* - 2x^3(x^*)^3 + 2x^2x^*x(x^*)^2 + 2x^2(x^*)^2xx^* + 2(xx^*)^3.$$

Then the non-commutative Motzkin polynomial in non-symmetric variables is

$$M := 1 + M_0 + M_0^* \in \mathbb{R}\langle x, x^*\rangle.$$

It is trace-positive but is not a sum of hermitian squares and commutators.

Life is somewhat easier in the constrained, bounded case. For instance, in the language of operator algebras we have:

Theorem 15 ([KS08a]). *For $f = f^* \in \mathbb{R}\langle x, x^*\rangle$ the following are equivalent:*

(i) $\operatorname{tr}\big(f(a, a^*)\big) \geq 0$ *for all finite von Neumann algebras $\mathcal{A}$ and all tuples of contractions $a \in \mathcal{A}^g$;*

(ii) *for every $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ is a sum of commutators and of an element from*
$\mathrm{QM}(1 - x_1^*x_1, \ldots, 1 - x_g^*x_g)$.

The big open question [Con76, KS08a] is whether (i) or (ii) is equivalent to

(iii) $\operatorname{tr}\big(f(X, X^*)\big) \geq 0$ for all $n \in \mathbb{N}$ and all tuples of contractions $X \in (\mathbb{R}^{n\times n})^g$.

An attempt at better understanding trace-positivity is made in [BK], where the duality between trace-positive polynomials and the *tracial moment problem* is exploited. The tracial moment problem is the following question: For which sequences (y_w) indexed by words w in x, x^*, does there exist $n \in \mathbb{N}$, and a positive Borel measure μ on $(\mathbb{S}\mathbb{R}^{n\times n})^n$ satisfying

$$y_w = \int w(A)\, d\mu(A)\,? \tag{5}$$

Such a sequence is a *tracial moment sequence*. If one is interested only in *finite* sequences (y_w), then this is the *truncated* tracial moment problem.

To a sequence $y = (y_w)$ one associated the (infinite) *Hankel matrix* $M(y)$, indexed by words, by $M(y)_{u,v} = y_{u^*v}$. One of the results in [BK] shows that if $M(y)$ is positive semidefinite and of finite rank, then y is a tracial moment sequence. In the truncated case a condition called "flatness" governs the existence of a representing measure, much like in the classical case. For details and proofs see [BK].

For the free non-commutative moment problem we refer the reader to [McC01].

6 Algebraic Software

This section briefly surveys existing software dealing with non-commutative convexity (Section 6.1) and positivity (Section 6.2).

6.1 NCAlgebra under Mathematica

Here is a list of software running under NCAlgebra [HdOSM10] (which runs under Mathematica) that implements and experiments on symbolic algorithms pertaining to non-commutative Convexity and LMIs.

NCAlgebra is available from `http://www.math.ucsd.edu/$\sim$ncalg`

- **Convexity Checker.** Camino, Helton, Skelton, Ye [CHSY03] have an (algebraic) algorithm for determining the region on which a rational expression is convex.
- **Classical Production of LMIs.** There are two Mathematica NCAlgebra notebooks by de Oliveira and Helton. The first is based on algorithms for implementing the 1997 approach of Skelton, Iwasaki and Grigonidas [SIG98] associating LMIs to more than a dozen control problems. The second (requires C++ and NCGB) produces LMIs by symbolically implementing the 1997 change of variables method of Scherer et al.
- **Schur Complement Representations of a non-commutative rational.** This computes a linear pencil whose Schur complement is the given nc rational function p using Shopple - Slinglend thesis algorithm. It is not known if p convex near 0 always leads to a monic pencil via this algorithm, but we never saw a counter example.

- **Determinantal Representations.** Finds Determinantal Representations of a given polynomial p. Shopple - Slinglend implement Slinglend's thesis algorithm plus the [HMV06] algorithm. Requires NCAlgebra.

See `http://www.math.ucsd.edu/~ncalg/surveydemo` for occurences of available demos.

6.2 NCSOStools under Matlab

NCSOStools [CKP] which runs under Matlab, implements and experiments on numeric algorithms pertaining to non-commutative positivity and sums of squares. Here is a sample of features available.

- **Non-commuting variables.** Basic symbolic computation with nc variables for Matlab has been implemented.
- **Matrix-positivity.** An nc polynomial p is matrix positive if and only if it is a sum of squares. This can be easily tested using a variant of the classical Gram matrix method. Indeed, $p \in \mathbb{R}\langle x\rangle_{2d}$ is a sum of squares if and only if $p = \langle x\rangle_d^* G \langle x\rangle_d$ for a positive semidefinite G. (Here, $\langle x\rangle_d$ denotes a (column) vector of all words in x of degree $\leq d$.) This can be easily formulated as a feasibility semidefinite program (SDP).
- **Eigenvalue optimization.** Again, using SDP we can compute the smallest eigenvalue $f^\star$ a symmetric $f \in \mathbb{R}\langle x\rangle$ can attain. That is,

  $$f^\star = \inf\{\langle f(A)v, v\rangle : A \text{ a } g\text{-tuple of symmetric matrices, } v \text{ a unit vector}\}.$$

  Hence $f^\star$ is the greatest lower bound on the eigenvalues $f(A)$ can attain for g-tuples of symmetric matrices A, i.e., $(f - f^\star)(A) \succeq 0$ for all n-tuples of symmetric matrices A, and $f^\star$ is the largest real number with this property. Given that a polynomial is matrix positive if and only if it is a sum of squares we can compute $f^\star$ efficiently with SDP:

  $$\begin{array}{rl} f^\star = \sup & \lambda \\ \text{s.t.} & f - \lambda \in \Sigma^2. \end{array}$$

- **Minimizer extraction.** Unlike in the commutative case, if $f^\star$ is attained, then minimizers (A, v) can always be computed. That is, A is a g-tuple of symmetric matrices and v is a unit eigenvector for $f(A)$ satisfying

  $$f^\star = \langle f(A)v, v\rangle.$$

  Of course, in general f will not be bounded from below. Another problem is that even if f is bounded, the infimum $f^\star$ need not be attained. The core ingredient of this minimizer extraction is the nc moment problem governed by a condition calles "*flatness*", together with the GNS construction.

- **Commutators and Cyclic equivalence.** Two polynomials are cyclically equivalent if their difference is a sum of commutators. This is easy to check.
- **Trace-positivity.** The sufficient condition for trace-positivity (i.e., sum of squares up to cyclic equivalence) is tested for using a variant of the Gram matrix method applied to matrix positivity.

NCSOStools is extensively documented and available at

`http://ncsostools.fis.unm.si`

References

[Arv69] W. Arveson. Subalgebras of C^*-algebras. *Acta Mathematica*, 123:141–224, 1969.

[Arv72] W. Arveson. Subalgebras of C^*-algebras. II. *Acta Mathematica*, 128(3-4):271–308, 1972.

[Arv08] W. Arveson. The noncommutative Choquet boundary. *Journal of the American Mathematical Society*, 21(4):1065–1084, 2008.

[BCR98] J. Bochnack, M. Coste, and M.-F. Roy. *Real algebraic geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer–Verlag, Berlin, 1998.

[BGM06] J.A. Ball, G. Groenewald, and T. Malakorn. Bounded real lemma for structured noncommutative multidimensional linear systems and robust control. *Multidimensional Systems And Signal Processing*, 17(2-3):119–150, July 2006.

[BK] S. Burgdorf and I. Klep. The truncated tracial moment problem. To appear in the *Journal of Operator Theory*, `http://arxiv.org/abs/1001.3679`.

[BLM04] D.P. Blecher and C. Le Merdy. *Operator algebras and their modules—an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2004. Oxford Science Publications.

[BMV75] D. Bessis, P. Moussa, and M. Villani. Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *Journal of Mathematical Physics*, 16(11):2318–2325, 1975.

[BTN02] A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12(3):811–833, 2002.

[CHSY03] J.F. Camino, J.W. Helton, R.E. Skelton, and Jieping Ye. Matrix inequalities: a symbolic procedure to determine convexity automatically. *Integral Equation and Operator Theory*, 46(4):399–454, 2003.

[CKP] K. Cafuta, I. Klep, and J. Povh. *NCSOStools: a computer algebra system for symbolic and numerical computation with noncommutative polynomials*. To appear in *Optimization methods and Software*. Available from `http://ncsostools.fis.unm.si`.

[Con76] A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Annals of Mathematics*, 104:73–115, 1976.

[EW97] E.G. Effros and S. Winkler. Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems. *Journal of Functional Analysis*, 144(1):117–152, 1997.

[HdOSM10] J.W. Helton, M.C. de Oliveira, M. Stankus, and R.L. Miller. *NCAlgebra*, 2010 release edition, 2010. Available from `http://math.ucsd.edu/~ncalg`.

[Hel02] J.W. Helton. 'Positive' noncommutative polynomials are sums of squares. *Annals Of Mathematics*, 156(2):675–694, September 2002.

[HHLM08] D.M. Hay, J.W. Helton, A. Lim, and S. McCullough. Non-commutative partial matrix convexity. *Indiana University Mathematics Journal*, 57(6):2815–2842, 2008.

[HKMa] J.W. Helton, I. Klep, and S. McCullough. Analytic mappings between noncommutative pencil balls. Preprint, available from `http://arxiv.org/abs/0908.0742`.

[HKMb] J.W. Helton, I. Klep, and S. McCullough. The matricial relaxation of a linear matrix inequality. Preprint, available from `http://arxiv.org/abs/1003.0908`.

[HKMc] J.W. Helton, I. Klep, and S. McCullough. Proper analytic free maps. Preprint, available from `http://arxiv.org/abs/1004.1381`.

[HKMS09] J.W. Helton, I. Klep, S. McCullough, and N. Slinglend. Noncommutative ball maps. *Journal of Functional Analysis*, 257(1):47–87, 2009.

[HM] J.W. Helton and S. McCullough. Every free basic convex semi-algebraic set has an LMI representation. Preprint, available from `http://arxiv.org/abs/0908.4352`.

[HM03] J.W. Helton and S. McCullough. Convex noncommutative polynomials have degree two or less. *SIAM Journal on Matrix Analysis and Applications*, 25(4):1124–1139, 2003.

[HM04] J.W. Helton and S. McCullough. A positivstellensatz for non-commutative polynomials. *Transactions of The American Mathematical Society*, 356(9):3721–3737, 2004.

[HMP04] J.W. Helton, S.A. McCullough, and M. Putinar. A non-commutative positivstellensatz on isometries. *Journal Für Die Reine Und Angewandte Mathematik*, 568:71–80, March 2004.

[HMP05] J.W. Helton, S. McCullough, and M. Putinar. Non-negative hereditary polynomials in a free *-algebra. *Mathematische Zeitschrift*, 250(3):515–522, July 2005.

[HMP07] J.W. Helton, S. McCullough, and M. Putinar. Strong majorization in a free *-algebra. *Mathematische Zeitschrift*, 255(3):579–596, March 2007.

[HMV06] J.W. Helton, S.A. McCullough, and V. Vinnikov. Noncommutative convexity arises from linear matrix inequalities. *Journal Of Functional Analysis*, 240(1):105–191, November 2006.

[HP07] J.W. Helton and M. Putinar. Positive polynomials in scalar and matrix variables, the spectral theorem and optimization. *Journal Operator Theory*, 2007. Special volume in memory of Tibi Constantinescu. Preprint.

[HV07] J.W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Communications On Pure And Applied Mathematics*, 60(5):654–674, May 2007.

[KS07] I. Klep and M. Schweighofer. A nichtnegativstellensatz for polynomials in noncommuting variables. *Israel Journal of Mathematics*, 161(1):17–27, 2007.

[KS08a] I. Klep and M. Schweighofer. Connes' embedding conjecture and sums of Hermitian squares. *Advances in Mathematics*, 217:1816–1837, 2008.

[KS08b] I. Klep and M. Schweighofer. Sums of Hermitian squares and the BMV conjecture. *Journal of Statistical Physics*, 133(4):739–760, 2008.

[KVV09] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. Singularities of rational functions and minimal factorizations: the noncommutative and the commutative setting. *Linear Algebra and its Applications*, 430(4):869–889, 2009.

[Las01] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[LPR05] A.S. Lewis, P.A. Parrilo, and M.V. Ramana. The Lax conjecture is true. *Proceedings of the American Mathematical Society*, 133(9):2495–2499, 2005.

[McC01] S. McCullough. Factorization of operator-valued polynomials in several non-commuting variables. *Linear Algebra and its Applications*, 326(1-3):193–203, 2001.

[NPA] M. Navascués, S. Pironio, and A. Acín. SDP relaxations for non-commutative polynomial optimization. this volume.

[Par00] P.A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization.* PhD thesis, California Institute of Technology, Passadena, CA, May 2000.

[Pau02] V. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[PD01] A. Prestel and C.N. Delzell. *Positive polynomials. From Hilbert's 17th problem to real algebra.* Springer Monographs in Mathematics. Springer, Berlin, 2001.

[Pis03] G. Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[PNA10] S. Pironio, M. Navascués, and A. Acín. Convergent relaxations of polynomial optimization problems with noncommuting variables. *SIAM Journal on Optimization*, 20(5):2157–2180, 2010.

[Pop09] G. Popescu. Noncommutative transforms and free pluriharmonic functions. *Advances in Mathematics*, 220(3):831–893, 2009.

[Pop10] G. Popescu. Free holomorphic functions on the unit ball of $B(H)^n$. II. *Journal of Functional Analysis*, 258(5):1513–1578, 2010.

[Put93] M. Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42(3):969–984, 1993.

[Sch05] K. Schmüdgen. A strict Positivstellensatz for the Weyl algebra. *Math. Ann.*, 331(4):779–794, 2005.

[SIG98] R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis. *A unified algebraic approach to linear control design.* The Taylor & Francis Systems and Control Book Series. Taylor & Francis Ltd., London, 1998.

[SS] K. Schmüdgen and Yu. Savchuk. Unbounded induced representations of $*$-algebras. Preprint, available from `http://arxiv.org/abs/0806.2428`.

[Voi] D. Voiculescu. Free analysis questions II: The Grassmannian completion and the series expansions at the origin. Preprint, available from http://arxiv.org/abs/0806.0361.

[Voi04] D. Voiculescu. Free analysis questions. I. Duality transform for the coalgebra of $\partial_{X\,:\,B}$. *International Mathematics Research Notices*, 16:793–822, 2004.

[Voi05] D. Voiculescu. Aspects of free probability. In *XIVth International Congress on Mathematical Physics*, pages 145–157. World Sci. Publ., Hackensack, NJ, 2005.